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A note on 4d kination and higher-dimensional uplifts

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Abstract This note expands on the details of the relationship between 4d kination solutions in string cosmology and Kasner solutions of 10d general relativity. It extends previous analyses of this relationship in IIB string theory to other string theories and also to 11d M-theory, while also providing extensive detail on the relationship between perturbations of the 10d Kasner metric and the presence of radiation and matter backgrounds in the dimensionally reduced 4d kination theory.

Contents

1 Introduction

An important task for string theory is to make connections to observational physics. One of the most promising places where such links may be made is in the context of cosmology. The normal approach to this starts through dimensional reduction of string theory into a 4-dimensional effective field theory; all actual calculations are then performed within this 4-dimensional theory and it is only rarely that the underlying 10-dimensional theory is considered explicitly. In this respect, a detailed recent review of string cosmology is [1] (see [2,3] for other reviews with a slightly different focus).

One potential non-standard cosmological epoch in the early universe is that of a kination epoch (the name first appears in [4]), an epoch dominated by the kinetic energy of a rolling scalar field. Such epochs naturally arise in string theory when moduli fields roll down steep potentials, for example when running from the centre of moduli space to the boundaries (a recent detailed analysis of this is [5]; see [6–10] for some other recent studies of kination epochs in string theory).

One intriguing aspect of kination epochs in the context of IIB string compactifications, realised in [11], is that the epoch also admits a direct 10-dimensional description in terms of a higher-dimensional Kasner solution. This provides an interesting contrast to most work in string cosmology, where no such 10-dimensional description exists. This provides one of the few examples where it is possible to go beyond the 4d description and work directly within the 10d theory (another

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famous example where a 10d description exists is the GKP stage of IIB flux compactifications [12], prior to the step of Kähler moduli stabilisation for which [13] and [14] are the best-known examples).

The purpose of this note is to build on the analysis of [11] and explore this link further. There are three sections. In the first, we extend the results of [11] to other string theories beyond the IIB orientifold examples considered there. In the second, we fill in more details on the perturbations of the 10d theory and their connection to moduli backgrounds in the 4d kination theory, going beyond the overall Kasner dynamics associated to the volume mode that was considered in [11]. In the final one, we explore the consequence of a gas of winding modes during the kination epoch.

2 Kination in general string theories with multiple rolling moduli

We first consider kination epochs in more general string scenarios than the type IIB orientifold examples with just a single rolling volume modulus considered in [11]. As a single framework to capture multiple examples, we consider a 4d EFT which contains two moduli, a volume modulus and a dilaton, both with vanishing potential.[1] The kinetic energy for these two moduli are determined by a Kähler potential which takes the form

$$K = -\log s^m u^n, \tag{2.1}$$

where m and n are integers which will vary for different string theories, u is (the real part of) the volume modulus and s is (the real part of) the 4d dilaton (which in general will differ from the 10d dilaton by factors of volume). The canonically normalised moduli are given by

$$S = \sqrt{\frac{m}{2}} \log s, \quad U = \sqrt{\frac{n}{2}} \log u, \tag{2.2}$$

in terms of which the Friedman equation and the scalar equations of motion are

$$\begin{aligned} 6H^2 = 6\left(\frac{\dot{a}}{a}\right)^2 &= \dot{S}^2 + \dot{U}^2, \\ \ddot{S} + 3H\dot{S} &= 0, \\ \ddot{U} + 3H\dot{U} &= 0. \end{aligned} \tag{2.3}$$

These are solved by

$$H(t_E) = \frac{1}{3t_E}, \quad S(t_E) = S_0 + \frac{\alpha}{\sqrt{2/m}} M_p \ln\left(\frac{t_E}{t_0}\right),$$

$$U(t_E) = U_0 + \frac{\beta}{\sqrt{2/n}} M_p \ln\left(\frac{t_E}{t_0}\right), \tag{2.4}$$

subject to the condition

$$m\alpha^2 + n\beta^2 = \frac{4}{3}. \tag{2.5}$$

We write t_E for the time coordinate to be explicit that this refers to a 4d Einstein frame metric. This 4-dimensional theory, in standard 4d Einstein frame, has a kination metric with $a(t_E) \propto t_E^{1/3}$,

$$ds_4^2 = -dt_E^2 + a^2(t_E) dx_i dx^i = -dt_E^2 + t_E^{2/3} dx_i dx^i. \quad i = 1, 2, 3 \tag{2.6}$$

Equations (2.4)–(2.6) describe the kination field and metric evolution in 4d Einstein frame.

In string theory compactifications, this 4d Einstein frame metric is related to the (4d components of the) 10d string frame metric as

$$g_{4d,Einstein} = e^{-2\phi} \mathcal{V} g_{4d,s}. \tag{2.7}$$

Based on this, we expect that the 4d kination solution (in 4d Einstein frame) should correspond to an uplifted 10-dimensional string frame solution of the form,

$$ds_{10}^2 = \left[e^{2\phi(t_E)} \mathcal{V}(t_E)^{-1}\right] \left(-dt_E^2 + t_E^{2/3} dx_i dx^i\right) + \left[\mathcal{V}^{1/3}(t_E)\right] ds_6^2, \tag{2.8}$$

where e^ϕ is the 10d rolling dilaton and $\mathcal{V}$ is the internal volume.

With an appropriate redefinition of the time coordinate, we will confirm that this is equivalent to a Mueller metric [15] which is a generalisation of the Kasner metric of the form

$$ds^2 = -dt^2 + \sum_i t^{2p_i} dx_i^2, \quad e^\phi = t^p \tag{2.9}$$

to include a rolling dilaton, satisfying the conditions

$$\sum_i p_i = 1 + 2p, \quad \sum_i p_i^2 = 1. \tag{2.10}$$

2.1 Kination and its 10d uplift in type IIA string theory

Here we make this connection explicit for type IIA string theory, where the dilaton and volume moduli s and u are defined by (for example, see [16])

$$s = e^{-\phi}\sqrt{\mathcal{V}}, \quad u = \mathcal{V}^{1/3}, \tag{2.11}$$

with the Kähler metric given by

$$K = -\log s^4 u^3. \tag{2.12}$$

Here the dilaton itself directly defines the relationship between the 4d Einstein frame metric and the string frame

[1] Even if the potential does not strictly vanish, if it is steep enough, fields enter a kination epoch as they roll down it and the potential can effectively be neglected.

metric. If t_E denotes the 4d Einstein frame metric with the standard kination scale factor $a(t_E) \sim t_E^{1/3}$, then the corresponding expected uplift of the 4d kination metric to a 10d string frame metric is

$$ds_{10}^2 = [s(t_E)]^{-2}(-dt_E^2 + a^2(t_E)dx_n dx^n) + [u(t_E)]ds_6^2. \tag{2.13}$$

As $s(t_E) \sim t_E^{\alpha}$ and $u(t_E) \sim t_E^{\beta}$, this gives

$$\begin{aligned} ds_{10}^2 &= -t_E^{-2\alpha} dt_E^2 + t_E^{\frac{2(1-3\alpha)}{3}} dx_n dx^n + t_E^{\beta} ds_6^2 \\ &= -d\tau^2 + \tau^{\frac{2(1-3\alpha)}{3(1-\alpha)}} dx_n dx^n + \tau^{\frac{\beta}{1-\alpha}} ds_6^2, \end{aligned} \tag{2.14}$$

where $\tau = t_E^{1-\alpha}$. The string coupling e^{ϕ} also depends on time as follows

$$e^{\phi(t_E)} = u(t_E)^{3/2} s(t_E)^{-1} = t_E^{\frac{1}{2}(3\beta-2\alpha)} = \tau^{\frac{3\beta-2\alpha}{2(1-\alpha)}}. \tag{2.15}$$

Equations (2.14) and (2.15) indeed represent a Mueller solution with

$$p_1 = \frac{1-3\alpha}{3(1-\alpha)}, \quad p_2 = \frac{\beta}{2(1-\alpha)}, \quad p = \frac{3\beta - 2\alpha}{2(1-\alpha)}. \tag{2.16}$$

These satisfy

$$\begin{aligned} &3p_1 + 6p_2 - 2p = 1, \\ &3p_1^2 + 6p_2^2 = 1, \end{aligned} \tag{2.17}$$

if and only if

$$4\alpha^2 + 3\beta^2 = \frac{4}{3}, \tag{2.18}$$

which is precisely the condition (2.5) applicable to the 4d kination solution. The α and β parameters are arbitrary (subject to the condition of Eq. (2.5)) and so can represent kinetic evolution in the direction of either or both of changing volume and changing dilaton.

2.2 Kination and its 10d uplift for other string theories

We can proceed similarly for other string theories, likewise obtaining 10d Mueller metrics for each case as the uplift of a kination solution involving rolling dilaton and volume moduli in the 4d effective field theory. The results are summarized in Table 1.

2.3 The Mueller solution and the 11d M-theory Kasner solution

Type IIA string theories have a direct uplift to M-theory, interpreting the string coupling as the size of the 11th dimensions (with strong string coupling corresponding to large radii). In this case, the 10d rolling dilaton Mueller solution of Eqs. (2.9) and (2.10), uplifted to an 11-dimensional M-theory solution, becomes

$$\begin{aligned} ds_{11}^2 &= e^{-\frac{2\phi}{3}} \left[-dt^2 + \sum_{i=1}^{9} t^{2p_i} dx_i^2 \right] + e^{\frac{4\phi}{3}} dz^2 \\ &= -t^{-\frac{2p}{3}} dt^2 + \sum_{i=1}^{9} t^{2p_i - \frac{2}{3}p} dx_i^2 + t^{\frac{4}{3}p} dz^2 \\ &= -d\tau^2 + \sum_{i=1}^{9} t^{\frac{2p_i - 2p/3}{1-p/3}} dx_i^2 + t^{\frac{4p/3}{1-p/3}} dz^2 \\ &\equiv -d\tau^2 + \sum_{i=1}^{9} \tau^{2\tilde{p}_i} dx_i^2 + \tau^{2\tilde{p}_{10}} dz^2, \end{aligned} \tag{2.19}$$

where z is the coordinate on the M-theory circle, τ is defined as $\tau = t^{1-p/3}$ and numerical prefactors are dropped.

This is indeed an 11d Kasner solution as from the first Mueller condition $\sum_{i=1}^{9} p_i = 1 + 2p$, it follows that

$$\sum_{i=1}^{10} \tilde{p}_i = 1. \tag{2.20}$$

Combining this with the second 10d condition $\sum_{i=1}^{9} p_i^2 = 1$, we also get

$$\sum_{i=1}^{10} \tilde{p}_i^2 = 1. \tag{2.21}$$

3 Perturbations of 10d Kasner solutions and their relation to 4d moduli backgrounds

Given the interest of Kasner solutions as a 10d completion of kination solutions, it is worthwhile understanding their perturbations better (the paper [17] describes gravitational waves in the context of Kasner spacetimes including the Unruh effect). From a 4-d perspective, such perturbations correspond to the presence of additional matter/radiation content in the 4d theory.

In this section we give a detailed analysis of the relationship between perturbations of the 10d Kasner solution and the moduli fields that are present in the 4d kination theory (the time-dependent nature of the Kasner solution is what differentiates this from the familiar flat space treatment).

3.1 10d Perturbations of the Kasner solution

We start with an analysis of the 10d solution. Anticipating Sect. 3.2 where we relate perturbations of the Kasner metric to moduli backgrounds in the kination solution, it is useful to express 10d quantities in terms of the kination conformal time

Table 1 10d uplift of the 4d kination metric $ds_4^2 = -dt^2 + t^{\frac{2}{3}} dx_i dx^i$ in different string theories with rolling dilaton and volume moduli: type IIA, type IIB heterotic, and type I string theory

Type IIA	$m=4, n=3$	$s = e^{-\phi}\mathcal{V}^{1/2}$	$u = \mathcal{V}^{1/3}$
	$p_1 = \frac{1-3\alpha}{3(1-\alpha)}, \quad p_2 = \frac{\beta}{2(1-\alpha)}, \quad p = \frac{3\beta-2\alpha}{2(1-\alpha)}$		
Type IIB	$m=1, n=3$	$s = e^{-\phi}$	$u = \mathcal{V}^{2/3}e^{-\phi}$
	$p_1 = \frac{3\alpha+9\beta-4}{3(\alpha+3\beta-4)}, \quad p_2 = \frac{\alpha-\beta}{\alpha+3\beta-4}, \quad p = \frac{4\alpha}{\alpha+3\beta-4}$		
Heterotic	$m=1, n=3$	$s = e^{-2\phi}\mathcal{V}$	$u = \mathcal{V}^{1/3}$
	$p_1 = \frac{2/3-\alpha}{2-\alpha}, \quad p_2 = \frac{\beta}{2-\alpha}, \quad p = \frac{3\beta-\alpha}{2-\alpha}$		
Type I	$m=1, n=3$	$s = e^{-\phi}\mathcal{V}$	$u = e^{-\phi}\mathcal{V}^{1/3}$
	$p_1 = \frac{1}{3}\frac{4-3(3\alpha+\beta)}{4-3\alpha-\beta}, \quad p_2 = \frac{2(\beta-\alpha)}{4-3\alpha-\beta}, \quad p = \frac{2(\beta-3\alpha)}{4-3\alpha-\beta}$		

η.[2] This relates to the time t of the 10-dimensional Kasner solution through $\eta = \frac{3}{4}t^{4/3}$, in which case the Kasner metric takes the form

$$ds^2_{\text{Kasner}} = \sqrt{\frac{3}{4\eta}}\left(-d\eta^2 + \sum_{i=1}^{3} dx_i^2\right) + \sqrt{\frac{4\eta}{3}}\sum_{\alpha=1}^{6} dy_\alpha^2. \tag{3.1}$$

Here x_i are the large noncompact coordinates, while y_j are the small compact coordinates. Working in the temporal gauge, where the off-diagonal temporal perturbations vanish, the massless perturbations of this 10-dimensional metric decouple at leading order into three types. From a lower-dimensional perspective, these correspond to complex structure perturbations, volume modulus perturbations, and radiation perturbations.

Complex structure perturbations

We first consider perturbations in the torus complex structure, expanding in a small parameter λ, which in general take the form

$$\begin{aligned} ds^2 = ds^2_{\text{Kasner}} &+ \lambda f(\eta, \boldsymbol{x}) dy_\alpha^2 + \lambda g(\eta, \boldsymbol{x}) dy_{\alpha+1}^2 \\ &+ 2\lambda h(\eta, \boldsymbol{x}) dy_\alpha dy_{\alpha+1}. \end{aligned} \tag{3.2}$$

To leading order in λ, the Einstein equations for diagonal and off-diagonal contributions decouple. Additionally, the only non-trivial position dependent solution for the diagonal components must satisfy $g(\eta, \boldsymbol{x}) = -f(\eta, \boldsymbol{x})$ (so that this is a complex structure perturbation and not a volume perturbation). The remaining equation of motion for both the diagonal and off-diagonal perturbations is

$$\nabla^2 f - \partial_\eta^2 f - \frac{1}{4\eta^2} f = 0, \tag{3.3}$$

with the same equation applying for $h(\eta, \boldsymbol{x})$. These have exact solutions in terms of Bessel functions,

$$f(\eta, \boldsymbol{x}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\sqrt{\eta}(c_1 J_0(k\eta) + c_2 Y_0(k\eta)), \tag{3.4}$$
$$h(\eta, \boldsymbol{x}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\sqrt{\eta}(d_1 J_0(k\eta) + d_2 Y_0(k\eta)), \tag{3.5}$$

where $k = |\boldsymbol{k}|$.

Volume modulus perturbations

To find volume modulus perturbations, we consider an ansatz of the form

$$\begin{aligned} ds^2 = ds^2_{\text{Kasner}} &+ \frac{3\lambda}{4\eta}\sum_{i,j=1}^{3} f_{ij}(\eta, \boldsymbol{x}) dx_i dx_j \\ &+ \lambda f(\eta, \boldsymbol{x}) \sum_{\alpha=1}^{6} m_\alpha dy_\alpha^2, \end{aligned} \tag{3.6}$$

with λ again a small numerical parameter and where m_α are numerical coefficients, with

$$f_{ij}(\eta, \boldsymbol{x}) = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}(\eta, \boldsymbol{x}). \tag{3.7}$$

This will be a volume perturbation of the compact tori if $\sum_\alpha m_\alpha \neq 0$. An analytic solution can be acquired by looking at late-time wavelike solutions, equivalent to large k wave solutions, where the highest order derivative terms dominate. The nonvanishing components of Einstein's vacuum equations then reduce down to three sets

$$\partial_\eta^2[(\Sigma_\alpha m_\alpha) f + \sum_i f_{ii}] = 0, \tag{3.8}$$

$$\begin{aligned} &\nabla^2 f_{ii} - \partial_\eta^2 f_{ii} + \partial_i^2\left[(\Sigma_\alpha m_\alpha) f + \sum_j f_{jj}\right] \\ &-2\sum_j \partial_i \partial_j f_{ij} = 0, \end{aligned} \tag{3.9}$$

$$\nabla^2 f - \partial_\eta^2 f = 0. \tag{3.10}$$

[2] As field perturbations in 4d theories are most naturally expressed in terms of the conformal time.

Equation 3.10 fixes $f(\eta, \boldsymbol{x})$ to the standard wave solution

$$f(\eta, \boldsymbol{x}) = Ae^{ik\eta + i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{3.11}$$

The remaining equations are solved by taking

$$f_{ij}(\eta, \boldsymbol{x}) = H_{ij}e^{ik\eta + i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{3.12}$$

where H_{ij} is now an amplitude matrix. Equation 3.8 requires that $\mathrm{tr}\, H = -(\Sigma_\alpha m_\alpha)A$, while Eq. 3.9 implies that $H_{ij}k_j = 0$. This shows that perturbations in the volume modulus are necessarily accompanied by perturbations in the noncompact part of the metric. During compactification and dimensional reduction, these noncompact perturbations correspond to the 4d dilaton, which is then re-absorbed into the final definition of the noncompact Einstein frame metric.

Radiation perturbations

Finally, consider the cross-coupled off-diagonal perturbation

$$ds^2 = ds^2_{\text{Kasner}} + \frac{2\lambda}{\sqrt{\eta}}\sum_i f_i(\eta, \boldsymbol{x})dx_i dy_\alpha \tag{3.13}$$

for some α. As per the original 5d Kaluza–Klein compactification, this would correspond to a $U(1)$ radiation mode in the 4d theory. The resulting equations to first order in λ are given by

$$\sum_i \partial_i[f_i - \eta\partial_\eta f_i] = 0, \tag{3.14}$$

$$\nabla^2 f_i - \partial_\eta^2 f_i - \sum_j \partial_i\partial_j f_j = 0. \tag{3.15}$$

This has an exact solution of the form

$$f_i(\eta, \boldsymbol{x}) = A_i e^{ik\eta + ik\cdot x}, \quad \boldsymbol{A}\cdot\boldsymbol{k} = 0, \tag{3.16}$$

where the polarization condition is equivalent to $\Sigma_i \partial_i f_i = 0$.

As a check on the above three types of perturbation, we can count the number of degrees of freedom. In the temporal gauge there are a total of 45 free metric components. The equations of motion impose six polarization conditions on the radiation perturbations $\boldsymbol{A}_\alpha\cdot\boldsymbol{k} = 0$, a single trace condition on volume perturbations $\sum_\alpha m_\alpha = -\,\mathrm{tr}\, H$, and three polarization conditions on the noncompact gravitational waves $k_i H_{ij} = 0$. This leaves $45 - 6 - 3 - 1 = 35$ degrees of freedom, matching the number of degrees of freedom of a graviton in ten dimensions.

Massive Kaluza–Klein perturbations

We can also look at perturbations of the Kasner metric which would correspond to massive modes from a 4-dimensional perspective. Such massive KK modes can be obtained using the metric

$$ds^2 = ds^2_{\text{Kasner}} + \frac{2\lambda}{\sqrt{\eta}}e^{2\pi i n\cdot y}f(\eta, \boldsymbol{x})\sum_{i=1}^{3}\sum_{\alpha=1}^{6} m_{i\alpha}dx_i dy_\alpha, \tag{3.17}$$

for some vector n_α. This has wavelike solutions with the late-time behaviour

$$f(\eta, \boldsymbol{x}) = Ae^{ik\eta + i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{3.18}$$

together with the conditions $\Sigma_i k_i m_{i\alpha} = 0$ for all α and $\Sigma_\alpha m_{i\alpha}n_\alpha = 0$ for all i.

3.2 10d Perturbations as 4d fields on the kination background

When expanding around a 10-dimensional background metric $\bar{g}_{MN}$ that solves the Einstein equation as $g_{MN} = \bar{g}_{MN} + H_{MN}$, then up to cubic contributions $\mathcal{O}(H^3)$ the Einstein–Hilbert action can be rewritten as the Fierz–Pauli action in curved spacetime [18,19]

$$\begin{aligned} S = -\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-\bar{g}}\Big[&\frac{1}{4}\bar{\nabla}_M H_{AB}\bar{\nabla}^M H^{AB} \\ &-\frac{1}{2}\bar{\nabla}_M H_{AB}\bar{\nabla}^A H^{MB} \\ &+\frac{1}{2}\bar{\nabla}_M H^{MN}\bar{\nabla}_N H - \frac{1}{4}\bar{\nabla}_M H\bar{\nabla}^M H\Big], \end{aligned} \tag{3.19}$$

where $H = H_A{}^A$. In this case, the background metric is the Kasner metric. The bar over the covariant derivative indicates that this is the ten dimensional covariant derivative as opposed to the four dimensional covariant derivative that will appear after compactification.

It is convenient to rewrite the perturbations as

$$\begin{aligned} H_{MN} &\equiv \begin{pmatrix} h_{\mu\nu} - g_{\mu\nu}\phi & \zeta A_{\mu i} \\ \zeta A_{\nu j} & 2\zeta\phi_{ij} \end{pmatrix}, \\ H^{MN} &\equiv \begin{pmatrix} h^{\mu\nu} - g^{\mu\nu}\phi & \zeta A^{\mu i} \\ \zeta A^{\nu j} & 2\zeta\phi^{ij} \end{pmatrix}, \end{aligned} \tag{3.20}$$

where $\phi = \sum_i \phi_{ii}$, $g_{\mu\nu} = \eta_{\mu\nu}\zeta^{-1}$, and $\zeta = \sqrt{4\eta/3}$. The Kasner compactification differs from a standard compactification on a torus by the time-dependent factors of ζ [20].

To extract an action for the massless perturbations, we assume that the perturbations are independent of the compact coordinates. In this case, we can directly integrate over the compact manifold to acquire a volume term $\mathcal{V}_6 = \mathcal{V}_{6,0}\zeta^3$. Meanwhile, expanding the Fierz–Pauli action yields what at first appears to be a mass term for the gauge field. However, this ends up forming part of a total derivative,

$$\frac{\mathcal{L}}{\sqrt{-g}} \supset \frac{2}{9}\zeta^2 A^2 - \frac{2}{3}\zeta^4 A^{\nu i}\nabla_0 A_{\nu i} \tag{3.21}$$

$$= -\frac{2}{9}\zeta^2 A^2 - \frac{2}{3}\left(\frac{1}{2}\nabla_0(\zeta^4 A^{\nu i} A_{\nu i}) - \zeta^2 A^2\right) \quad (3.22)$$
$$= -\frac{2}{9}\zeta^2 A^2 - \frac{2}{3}\left(\frac{1}{2}\nabla_\mu X^\mu - \frac{1}{3}\zeta^2 A^2\right) \quad (3.23)$$
$$= -\frac{1}{3}\nabla_\mu X^\mu, \quad (3.24)$$

where $A^2 = A^{\nu i}A_{\nu i}$ and $X^\mu = \delta_0^\mu \zeta^4 A^2$. Dropping this total derivative leaves, as expected, a massless gauge field. A similar argument shows that the terms $\phi\nabla_0\phi$ and $\phi^{ij}\nabla_0\phi_{ij}$ also drop out the action as total derivatives.

The resulting Lagrangian is given by

$$\begin{aligned}\mathcal{L} = \sqrt{-g}\mathcal{V}_{6,0}\zeta^3\Big[&\tfrac{1}{4}\partial_\mu h\partial^\mu h - \tfrac{1}{2}\nabla_\mu h^{\mu\alpha}\nabla_\alpha h\\ &+\tfrac{1}{2}\nabla_\mu h_{\nu\rho}\nabla^\nu h^{\mu\rho}\\ &-\tfrac{1}{4}\nabla_\mu h_{\nu\rho}\nabla^\mu h^{\nu\rho}\\ &-\tfrac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \tfrac{1}{4}\zeta^2 g^{ij}F_{i,\mu\nu}F_j^{\mu\nu}\\ &-\zeta^2 g^{ik}g^{jp}\nabla_\mu\phi_{ij}\nabla^\mu\phi_{kp}\\ &-\tfrac{4}{3}\zeta^{-1}\phi\nabla_0 h + \zeta^{-1}h_0^\mu\nabla_\mu h - \tfrac{8}{3}\zeta^{-1}h_0^\mu\nabla_\mu\phi\Big], \end{aligned} \quad (3.25)$$

where $h = h^\mu{}_\mu$ and $F_{i,\mu\nu}$ are the field-strength tensors for the $A_{i,\mu}$ gauge fields. To go into the Einstein frame requires performing a Weyl transformation on the metric $g_{\mu\nu} \to \mathcal{V}_6^{-1}g_{\mu\nu}$ and its perturbations $h_{\mu\nu}$, giving

$$\begin{aligned}S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[&\tfrac{1}{4}\partial_\mu h\partial^\mu h\\ &-\tfrac{1}{2}\nabla_\mu h^{\mu\alpha}\nabla_\alpha h + \tfrac{1}{2}\nabla_\mu h_{\nu\rho}\nabla^\nu h^{\mu\rho} - \tfrac{1}{4}\nabla_\mu h_{\nu\rho}\nabla^\mu h^{\nu\rho}\\ &-\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{4\tilde{g}^2}\sum_{i=1}^{6}F_i^2\\ &-\sum_{ij}\partial_\mu\phi_{ij}\partial^\mu\phi_{ij}\\ &-\tfrac{4}{3}\phi B^\mu\partial_\mu h + B^\mu h_\mu^\nu\partial_\nu h - \tfrac{8}{3}B^\mu h_\mu^\nu\partial_\nu\phi\Big], \end{aligned} \quad (3.26)$$

where $B^\mu = (\mathcal{V}_{6,0}^{-1}\zeta^{-4}, 0, 0, 0)$ is a non-dynamical background vector field that goes to zero at late times and $\tilde{g}^2 = \zeta^{-1}\mathcal{V}_6^{-1} \sim \eta^{-2}$ is a time-dependent coupling for the gauge field.

In order to simplify the equations, we next separate out the 4d dilaton from the complex structure scalars ϕ_{ij}, which can be done via a field redefinition

$$\begin{aligned}\varphi &= \tfrac{2}{\sqrt{3}}\phi,\\ \Delta_1 &= \phi_{11} - \phi_{22},\\ \Delta_2 &= \phi_{33} - \phi_{44},\\ \Delta_3 &= \phi_{55} - \phi_{66},\\ \Delta_4 &= \tfrac{1}{\sqrt{2}}(\phi_{33} + \phi_{44} - \phi_{11} - \phi_{22}),\\ \Delta_5 &= \tfrac{1}{\sqrt{6}}(2\phi_{33} + 2\phi_{44} - \phi_{11} - \phi_{22} - \phi_{55} - \phi_{66}). \end{aligned} \quad (3.27)$$

The final form of the 4d Einstein frame action is then given by

$$\begin{aligned}S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[&\tfrac{1}{4}\partial_\mu h\partial^\mu h - \tfrac{1}{2}\nabla_\mu h^{\mu\alpha}\nabla_\alpha h\\ &+\tfrac{1}{2}\nabla_\mu h_{\nu\rho}\nabla^\nu h^{\mu\rho} - \tfrac{1}{4}\nabla_\mu h_{\nu\rho}\nabla^\mu h^{\nu\rho}\\ &-\tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \tfrac{1}{4\tilde{g}^2}\sum_{i=1}^{6}F_i^2\\ &-\tfrac{1}{2}\sum_{i=1}^{5}\partial_\mu\Delta_i\partial^\mu\Delta_i - \sum_{i\neq j}\partial_\mu\phi_{ij}\partial^\mu\phi_{ij}\\ &-\tfrac{2}{3}\varphi B^\mu\partial_\mu h + B^\mu h_\mu^\nu\partial_\nu h - \tfrac{4}{3}B^\mu h_\mu^\nu\partial_\nu\varphi\Big]. \end{aligned} \quad (3.28)$$

At small scales or at late times the last line in action goes to zero since $B^\mu \to 0$, while the first line reduces to the leading order perturbation of the Ricci scalar around a flat metric since the covariant derivatives reduce to regular derivatives. This corresponds to letting the Hubble scale become small relative to the energy scale associated with the moduli fields in which case the perturbations behave like perturbations in a standard toroidal compactification.

Gauge transformations in 10d and 4d

The ten dimensional gauge transformation of the Fierz–Pauli action

$$H_{MN} \to H_{MN} + \nabla_M\xi_N + \nabla_N\xi_M, \quad (3.29)$$

results, after the Weyl transformation, in the simultaneous gauge transformations

$$\begin{aligned}h_{\mu\nu} &\to h_{\mu\nu} + \zeta^3\nabla_\mu\xi_\nu + \zeta^3\nabla_\nu\xi_\mu - 2\zeta^{-1}g_{\mu\nu}\xi_0,\\ A_{\mu i} &\to A_{\mu i} + \zeta^{-1}\partial_\mu\xi_i - \tfrac{2}{3}\zeta^{-3}\delta_{\mu 0}\xi_i,\\ \varphi &\to \varphi - \tfrac{4}{\sqrt{3}}\zeta^{-1}\xi_0. \end{aligned} \quad (3.30)$$

The ξ_i components result in gauge transformations for the gauge field and leave the F_i^2 term invariant. Meanwhile, the ξ_μ transformations act nontrivially on the 4d dilaton and graviton. In the appendix, we illustrate the explicit gauge invariance of 4d action for one of these transformations.

Equations of motion

In conformal time, the 4d kination metric takes the form

$$ds^2_{\text{Kination}} = \frac{4\eta}{3}(-d\eta^2 + dx_i^2). \quad (3.31)$$

In the late-time limit where the mixing between the 4d dilaton and the graviton, the last line of (3.28), become negligible, all massless scalar fields satisfy the equation of motion

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = 0, \tag{3.32}$$

which gives the equation

$$\partial_\eta^2\phi + \eta^{-1}\partial_\eta\phi = \nabla^2\phi. \tag{3.33}$$

This equation can be solved in terms of Bessel functions to give

$$\phi(\eta, \boldsymbol{x}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}(c_1 J_0(k\eta) + c_2 Y_0(k\eta)). \tag{3.34}$$

For the massless gauge field arising from dimensional reduction of the mixed off-diagonal compact-noncompact part of the metric, its equation of motion in the temporal gauge is given by

$$\tfrac{2}{\eta}\partial_0 A_i + \partial_0^2 A_i - \nabla^2 A_i = 0, \tag{3.35}$$

which has a solution of the form

$$A_i(\eta, \boldsymbol{x}) = \tfrac{C_i}{\eta}e^{ik\eta + i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{3.36}$$

together with $\boldsymbol{C}\cdot\boldsymbol{k} = 0$. The η^{-1} prefactor arises due to the time-dependent coupling constant $\tilde{g}$. It can be eliminated through a field redefinition at the expense of introducing terms in the action that are negligible in the late time/small distance limit.

Energy density

The energy density of the massless KK gauge fields for a normal observer $n_\mu = (-\zeta, 0, 0, 0)$ can be calculated from the action by deriving the stress–energy tensor

$$\rho = n_\mu n_\nu T^{\mu\nu} \sim -\frac{\zeta^2}{\tilde{g}^2}(F^{0\mu}F^0{}_\mu - \tfrac{1}{4}g^{00}F^2). \tag{3.37}$$

Plugging in our solution to the gauge field and averaging over the oscillations yields a energy density that behaves as

$$\rho \propto c_1 k^2 a^{-4} + c_2 a^{-8}. \tag{3.38}$$

This shows that the energy density indeed behaves as radiation for high energy modes $k \gg 1$ (which are well within the horizon and admit an ordinary particle interpretation) but behaves as $\rho \propto a^{-8}$ for low energy modes (outside the horizon). A similar calculation for the scalar field perturbations shows that at late times these also behave as radiation $\rho \propto k^2 a^{-4}$.

In the string frame the mass of heavy KK modes goes as $m_{KK} \propto \zeta^{-1/2}$ which results in a mass in the Einstein frame of

$$m_{KK} \propto \mathcal{V}_6^{-2/3} \propto a^{-2}. \tag{3.39}$$

The winding modes in the meantime have a time dependence of the form $m_w \propto \zeta^{1/2}$, which in the Einstein frame is

$$m_w \propto \mathcal{V}_6^{-1/3} \propto a^{-1}. \tag{3.40}$$

The energy density goes as $\rho = nm$ where n is the number density that scales as the inverse volume, giving

$$\rho_{KK} \propto a^{-5}, \quad \rho_w \propto a^{-4}. \tag{3.41}$$

We thus see that winding mode states decay as radiation while KK modes decay even faster (a consequence of the fact that the mass of these particles also evolves with time and so these heavy modes cannot be treated as simply dust).

4 Winding mode domination

Modes with non-trivial winding in the extra dimensions are an interesting example of states whose mass grows relative to the string scale as the size of the extra dimensions increases (note that although non-toroidal Calabi–Yau compactifications do not allow for winding modes with integer quantum numbers, discrete $\mathbb{Z}_2$ winding modes are still possible). We explore here the evolution of the 10-dimensional Kasner solution in the presence of a gas of these. We assume that the compact dimensions contain an averaged gas of such modes, such that we can roughly deal with the effect on the winding strings semi-classically. We also assume, for simplicity, a toroidal geometry for the extra dimensions.

Classical cosmic strings with tension μ pointing in the y_6 direction have a stress energy tensor of

$$T^\mu{}_\nu = \mu\,\mathrm{diag}[1, 0, 0, 0, \ldots, -1]\delta(x_1)\delta(x_2)\cdots\delta(y_5). \tag{4.1}$$

If we have a large number of such winding modes with a number density n along each direction that we can average over, and assuming a metric of the form

$$ds_{10}^2 = -dt^2 + p(t)d\boldsymbol{x}^2 + q(t)d\boldsymbol{y}^2, \tag{4.2}$$

then the indices-lowered stress energy tensor is

$$T_{\mu\nu} = \mu n\,\mathrm{diag}[6, 0, 0, 0, -q, -q, -q, -q, -q, -q]. \tag{4.3}$$

Writing $p(t) = e^{\alpha(t)}$ and $q(t) = e^{\beta(t)}$, the 10d Einstein equations reduce to

$$\dot{\alpha}^2 + 6\dot{\alpha}\dot{\beta} + 5\dot{\beta}^2 = 64\pi\mu n_0 e^{-\frac{3}{2}\alpha - \frac{5}{2}\beta}, \tag{4.4}$$

$$3\dot{\alpha}^2 + 12\dot{\alpha}\dot{\beta} + 21\dot{\beta}^2 + 4\ddot{\alpha} + 12\ddot{\beta} = 0, \tag{4.5}$$

$$6\dot{\alpha}^2 + 15\dot{\alpha}\dot{\beta} + 15\dot{\beta}^2 + 6\ddot{\alpha} + 10\ddot{\beta} = 32\pi\mu n_0 e^{-\frac{3}{2}\alpha - \frac{5}{2}\beta}. \tag{4.6}$$

These allow us to solve for α and β along with fixing $n = n_0/\sqrt{p^3 q^5}$, which is the scaling relation required for winding modes whose number density does not change in the direction that they are winding.

These equations can be solved for in the early and late-time limits. Early times with $n_0 \sim 0$ give a 10 Kasner/4d kination solution. Numerically one can see that at late times $\dot{\alpha} = 3\dot{\beta}$, which holds for any n_0. This then implies that $\ddot{\beta} = -7\dot{\beta}^2/2$ and $\ddot{\alpha} = -7\dot{\alpha}^2/6$. With this the equations simplify to

$$\ddot{\beta} = 2\pi \mu n, \tag{4.7}$$

with $\beta = \alpha/3 - c/3$, where c is an integration constant. We can then solve these equations to get

$$q(t) = \left[\frac{7}{2}\sqrt{2\pi \mu n_0 e^{-3c}}\, t + \tilde{c}\right]^{2/7}, \tag{4.8}$$

$$p(t) = e^c \left[\frac{7}{2}\sqrt{2\pi \mu n_0 e^{-3c}}\, t + \tilde{c}\right]^{6/7}. \tag{4.9}$$

Roughly, the late-time behaviour here is $q(t) \sim t^{2/7}$ and $p(t) \sim t^{6/7}$.

Compactification of the metric gives an Einstein frame metric of

$$ds_E^2 = -q^3 dt^2 + q^3 p d\boldsymbol{x}^2, \tag{4.10}$$

hence $a(t) = \sqrt{q^3 p} \sim t^{6/7}$. To get the metric into the FRW form we need to perform a change of temporal coordinates $\tau = f(t)$ with $q^3 = \dot{f}^2$. Then the Hubble constant is defined by

$$H(\tau) = \frac{1}{a}\frac{d}{d\tau}a(\tau) = \frac{\dot{a}}{a}\frac{d\tau}{dt} = \frac{\dot{a}}{a\dot{f}} = \frac{\dot{a}}{aq^{3/2}}. \tag{4.11}$$

The energy density during this winding mode domination behaves as $\rho \propto H^2$, giving

$$\rho \propto a^{-10/3}. \tag{4.12}$$

Notice that this scales slower than radiation but faster than matter. The same result can also be understood more directly by noting that the energy density of winding modes behaves as $\rho = m_w n_3 \propto m_w/a^3$. Compactification of the action shows that the 10d mass $m_{10} = \mathcal{V}^{1/6}\mu$ gets rescaled due to the Weyl transformation to $m_w = \mu \mathcal{V}^{-1/3} \propto q^{-1}$. Since at late times $q \propto a^{1/3}$, this gives the same energy density behaviour.

From the Friedmann equation one can see that the resulting winding mode domination cosmology is equivalent to one with a cosmological fluid with $w = 1/9$ (although we would not expect such modes to remain stable; they should ultimately decay to lighter modes).

5 Conclusions

This note has extended the previous work of [11] on the relationship between kination and Kasner solutions in type IIB compactifications. Here we have provided much more detail on the kination-Kasner relationship. In the first part of this paper, we extended the earlier analysis of [11] to multiple moduli and other corners of the M-theory duality web, such as type I, heterotic, IIA and also M-theory.

As $\rho_{kin} \sim a^{-6}$ whereas $\rho_\gamma \sim a^{-4}$, kination backgrounds are unstable against a small initial radiation fraction. Moduli excitations in the 4d theory correspond to perturbations of the 10d metric. To understand these better, in the second part of the paper we went into very explicit detail on the correspondence between perturbations of the 10d Kasner solution and the moduli fields in the 4d kination solution.

Post-inflationary kination epochs are well motivated in string theory and offer an appealing way to modify the standard cosmology. By providing more explicit detail on them, and their relationship to higher-dimensional Kasner solutions, in this note we hope to have provided foundations that can subsequently be used to improve our understanding of such epochs, with the ultimate goal of extracting testable predictions.

Key to contact with data would be observational signals that would give clear evidence of a kination epoch in the very early universe. The ideal form of such evidence would be a legacy that could originate in the kination epoch and survive to the present day. Gravitational waves are probably the best candidate for such a legacy: the universe remains transparent to gravitational waves throughout its existence, and so a primordial gravitational wave signal could survive to the present day from an early kination epoch. Furthermore, as kination enhances the relevant fraction of radiation, gravitational waves from a kination epoch would be blueshifted (which would, potentially, be the signal of the kination epoch).

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Appendix

Here we provide various formulae that may be useful as reference but would obstruct the flow of the main text.

Formulae for covariant derivatives of the metric

Here we include various formulae for the covariant derivatives of the metric that are useful for summing the terms in the Fierz–Pauli action and deriving Eq. 3.25.

$$\begin{aligned}
\bar\nabla_\mu H_{\nu\rho} &= \nabla_\mu h_{\nu\rho} - g_{\nu\rho}\nabla_\mu\phi,\\
\bar\nabla_\mu H_{\nu i} &= \zeta\nabla_\mu A_{\nu i} - \tfrac{1}{3}g_{\mu 0}A_{\nu i}\\
\bar\nabla_\mu H_{ij} &= 2\zeta\nabla_\mu\phi_{ij},\\
\bar\nabla_i H_{\mu\nu} &= \partial_i h_{\mu\nu} - g_{\mu\nu}\partial_i\phi + \tfrac{1}{3}g_{\mu 0}A_{\nu i} + \tfrac{1}{3}g_{\nu 0}A_{\mu i},\\
\bar\nabla_i H_{j\mu} &= \zeta\partial_i A_{\mu j} + \tfrac{1}{3}g_{\mu 0}[\phi\zeta^{-1}g_{ij} + 2\phi_{ij}],\\
\bar\nabla_i H_{jk} &= 2\zeta\partial_i\phi_{jk},
\end{aligned} \tag{A.1}$$

and

$$\begin{aligned}
\bar\nabla^\mu H^{\nu\rho} &= \nabla^\mu h^{\nu\rho} - g^{\nu\rho}\nabla^\mu\phi,\\
\bar\nabla^\mu H^{\nu i} &= \zeta\nabla^\mu A^{\nu i} - \delta^\mu_0 A^{\nu i} = \zeta g^{ij}\nabla^\mu A^\nu_j - \tfrac{1}{3}\delta^\mu_0 A^{\nu i}\\
\bar\nabla^\mu H^{ij} &= 2\zeta g^{ik}g^{jm}\nabla^\mu\phi_{km},\\
\bar\nabla^i H^{\mu\nu} &= \partial^i h^{\mu\nu} - g^{\mu\nu}\partial^i\phi + \tfrac{1}{3}\delta^\mu_0 A^{\nu i} + \tfrac{1}{3}\delta^\nu_0 A^{\mu i},\\
\bar\nabla^i H^{j\mu} &= \zeta\partial^i A^{\mu j} + \tfrac{1}{3}\delta^\mu_0[\phi\zeta^{-1}g^{ij} + 2\phi^{ij}],\\
\bar\nabla^i H^{jk} &= 2\zeta\partial^i\phi^{jk}.
\end{aligned} \tag{A.2}$$

Next, $H = h - 2\phi$, where $h = \zeta\eta^{\mu\nu}h_{\mu\nu} = g^{\mu\nu}h_{\mu\nu}$, and so

$$\begin{aligned}
-\tfrac{1}{4}\bar\nabla_M H\bar\nabla^M H &= -\tfrac{1}{4}\partial_\mu h\partial^\mu h + \partial_\mu h\partial^\mu\phi - \partial_\mu\phi\partial^\mu\phi\\
&\quad -\tfrac{1}{4}\partial_i h\partial^i h + \partial_i h\partial^i\phi - \partial_i\phi\partial^i\phi.
\end{aligned} \tag{A.3}$$

It is also useful to evaluate

$$\begin{aligned}
\tfrac{1}{2}\bar\nabla_M H^{MN}\bar\nabla_N H &= \tfrac{1}{2}\nabla_\mu h^{\mu\alpha}\nabla_\alpha h - \nabla_\mu h^{\mu\alpha}\nabla_\alpha\phi\\
&\quad -\tfrac{1}{2}\nabla_\mu h\nabla^\mu\phi + \nabla_\mu\phi\nabla^\mu\phi\\
&\quad +\tfrac{4}{3}\zeta^{-1}\phi\nabla_0 h - \tfrac{8}{3}\zeta^{-1}\phi\nabla_0\phi\\
&\quad +\tfrac{1}{2}\zeta\nabla_\mu A^\mu_i\partial^i h - \zeta\nabla_\mu A^\mu_i\partial^i\phi\\
&\quad +\tfrac{1}{2}\zeta\partial_i A^{\mu i}\nabla_\mu h - \zeta\partial_i A^{\mu i}\nabla_\mu\phi\\
&\quad +\zeta\partial_i\phi^{ij}\partial_j h - 2\zeta\partial_i\phi^{ij}\partial_j\phi,
\end{aligned} \tag{A.4}$$

and also

$$\begin{aligned}
&-\tfrac{1}{2}\bar\nabla_M H_{AB}\bar\nabla^A H^{MB}\\
&\quad = -\tfrac{1}{2}\nabla_\mu h_{\nu\rho}\nabla^\nu h^{\mu\rho} + \nabla_\mu h^{\mu\nu}\nabla_\nu\phi\\
&\quad -\tfrac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \tfrac{1}{2}\zeta^2 g^{ij}\nabla_\mu A_{\nu i}\nabla^\nu A^\mu_j\\
&\quad -\tfrac{1}{9}\zeta^{-1}A_{\nu i}A^{\nu i} + \tfrac{1}{9}\zeta^{-1}[5\zeta^{-2}\phi^2 + 2\phi_{ij}\phi^{ij}]\\
&\quad -\tfrac{1}{3}\zeta A^{\nu i}\nabla_0 A_{\nu i} - \tfrac{2}{3}\zeta^{-1}\phi\nabla_0\phi - \tfrac{4}{3}\zeta\phi^{ij}\nabla_0\phi_{ij}\\
&\quad -2\zeta^2\partial_i\phi_{jk}\partial^j\phi^{ik} - \tfrac{1}{2}\zeta^2\partial_i A_{\mu j}\partial^j A^{\mu i}\\
&\quad -\zeta\nabla_\mu A_{\nu i}\partial^i h^{\mu\nu} + \zeta\partial^i\phi\nabla_\mu A^\mu_i\\
&\quad -2\zeta^2\nabla_\mu\phi_{ij}\partial^i A^{\mu j},
\end{aligned} \tag{A.5}$$

and finally,

$$\begin{aligned}
&\tfrac{1}{4}\bar\nabla_M H_{AB}\bar\nabla^M H^{AB}\\
&\quad = \tfrac{1}{4}\nabla_\mu h_{\nu\rho}\nabla^\mu h^{\nu\rho} - \tfrac{1}{2}\nabla_\mu h\nabla^\mu\phi\\
&\quad +\nabla_\mu\phi\nabla^\mu\phi + \tfrac{1}{2}\zeta^2 g^{ij}\nabla_\mu A_{\nu i}\nabla^\mu A^\nu_j\\
&\quad +\zeta^2 g^{ik}g^{jp}\nabla_\mu\phi_{ij}\nabla^\mu\phi_{kp}\\
&\quad -\tfrac{1}{9}\zeta^{-1}A_{\nu i}A^{\nu i} - \tfrac{1}{9}\zeta^{-1}[5\phi^2\zeta^{-2} + 2\phi_{ij}\phi^{ij}]\\
&\quad -\tfrac{1}{3}\zeta A^{\nu i}\nabla_0 A_{\nu i}\\
&\quad +\tfrac{1}{4}\partial_i h_{\mu\nu}\partial^i h^{\mu\nu} - \tfrac{1}{2}\partial_i h\partial^i\phi + \partial_i\phi\partial^i\phi\\
&\quad +\tfrac{1}{2}\zeta^2\partial_i A_{\mu j}\partial^i A^{\mu j} + \zeta^2\partial_i\phi_{jk}\partial^i\phi^{jk}.
\end{aligned} \tag{A.6}$$

Gauge invariance of 4d action

It is instructive to verify that the action is indeed gauge invariant. We do this by using the form of the transformations before the Weyl transformation was performed (i.e starting with Eq. 3.25) and focusing for simplicity only on the variation that will be linear in the dilaton. The transformations act on the dilaton and scalar kinetic terms as

$$\begin{aligned}
\delta\mathcal{L}_{\phi\phi} &= \tfrac{8}{3}\zeta^3\nabla^\mu\phi\nabla_\mu(\zeta^{-1}\xi_0)\\
&= \tfrac{16}{9}\zeta\xi_0\nabla_0\phi + \tfrac{8}{3}\zeta^2\nabla^\mu\phi\nabla_\mu\xi_0\\
&= \tfrac{32}{27}\zeta^{-1}\phi\xi_0 + \tfrac{16}{9}\zeta\phi\nabla_0\xi_0 - \tfrac{8}{3}\zeta^2\phi\nabla_\mu\nabla^\mu\xi_0,
\end{aligned} \tag{A.7}$$

where the first term on the second line is equivalent to $B^\mu\nabla_\mu\phi$. Meanwhile, the term in the action that we need to vary is

$$\mathcal{L}_{h\phi} = -\tfrac{4}{3}\zeta^2\phi\nabla_0 h - \tfrac{8}{3}\zeta^2 h^\mu_0\nabla_\mu\phi$$

$$= \tfrac{4}{3}\zeta^2 h\nabla_0\phi - \tfrac{32}{9}\zeta\phi h_{00} + \tfrac{8}{3}\zeta^2\phi\nabla_\mu h^\mu_0. \tag{A.8}$$

The variation has to be done carefully with

$$\delta h_{00} = B^\mu B^\nu \delta h_{\mu\nu} = 2B^\mu B^\nu \nabla_\mu \xi_\nu = 2\nabla_0\xi_0 + \tfrac{8}{3}\zeta^{-2}\xi_0, \tag{A.9}$$

as well as

$$\delta h = 2\nabla_\rho \xi^\rho - 8\zeta^{-1}\xi_0, \tag{A.10}$$

and additionally

$$\begin{aligned}\delta(\nabla_\mu h^\mu_0) &= \nabla_\mu B^\alpha \nabla_\alpha \xi^\mu + \nabla_\mu B^\alpha \nabla^\mu \xi_\alpha - 2\nabla_\mu (B^\mu \zeta^{-1}\xi_0)\\ &= B^\alpha \nabla_\mu \nabla_\alpha \xi^\mu + \Gamma^\alpha_{0\mu}\nabla_\alpha \xi^\mu + \nabla_\mu \nabla^\mu \xi_0\\ &\quad + \tfrac{1}{3}\nabla_\mu(\zeta^{-2}\xi^\mu) + 4\zeta^{-3}\xi_0 - 2\zeta^{-1}\nabla_0\xi_0\\ &= B^\alpha \nabla_\mu \nabla_\alpha \xi^\mu + \nabla_\mu \nabla^\mu \xi_0 + \tfrac{40}{9}\zeta^{-3}\xi_0\\ &\quad -2\zeta^{-1}\nabla_0\xi_0. \end{aligned} \tag{A.11}$$

Using these we see that

$$\begin{aligned}\delta\mathcal{L}_{h\phi} &= -\tfrac{8}{3}\zeta^2\phi B^\alpha \nabla_\alpha \nabla_\rho \xi^\rho - \tfrac{64}{9}\zeta^{-1}\phi\xi_0 + \tfrac{32}{3}\zeta\phi\nabla_0\xi_0\\ &\quad -\tfrac{64}{9}\zeta\phi\nabla_0\xi_0 - \tfrac{256}{27}\zeta^{-1}\phi\xi_0\\ &\quad +\tfrac{8}{3}\zeta^2\phi B^\alpha \nabla_\mu\nabla_\alpha\xi^\mu + \tfrac{8}{3}\zeta^2\phi\nabla_\mu\nabla^\mu\xi_0\\ &\quad +\tfrac{320}{27}\zeta^{-1}\phi\xi_0 - \tfrac{16}{3}\zeta\phi\nabla_0\xi_0\\ &= -\tfrac{8}{3}\zeta^2\phi B^\alpha\nabla_\alpha\nabla_\rho\xi^\rho - \tfrac{64}{9}\zeta\phi\nabla_0\xi_0 - \tfrac{128}{27}\zeta^{-1}\phi\xi_0\\ &\quad +\tfrac{8}{3}\zeta^2\phi B^\alpha\nabla_\mu\nabla_\alpha\xi^\mu + \tfrac{8}{3}\zeta^2\phi\nabla_\mu\nabla^\mu\xi_0\\ &= \tfrac{8}{3}\zeta^2\phi R_{\rho 0}\xi^\rho - \tfrac{128}{27}\zeta^{-1}\phi\xi_0 - \tfrac{16}{9}\zeta\phi\nabla_0\xi_0\\ &\quad +\tfrac{8}{3}\zeta^2\phi\nabla_\mu\nabla^\mu\xi_0\\ &= -\tfrac{32}{27}\zeta^{-1}\phi\xi_0 - \tfrac{16}{9}\zeta\phi\nabla_0\xi_0 + \tfrac{8}{3}\zeta^2\phi\nabla_\mu\nabla^\mu\xi_0, \end{aligned} \tag{A.12}$$

where we used that $R_{0\rho}\xi^\rho = \frac{4}{3}\zeta^{-3}\xi_0$. Now we see that $\delta\mathcal{L}_{h\phi} + \delta\mathcal{L}_{\phi\phi} = 0$ for the variation that is proportional to the dilaton. One can similarly show that the other variations also vanish.

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