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Supergravity duals to five-dimensional supersymmetric gauge theories.

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Declaration

I confirm that this thesis is an original work of my authorship. It contains parts that have been produced in collaboration with Professor Fernando Alday, Doctor Martin Fluder, Doctor Paul Richmond and Professor James Sparks, published in the papers [1, 2], to which I made a substantial contribution.

I also declare that the work here presented has not been submitted, in whole or in part, for any other degree or professional qualification.

Dated

Signed

Abstract

In this thesis we study gauge/gravity duals in the $5d/6d$ AdS/CFT correspondence. We start with field theories defined on squashed five-spheres with $SU(3) \times U(1)$ symmetry. These five-sphere backgrounds are continuously connected to the round sphere. We find a one-parameter family of 3/4 BPS deformations and a two-parameter family of (generically) 1/4 BPS deformations. The gravity duals are constructed in Euclidean Romans $F(4)$ gauged supergravity in six dimensions, and uplift to massive type IIA supergravity. We holographically renormalize the Romans theory, and use our general result to compute the renormalized on-shell actions for the solutions. The results agree perfectly with the large N limit of the dual gauge theory partition function, which we compute using large N matrix model techniques. In addition we compute BPS Wilson loops in these backgrounds, both in supergravity and in the large N matrix model, again finding precise agreement. We conjecture a general formula for the partition function on any five-sphere background, which for fixed gauge theory depends only on a certain supersymmetric Killing vector. We then proceed to study Euclidean Romans supergravity in six dimensions with a non-trivial Abelian R-symmetry gauge field. We show that supersymmetric solutions are in one-to-one correspondence with solutions to a set of differential constraints on an $SU(2)$ structure. As an application of our results we (i) show that this structure reduces at a conformal boundary to the five-dimensional rigid supersymmetric geometry previously studied, (ii) find a general expression for the holographic dual of the VEV of a BPS Wilson loop, matching an exact field theory computation, (iii) construct holographic duals to squashed Sasaki-Einstein backgrounds, again matching to a field theory computation, and (iv) find new analytic solutions to the squashed five-sphere background. We also analyse the classification of gravity duals with zero B-field.

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Contents

Chapter 1

Introduction

Over the last few years there has been increasing interest in defining and studying supersymmetric gauge theories on curved backgrounds. Such constructions lead to interesting classes of observables that can be computed exactly, which may in turn be used to test and explore conjectured dualities. Here we focus on the case of five-dimensional gauge theories. These have been defined on round spheres [3, 4, 5, 6, 7], as well as on certain continuous deformations thereof [8, 9], referred to as squashed five-spheres. The main observable that can be computed exactly in these theories is the partition function Z, which depends non-trivially on the background geometry. A particular class of five-dimensional superconformal gauge theories, with gauge group $USp(2N)$ and arising from a $D4-D8$-system, is expected to have a large N description in terms of massive type IIA supergravity [10, 11, 12]. In [7] the large N limit of the partition function of these theories on the *round* sphere was computed and successfully compared to the entanglement entropy of the dual warped $\mathrm{AdS}_6 \times S^4$ supergravity solution.

Here we shall present the first construction of gravity duals to gauge theories on non-conformally flat backgrounds (starting with certain families of squashed five-spheres). As we shall explain, we may effectively work in six-dimensional Romans $F(4)$ supergravity [13], which is a consistent truncation of massive IIA supergravity on S^4 [14]. In particular the computation of [7] effectively determines the six-dimensional Newton constant. Having constructed supergravity solutions that have squashed five-sphere conformal boundaries, we compute the holographic free energy $\mathcal{F} = -\log Z$ by holographically renormalizing the on-shell Euclidean action. The holographic renormalization of the Romans theory is a new result. Due to its exotic character, namely, the presence of a massive B-field and Chern-Simons terms, the renormalization of this particular supergravity theory had never been done before. More specifically, we construct families of solutions with different numbers of preserved supercharges. Two of these families are shown to be dual to the 1/4 BPS and 3/4 BPS gauge theories defined in [9]. The perturbative partition function for these theories has been computed in [8] and we explicitly show that the large N limit of these partition functions is in precise agreement with the holographic free energies of our supergravity solutions. We also present more general solutions (and in particular a 1/2 BPS solution) which have not previously been considered from the gauge theory side.

From the Killing spinors of a supersymmetric supergravity solution one can often construct a certain Killing vector K. For all solutions presented here, the free energy is only sensitive to this Killing vector $\mathcal{F} = \mathcal{F}(K)$, and not to other parameters of the solution. It is natural to conjecture that this is also the case for more general solutions, extending what happens in four dimensions [15]. In addition we compute the expectation values of BPS Wilson loops in these backgrounds, both in supergravity and in the large N matrix model, finding precise agreement. Again the expectation value depends only on the Killing vector K.

Rigid supersymmetric gauge theories in five-dimensional curved backgrounds have been constructed and studied in a series of papers [3, 4, 5, 6, 8, 9, 16, 17, 18, 19, 20, 21, 22, 23, 24]. In the approach of [21] these rigid backgrounds are equipped with a transversely holomorphic foliation. Inspired by the lower-dimensional results of [25, 26] it was conjectured that supersymmetric observables depend only on this foliation. We then systematically study supersymmetric solutions to Euclidean Romans supergravity in six dimensions. Our aim is to compute observables of interest for gauge/gravity duality, and in particular understand the conjecture of [21] from a holographic perspective.

We show that real Euclidean supersymmetric solutions to Romans $F(4)$ gauged supergravity, with a non-trivial Abelian R-symmetry gauge field, have a canonical $SU(2)$ structure determined by the Killing spinor. More precisely we show that supersymmetry together with the equations of motion are equivalent to a set of differential constraints on this $SU(2)$ structure. This geometric formulation then leads to a number of interesting applications. First, we show that this structure extends into the bulk the conformal boundary $SU(2)$ structure studied in [21]. This allows for the construction of gravity duals to families of five-dimensional gauge theories on rigid backgrounds. As another application we extend several of the previous results ([28, 1]). We extend these results to new families of solutions, in general with different topology. In particular this includes squashed Sasaki-Einstein conformal boundaries, together with new analytic solutions. We then take the B-field to be zero and classify this type of solutions.

This dissertation is organised as follows. In chapter 2, we discuss supersymmetric gauge theories defined on squashed five-spheres, their exact partition function and the large N limit. In chapter 3, we change focus and describe the Romans $F(4)$ supergravity theory we will work with. Then in chapter 4, we present our supergravity solutions dual to the squashed five-sphere backgrounds. In chapter 5, we apply holographic renormalization to the Romans $F(4)$ supergravity theory and use this to compute the holographic free energy of our solutions. Another exact observable that can be computed both in supersymmetric gauge theories and in supergravity are Wilson loops, which are the subject of chapter 6. In chapter 7, we examine the supersymmetry conditions which arise at the conformal boundary for the Romans supergravity theory. Chapter 8 contains a general analysis of Euclidean supersymmetric solutions to Romans supergravity, recasting the conditions in terms of a canonical local $SU(2)$ structure. In chapter 9 we present a number of applications of our formalism. In chapter 10 we present new solutions found using this new set of equations. Chapter 11 contains the classification of solutions with zero B-field. Our conclusions are presented in chapter 12. A number of technical details have been included in the appendices.

Chapter 2

Supersymmetric gauge theories on squashed five-spheres

We begin this chapter by describing supersymmetric gauge theories in five dimensions. We proceed then to study the squashed five-sphere backgrounds of interest [8]. One can define a supersymmetric gauge theory with general matter content on such a background. The perturbative partition function was computed in [9] via a twisted reduction of the supersymmetric index in six dimensions. A particular class of five-dimensional gauge theories, with gauge group $USp(2N)$ and arising from a $D4-D8$ system in massive type IIA string theory, is expected to have a large N limit with a gravity dual. In section 2.4 we compute the large N limit of the partition function for these theories using matrix model techniques.

2.1 Supersymmetric gauge theories in five dimensions

In five dimensional supersymmetric gauge theories in flat space, the spinor representation of $SO(4,1)$ is four dimensional and pseudoreal. The minimal supersymmetry will contain two supercharges, as the vector of $SO(4,1)$ is in the antisymmetric product of two spinors. The two supercharges are a doublet in the $SU(2)_R$ automorphism of the five dimensional SUSY algebra.

In five dimensions it is difficult to construct conformally invariant classical Lagrangians other than the one for a free theory. This is the only conformally invariant Lagrangian known in five dimensions at the classical level. Yang-Mills theory is only classically conformally invariant in four dimensions; in five dimensions the gauge coupling becomes dimensionful, and Yang-Mills is no longer a classical CFT.

As a result of renormalization, things one can measure depend on the energy scale. How the dynamics of a system changes once we change the energy scale is investigated by the renormalization group. This makes sure that one can obtain sensible answers, i.e., that measurements do not depend on how the theory is renormalized.

The β-function determines how the coupling constant depends on the energy scale. A CFT always has $\beta = 0$. To study interesting quantum field theories, one starts with a free theory and then adds operators. The renormalization group flow takes care of the constants sitting in front of the couplings in the Lagrangian, and those will run with the couplings. Given the β-function in the space of the coupling constants, one hopes to find the surfaces where $\beta = 0$, i.e., the conformally invariant part of the theory. This is not always easy, and one can land in a portion of the theory that is strongly interacting, where computing quantities is quite hard. New localization techniques were created for that purpose, and allow us to compute the path integral exactly.

The massless representations of the Poincaré group are given by two types of multiplets. For an arbitrary gauge group, we have a vectormultiplet, composed by a vector A_m, a real scalar σ, a triplet of auxiliary scalars D_{IJ} and a Majorana spinor λ_I; and a hypermultiplet, composed by scalars q_I, fermions ψ and auxiliary scalars F_I.

The variations of the fields for each of the representations in the flat five dimensional space are given as follows [4]. For the vectormultiplet, one has

$$\begin{aligned}
\delta_\xi A_m &= \mathrm{i}\varepsilon^{IJ}\xi_I\Gamma_m\lambda_J \ , \\
\delta_\xi\sigma &= \mathrm{i}\varepsilon^{IJ}\xi_I\lambda_J \ , \\
\delta_\xi\lambda_I &= -\tfrac{1}{2}\Gamma^{mn}\xi_I F_{mn} + \Gamma^m\xi_I D_m\sigma + \xi_J D_{KI}\varepsilon^{JK} \ , \\
\delta_\xi D_{IJ} &= -\mathrm{i}(\xi_I\Gamma^m D_m\lambda_J + \xi_J\Gamma^m D_m\lambda_I) + [\sigma, \xi_I\lambda_J + \xi_J\lambda_I] \ ,
\end{aligned} \tag{2.1.1}$$

where D_m is the covariant derivative with respect to m. This vectormultiplet has an invariant Lagrangian given by

$$\mathcal{L}_{SYM} = tr\left[\tfrac{1}{2}F_{mn}F^{mn} - D_m\sigma D^m\sigma - \tfrac{1}{2}D_{IJ}D^{IJ} + \mathrm{i}\varepsilon^{IJ}\lambda_I\Gamma^m D_m\lambda_J - \varepsilon^{IJ}\lambda_I[\sigma,\lambda_J]\right] \ . \tag{2.1.2}$$

And for the hypermultiplet, one has

$$\begin{aligned}
\delta q_I &= -2\mathrm{i}\xi_I\psi \ , \\
\delta\psi &= \varepsilon^{IJ}\Gamma^m\xi_I D_m q_J + \mathrm{i}\varepsilon^{IJ}\xi_I\sigma q_J + \varepsilon^{I'J'}\hat{\xi}_{I'}F_{J'} \ , \\
\delta F_{I'} &= 2\hat{\xi}_{I'}(\mathrm{i}\Gamma^m D_m\psi + \sigma\psi + \varepsilon^{KL}\lambda_K q_L) \ ,
\end{aligned} \tag{2.1.3}$$

with an invariant Lagrangian given by

$$\begin{aligned}
\mathcal{L} =& \varepsilon^{IJ}(D_m\bar{q}_I D^m q_J - \bar{q}_I\sigma^2 q_J) - 2(\mathrm{i}\bar{\psi}\Gamma^m D_m\psi + \bar{\psi}\sigma\psi) \\
&- \mathrm{i}\bar{q}_I D^{IJ}q_J - 4\varepsilon^{IJ}\bar{\psi}\lambda_I q_J - \varepsilon^{I'J'}\bar{F}_{I'}F_{J'} \ .
\end{aligned} \tag{2.1.4}$$

Changing the background from a flat space to a sphere was not considered until recently, although studying a gauge theory on a curved space could have been done since the eighties, after Witten's work on topological field theories [29]. The first paper analysing supersymmetric gauge theories on a sphere was written by Pestun, in 2007 [30], considering a sphere in four dimensions. In 2009, Kapustin et al did the same for a three-sphere [31]. Only in 2012, Kählen and Zabzine analysed Yang-Mills theory on a five-sphere [3]. Although pioneers, the procedure used in these papers to take a field theory from a flat space to a curved one is rather *ad hoc*. A more systematic way for analysing the theory in curved backgrounds was later analysed by Festuccia and Seiberg [32].

2.2 $SU(3)\times U(1)$ squashed five-sphere

The squashed S^5 backgrounds of interest are homogeneous spaces with symmetry $SU(3)\times U(1)$. In particular this is the isometry group of the metric

$$\begin{aligned} \mathrm{d}s_5^2 &= \frac{1}{s^2}(\mathrm{d}\tau + C)^2 + \mathrm{d}\sigma^2 + \frac{1}{4}\sin^2\sigma(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \\ &\quad + \frac{1}{4}\cos^2\sigma\sin^2\sigma(\mathrm{d}\psi + \cos\theta\mathrm{d}\varphi)^2 \ , \end{aligned} \tag{2.2.1}$$

where we have defined the (local) one-form

$$C = -\frac{1}{2}\sin^2\sigma(\mathrm{d}\psi + \cos\theta\mathrm{d}\varphi) \ . \tag{2.2.2}$$

We refer to the parameter s as a squashing parameter, and note that $s=1$ is the round sphere. The coordinates in (2.2.1) realize the five-sphere as the total space of the Hopf circle bundle over $\mathbb{CP}^2$, where τ is a 2π-period coordinate along the circle fibre. The coordinates $\sigma,\psi,\theta,\varphi$ are then coordinates on the base $\mathbb{CP}^2$, with ψ having period 4π, φ having period 2π, while $\sigma\in[0,\frac{\pi}{2}]$, $\theta\in[0,\pi]$. The local one-form C in (2.2.2) satisfies

$$\mathrm{d}C \equiv 2\omega = -\sin\sigma\cos\sigma\mathrm{d}\sigma\wedge(\mathrm{d}\psi+\cos\theta\mathrm{d}\varphi) + \frac{1}{2}\sin^2\sigma\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\varphi, \tag{2.2.3}$$

where ω is the Kähler two-form on $\mathbb{CP}^2$.

In order to preserve supersymmetry one must also turn on other backgrounds fields. In particular in [8] it was shown that one can define general supersymmetric gauge theories on the above squashed five-sphere, provided one turns on a background $SU(2)_R$ gauge field

$$\mathcal{A} = \frac{(1+Q\sqrt{1-s^2})\sqrt{1-s^2}}{s^2}(\mathrm{d}\tau + C) \ , \tag{2.2.4}$$

where we have embedded $U(1)_R\subset SU(2)_R$. More precisely, writing the $SU(2)_R\sim SO(3)_R$ gauge field as a triplet of one-forms $\mathcal{A}^i$, $i=1,2,3$, we have $\mathcal{A}^1=\mathcal{A}^2=0$, while $\mathcal{A}^3=\mathcal{A}$ is given by (2.2.4). For supersymmetric backgrounds the parameter Q takes the values $Q=1$ and $Q=-3$, which lead to 3/4 BPS and 1/4 BPS solutions, respectively. Notice that the gauge field (2.2.4) is also invariant under $SU(3)\times U(1)$, and is real when $|s|<1$ but complex for $|s|>1$.

A supersymmetric background of course admits an appropriate Killing spinor, which then enters the supersymmetry transformations of a supersymmetric gauge theory defined on the background. Recall that a Killing spinor χ on the round S^5 with $s=1$, solving $\nabla_m\chi = -\frac{\mathrm{i}}{2}\gamma_m\chi$ where γ_m generate the Clifford algebra Cliff$(5,0)$ in an orthonormal frame, transforms in the $\mathbf{4}$ of the $SU(4)\sim SO(6)$ isometry. The squashing breaks this symmetry to $SU(3)\times U(1)$, and for $Q=1$ the resulting Killing spinor transforms as $\mathbf{3_{+1}}$, while for $Q=-3$ the resulting Killing spinor instead transforms as $\mathbf{1_{-3}}$. Similarly, solutions to $\nabla_m\chi = \frac{\mathrm{i}}{2}\gamma_m\chi$ transform in the $\bar{\mathbf{4}}$ of $SU(4)$, which is broken to $\bar{\mathbf{3}}_{\mathbf{-1}}$ and $\mathbf{1_{+3}}$ in the two cases, respectively.

The corresponding Killing spinor equation for the squashed S^5 was obtained in [8] via a twisted reduction (described in the next section) of a standard Killing spinor equation in six dimensions. In order to write this down, we first introduce an orthonormal frame for the metric (2.2.1)

$$\begin{aligned} e^1_{(5)} &= \frac{1}{s}(\mathrm{d}\tau + C)\ , \qquad e^2_{(5)} = \mathrm{d}\sigma\ , \qquad e^3_{(5)} = \frac{1}{2}\sin\sigma\cos\sigma\tau_3\ ,\\ e^4_{(5)} &= \frac{1}{2}\sin\sigma\tau_2\ , \qquad e^5_{(5)} = \frac{1}{2}\sin\sigma\tau_1\ , \end{aligned} \tag{2.2.5}$$

where τ_i, $i=1,2,3$, are left-invariant one-forms on $SU(2)$. These are parametrized in terms of the Euler angles as

$$\tau_1 + \mathrm{i}\tau_2 = \mathrm{e}^{-\mathrm{i}\psi}(\mathrm{d}\theta + \mathrm{i}\sin\theta\mathrm{d}\varphi)\ , \qquad \tau_3 = \mathrm{d}\psi + \cos\theta\mathrm{d}\varphi\ . \tag{2.2.6}$$

The Killing spinor equation then reads

$$\begin{aligned} \nabla_m\chi_I + \tfrac{\mathrm{i}}{2}\mathcal{A}^i_m(\sigma^i)_I{}^J\chi_J &= -\frac{\mathrm{i}\left(1+Q\sqrt{1-s^2}\right)}{2s}(\sigma^3)_I{}^J\gamma_m\chi_J\\ &\quad + \frac{\sqrt{1-s^2}}{4s}\left(3\gamma_m\not{\partial} - \not{\partial}\gamma_m\right)\chi_I\ , \end{aligned} \tag{2.2.7}$$

which is supplemented by the following algebraic equation

$$Q\sqrt{1-s^2}\chi_I = -\sqrt{1-s^2}\gamma_1\chi_I - \mathrm{i}\sqrt{1-s^2}(\sigma^3)_I{}^J\not{\partial}\chi_J\ . \tag{2.2.8}$$

Here χ_I, $I=1,2$, form a doublet under the $SU(2)_R$ symmetry, γ_m generate the Clifford algebra Cliff$(5,0)$ in the orthonormal frame (2.2.5), and $(\sigma^i)_I{}^J$ denote the Pauli matrices. Recall also that ω denotes the Kähler form on $\mathbb{CP}^2$, given by (2.2.3), and if α is a p-form we denote $\alpha\!\!\!/ \equiv \frac{1}{p!}\alpha_{m_1\cdots m_p}\gamma^{m_1\cdots m_p}$.

Of course in the case at hand we have that the $SU(2)_R$ gauge field $\mathcal{A}^i$ is only turned on in the $i = 3$ direction, with $\mathcal{A}^3 = \mathcal{A}$ given by (2.2.4), and we may also write (2.2.7) and (2.2.8) as

$$\begin{aligned} \nabla_m\chi_\pm \pm \tfrac{\mathrm{i}}{2}\mathcal{A}_m\chi_\pm &= \mp \frac{\mathrm{i}\left(1+Q\sqrt{1-s^2}\right)}{2s}\gamma_m\chi_\pm + \frac{\sqrt{1-s^2}}{4s}\left(3\gamma_m\omega\!\!\!/ - \omega\!\!\!/\gamma_m\right)\chi_\pm \,, && (2.2.9) \\ Q\sqrt{1-s^2}\chi_\pm &= -\sqrt{1-s^2}\gamma_1\chi_\pm \mp \mathrm{i}\sqrt{1-s^2}\,\omega\!\!\!/\chi_\pm \,, && (2.2.10) \end{aligned}$$

where $\chi_+ = \chi_1$, $\chi_- = \chi_2$. Provided the background fields are real, meaning in particular that the metric and $\mathcal{A}$ are real and $|s| < 1$, then notice that the equations for χ_- are simply the charge conjugates of the χ_+ equations, where we define the charge conjugate as

$$\chi^c \equiv \mathcal{C}_5\chi^* \,, \qquad (2.2.11)$$

and the charge conjugation matrix $\mathcal{C}_5$ satisfies $\mathcal{C}_5^{-1}\gamma_m\mathcal{C}_5 = \gamma_m^*$. In particular it is then consistent to impose the symplectic Majorana condition $\chi_- = \chi_+^c$, or equivalently $\varepsilon_I{}^J\chi_J = \mathcal{C}_5\chi_I^*$, as we shall see below.

Notice that in setting $s = 1$ to obtain the round sphere one has that (2.2.8) is trivially satisfied, while the Killing spinor equation (2.2.7) implies that χ_1 and χ_2 transform in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of the enhanced $SU(4) \sim SO(6)$ symmetry, respectively. In order to present the general solution to (2.2.7), (2.2.8) (which is not written in [8]), we first introduce the following basis of Cliff$(5,0)$

$$\begin{aligned} \gamma_1 &= \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} , \quad \gamma_2 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} , \quad \gamma_3 = \begin{pmatrix} 0 & \mathrm{i}\sigma^3 \\ -\mathrm{i}\sigma^3 & 0 \end{pmatrix} , \\ \gamma_4 &= \begin{pmatrix} 0 & \mathrm{i}\sigma^2 \\ -\mathrm{i}\sigma^2 & 0 \end{pmatrix} , \quad \gamma_5 = \begin{pmatrix} 0 & \mathrm{i}\sigma^1 \\ -\mathrm{i}\sigma^1 & 0 \end{pmatrix} , \end{aligned} \qquad (2.2.12)$$

where as above σ^i, $i = 1, 2, 3$ denote the Pauli matrices, and 1_2 is the 2×2 identity matrix. A choice of the charge conjugation matrix in this basis is

$$\mathcal{C}_5 \;=\; \begin{pmatrix} -\mathrm{i}\sigma_2 & 0 \\ 0 & -\mathrm{i}\sigma_2 \end{pmatrix} . \tag{2.2.13}$$

Then for the 1/4 BPS background we find the general solution to (2.2.7), (2.2.8) (or equivalently (2.2.9), (2.2.10)) is given by

$$\chi_+ \;=\; c_+ \mathrm{e}^{-\frac{3\mathrm{i}\tau}{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \qquad \chi_- \;=\; c_- \mathrm{e}^{\frac{3\mathrm{i}\tau}{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \tag{2.2.14}$$

where $c_\pm$ are integration constants. In particular then notice that the symplectic Majorana condition $\chi_- = \chi_+^c$ simply imposes $c_- = c_+^*$.

For the 3/4 BPS background the solution is a little more complicated. One finds

$$\chi_+ \;=\; a_+^{(1)} \mathrm{e}^{\mathrm{i}\frac{\tau}{2}} \begin{pmatrix} \cos\sigma + \mathrm{i}\lambda_+(s) \mathrm{e}^{\mathrm{i}\frac{\psi}{2}} S_+^{(1)} \sin\sigma \\ 0 \\ \mathrm{i}\lambda_-(s)\sin\sigma - \mathrm{e}^{\mathrm{i}\frac{\psi}{2}} S_+^{(1)} \cos\sigma \\ -\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}} S_+^{(2)} \end{pmatrix} , \tag{2.2.15}$$

where

$$\begin{aligned} S_\pm^{(1)} \;&=\; S_\pm^{(1)}(\theta,\varphi) \;=\; a_\pm^{(3)} \mathrm{e}^{\pm\mathrm{i}\frac{\varphi}{2}} \cos\frac{\theta}{2} - a_\pm^{(2)} \mathrm{e}^{\mp\mathrm{i}\frac{\varphi}{2}} \sin\frac{\theta}{2} , \\ S_\pm^{(2)} \;&=\; S_\pm^{(2)}(\theta,\varphi) \;=\; a_\pm^{(2)} \mathrm{e}^{\mp\mathrm{i}\frac{\varphi}{2}} \cos\frac{\theta}{2} + a_\pm^{(3)} \mathrm{e}^{\pm\mathrm{i}\frac{\varphi}{2}} \sin\frac{\theta}{2} , \end{aligned} \tag{2.2.16}$$

and where we have introduced $\lambda_\pm(s) \equiv (\pm 1 + \sqrt{1-s^2})/s$. As expected, the solution depends on three integration constants $a_+^{(1)}, a_+^{(2)}, a_+^{(3)}$. Similarly, one finds

$$\chi_- \;=\; a_-^{(1)}\, \mathrm{e}^{-\mathrm{i}\frac{\tau}{2}} \begin{pmatrix} 0 \\ \cos\sigma - \mathrm{i}\lambda_+(s)\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}} S_-^{(1)} \sin\sigma \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\psi}{2}} S_-^{(2)} \\ -\mathrm{i}\lambda_-(s)\sin\sigma - \mathrm{e}^{-\mathrm{i}\frac{\psi}{2}} S_-^{(1)} \cos\sigma \end{pmatrix} , \qquad (2.2.17)$$

where $a_-^{(i)}$ are integration constants. One can once again impose the symplectic Majorana condition, which leads to the relation $(a_-^{(i)})^* = a_+^{(i)}$ for $i = 1, 2, 3$.

2.3 Twisted reduction and the partition function

The backgrounds above may be obtained via a twisted reduction of $\mathbb{R} \times S^5$, starting from the *round* metric on S^5. This is important, as the perturbative partition function on the squashed five-spheres was computed in [9] indirectly, by taking a limit of the supersymmetric index of a corresponding six-dimensional theory on $\mathbb{R} \times S^5$.

We thus begin with the product metric on $\mathbb{R}$ times the round S^5

$$\mathrm{d}s^2_{\mathbb{R}\times S^5} \;=\; \mathrm{d}t^2 + \sum_{i=1}^{3} |\mathrm{d}w_i|^2 , \qquad (2.3.1)$$

where the complex coordinates w_i on $\mathbb{C}^3 \cong \mathbb{R}^6$, $i = 1, 2, 3$, satisfy the constraint $\sum_{i=1}^{3} |w_i|^2 = 1$. We then compactify this space by identifying

$$(t, w_i) \sim (t + \beta, \mathrm{e}^{\mathrm{i}\mu_i\beta} w_i) , \qquad (2.3.2)$$

where $\beta > 0$ and the μ_i are also sometimes referred to as squashing parameters. Notice that (2.3.2) is an isometry for $\mu_i \in \mathbb{R}$. We may then change coordinates

$$\rho_i \mathrm{e}^{\mathrm{i}\varphi_i} \;\equiv\; \mathrm{e}^{-\mathrm{i}\mu_i t} w_i , \qquad (2.3.3)$$

where $\rho_i \geq 0$ and the φ_i have period 2π. In terms of these new coordinates the identification (2.3.2) reads $(t, \rho_i, \varphi_i) \sim (t + \beta, \rho_i, \varphi_i)$. We then dimensionally reduce along the t-direction to obtain the five-dimensional metric

$$\mathrm{d}s_5^2 \;=\; \sum_{i=1}^{3}(\mathrm{d}\rho_i^2+\rho_i^2\mathrm{d}\varphi_i^2)-\frac{1}{1+\sum_{i=1}^3\mu_i^2\rho_i^2}\left(\sum_{i=1}^{3}\mu_i\rho_i^2\mathrm{d}\varphi_i\right)^2 . \tag{2.3.4}$$

Notice that, via the constraint $\sum_{i=1}^3\rho_i^2=1$, the first term in (2.3.4) is the round metric on S^5.

One then makes contact with the previous section by choosing

$$\begin{aligned} -\mu_1 \;&=\; \mu_2 \;=\; \mu_3 \;=\; \mathrm{i}\sqrt{1-s^2}\ , \qquad && 3/4 \text{ BPS}\ , \\ \mu_1 \;&=\; \mu_2 \;=\; \mu_3 \;=\; -\mathrm{i}\sqrt{1-s^2}\ , \qquad && 1/4 \text{ BPS}\ . \end{aligned} \tag{2.3.5}$$

Notice these are real only if $|s|\geq 1$. The metric (2.3.4) then agrees with the metric (2.2.1) on making the standard polar coordinate identifications

$$\rho_1 \;=\; \cos\sigma\ , \qquad \rho_2 \;=\; \sin\sigma\cos\frac{\theta}{2}\ , \qquad \rho_3 \;=\; \sin\sigma\sin\frac{\theta}{2}\ , \tag{2.3.6}$$

together with

$$\begin{aligned} \varphi_1 \;&=\; -\tau\ , \quad \varphi_2 \;=\; \tau-\frac{1}{2}(\psi+\varphi)\ , \quad \varphi_3 \;=\; \tau-\frac{1}{2}(\psi-\varphi)\ , \quad 3/4 \text{ BPS}\ , \\ \varphi_1 \;&=\; \tau\ , \quad \varphi_2 \;=\; \tau-\frac{1}{2}(\psi+\varphi)\ , \quad \varphi_3 \;=\; \tau-\frac{1}{2}(\psi-\varphi)\ , \quad 1/4 \text{ BPS}\ . \end{aligned} \tag{2.3.7}$$

The Killing spinor equation (2.2.7) and algebraic equation (2.2.8) were then obtained in [8] by dimensionally reducing a standard Killing spinor equation on the $\mathbb{R}\times S^5$ background (2.3.1).

In practice the perturbative contribution to the squashed S^5 partition function, with more general squashed metric (2.3.4), was computed in [9] by dimensionally reducing the superconformal index of a corresponding six-dimensional theory on the $\mathbb{R}\times S^5$ background (2.3.1) with twisted identification (2.3.2), and then taking the limit $\beta\to 0$, so that the radius of the circle we reduced on to obtain (2.3.4) is sent to zero. For a gauge theory with gauge group G, prepotential $\mathscr{F}$, which is a cubic polynomial in the scalar σ in the vector multiplet, and matter in the real representation $\mathbf{R}\oplus\bar{\mathbf{R}}$ of G, the result is

$$Z_{\text{pert}} \;=\; C(\mathbf{b})\prod_{a=1}^{\text{rank}\,G}\int_{-\infty}^{\infty}\mathrm{d}\sigma_a\,\mathrm{e}^{-\frac{(2\pi)^3}{b_1b_2b_3}\mathscr{F}(\sigma)}\frac{\prod_\alpha S_3\left(-\mathrm{i}\alpha(\sigma)\mid\mathbf{b}\right)}{\prod_\rho S_3\left(-\mathrm{i}\rho(\sigma)+\frac{1}{2}(b_1+b_2+b_3)\mid\mathbf{b}\right)}\ . \tag{2.3.8}$$

Here we have introduced

$$\mathbf{b} \;=\; (b_1, b_2, b_3) \;, \qquad \text{where} \qquad b_i \;=\; 1 + \mathrm{i}\mu_i \;, \tag{2.3.9}$$

and the prefactor $C(\mathbf{b})$ in (2.3.8) depends only on (b_1, b_2, b_3), and in particular will not contribute to the large N limit of interest in the next chapter.[1] The perturbative partition function thus localizes onto field configurations in which the only non-zero field is a constant mode for the scalar σ in the vector multiplet, and this is then integrated over in (2.3.8). As usual in such expressions the product over α in the numerator is over roots of G, while the product over ρ in the denominator is over weights in a weight space decomposition of $\mathbf{R}$. Finally, $S_3\,(z \mid \mathbf{b})$ is the triple sine function, which is a special case of the multiple sine functions defined by

$$\begin{aligned} S_{\mathcal{N}}\,(z \mid \mathbf{b}) \;&\equiv\; \Gamma_{\mathcal{N}}(z \mid \mathbf{b})^{-1}\;\Gamma_{\mathcal{N}}(b_{\mathrm{tot}} - z \mid \mathbf{b})^{(-1)^{\mathcal{N}}} && (2.3.10)\\ &=\; \prod_{n_1,\ldots,n_{\mathcal{N}}=0}^{\infty}\left[\sum_{i=1}^{\mathcal{N}} n_i b_i + z\right]\prod_{n_1,\ldots,n_{\mathcal{N}}=1}^{\infty}\left[\sum_{i=1}^{\mathcal{N}} n_i b_i - z\right]^{(-1)^{\mathcal{N}-1}} \;, && (2.3.11) \end{aligned}$$

where we have written $\mathbf{b} = (b_1, \ldots, b_{\mathcal{N}})$ and defined $b_{\mathrm{tot}} = \sum_{i=1}^{\mathcal{N}} b_i$. The function $\Gamma_{\mathcal{N}}(z \mid \mathbf{b})$ is the so-called Barnes' multiple gamma function

$$\Gamma_{\mathcal{N}}(z \mid \mathbf{b}) \;\equiv\; \prod_{n_1,\ldots,n_{\mathcal{N}}=0}^{\infty}\left[\sum_{i=1}^{\mathcal{N}} n_i b_i + z\right]^{-1} \;. \tag{2.3.12}$$

We conclude this chapter by noting from (2.3.5) and (2.3.9) that for the $SU(3)\times U(1)$ squashed five-spheres in chapter 2.2

$$\begin{aligned} b_1 \;&=\; 1 + \sqrt{1-s^2}\;, \qquad b_2 \;=\; b_3 \;=\; 1 - \sqrt{1-s^2}\;, \qquad && 3/4 \text{ BPS}\;,\\ b_1 \;&=\; b_2 \;=\; b_3 \;=\; 1 + \sqrt{1-s^2}\;, && 1/4 \text{ BPS}\;. \end{aligned} \tag{2.3.13}$$

In particular it is straightforward to see [9] that in the 1/4 BPS case the perturbative partition function (2.3.8) is independent of the squashing parameter s.

It is interesting to note that (2.3.2) is an isometry of the original six-dimensional $\mathbb{R}\times S^5$ background only for real μ_i, which via (2.3.5) one sees corresponds to $|s| \geq 1$. On the other hand

[1] The precise formula for $C(\mathbf{b})$ may be found in [9].

from (2.3.13) we see that the parameters b_i are real (and then positive) only if $|s| \leq 1$. The dual six-dimensional supergravity backgrounds we shall construct in chapter 4 will correspondingly be real for $|s| \leq 1$.

2.4 The large N limit

The result for the perturbative partition function (2.3.8) in the previous section is valid for a general supersymmetric gauge theory in five dimensions, but we now focus on a particular class of theories with gauge group $G = USp(2N)$, that arises from a system of N D4-branes and some number of D8-branes and orientifold planes in massive type IIA string theory. These theories are expected to have a large N limit that has a dual description in massive type IIA supergravity [10, 11, 12]. Indeed, in [7] the large N limit of the partition function of these theories on the *round* five-sphere was computed and successfully compared to the entanglement entropy of the dual warped $\mathrm{AdS}_6 \times S^4$ supergravity solution. Here the gauge theories flow to a UV superconformal fixed point, and in particular the localization computation in the IR supersymmetric Yang-Mills theory coupled to matter theory successfully reproduces the expected $N^{5/2}$ scaling of the number of degrees of freedom.

In general one certainly expects non-perturbative contributions to the full partition function Z, in addition to the perturbative result (2.3.8). In particular in the localization computation of [5] on the round five-sphere one finds that the gauge multiplet localizes onto instanton configurations on $\mathbb{CP}^2$. There is thus a non-perturbative contribution to Z involving a sum over the instanton number. For fixed instanton number $n \neq 0$ and fixed choice of instanton, in addition to the classical instanton action there will also be one-loop determinant contributions around that instanton, plus an integral over the instanton moduli space with fixed n. In general this expression will be very difficult to evaluate. However, in [7] it was argued that in the large N limit these instanton contributions should be suppressed. We shall also assume this to be the case on the squashed five-sphere, although clearly this issue deserves further study. In particular, for general choice of the vector $\mathbf{b} = (b_1, b_2, b_3)$ we expect to find instantons not on $\mathbb{CP}^2$, but rather instantons transverse to the Killing vector $K = \sum_{i=1}^{3} b_i \partial_{\varphi_i}$, as in [16]. These *contact instantons* were discussed in the latter reference in the context of the partition function on Sasaki-Einstein manifolds. In any case, we leave this issue open for future investigation.

Our task thus reduces to computing the large N limit of the perturbative result (2.3.8), for the $USp(2N)$ gauge theories of interest. This may be carried out using the matrix model saddle point method originally introduced in [33], and subsequently applied to the round S^5 partition function in [7]. As in the latter reference, we also set the Chern-Simons level for the theory $k = 0$ (thus setting the cubic terms in the prepotential $\mathscr{F}(\sigma)$ to zero). The quadratic and linear terms of $\mathscr{F}(\sigma)$ will only contribute to subleading order in the large N limit. This is because the leading contribution to the free energy arises from the scaling $\sigma = \mathcal{O}(N^{1/2})$. Such a behaviour for σ leads to an $\mathcal{O}(N^2)$ contribution for the classical parts in the perturbative partition function (2.3.8). Thus in the limit of large N we only have to analyse the behaviour of the two one-loop determinants from the vector and matter multiplets. In particular, for a given theory we will have to find the expansion of the logarithm of the triple sine function entering (2.3.8).

The $USp(2N)$ gauge theories have N_f matter fields in the fundamental and a single hypermultiplet in the antisymmetric representation of the gauge group. Let us denote an element in the Cartan subalgebra for $USp(2N)$ as $\{\lambda_1, \ldots, \lambda_N\}$, so that $\sigma = \mathrm{diag}(\lambda_1, \ldots, \lambda_N, -\lambda_1, \ldots, -\lambda_N)$. The Weyl group acts as $\lambda_i \to -\lambda_i$ for each i, and also permutes the λ_i. If the normalized weights of the fundamental representation are given by $\pm e_i$, where $\{e_1, \ldots, e_N\}$ is a basis of $\mathbb{R}^N$, then the antisymmetric representation has weights $\{e_i \pm e_j\}_{i \neq j}$ and the adjoint representation has weights $\{e_i \pm e_j\}_{i \neq j} \cup \{\pm 2e_i\}_{i=1}^{N}$. Therefore we can write the free energy for this theory as

$$\begin{aligned} \mathcal{F}(\lambda_i) = & \sum_{\substack{i,j=1 \\ i \neq j}}^{N} G_V(\lambda_i + \lambda_j \mid \mathbf{b}) + G_V(\lambda_i - \lambda_j \mid \mathbf{b}) + G_H(\lambda_i + \lambda_j \mid \mathbf{b}) + G_H(\lambda_i - \lambda_j \mid \mathbf{b}) \\ & + \sum_{i=1}^{N} G_V\left(2\lambda_i \mid \mathbf{b}\right) + G_V\left(-2\lambda_i \mid \mathbf{b}\right) + N_f \left[G_H\left(\lambda_i \mid \mathbf{b}\right) + G_H\left(-\lambda_i \mid \mathbf{b}\right)\right] , \end{aligned} \tag{2.4.1}$$

where G_V and G_H are the logarithms of the triple sine functions in the numerator and denominator of (2.3.8) for the vector and the hypermultiplets, respectively. We are interested in their asymptotics for large λ_i only, because we assume that the eigenvalues scale with N^α for some $\alpha > 0$. These asymptotics are explicitly computed in appendix C, and here we simply quote the results:

$$\begin{aligned} G_V(x \mid \mathbf{b}) + G_V(-x \mid \mathbf{b}) &= -\log S_3(-\mathrm{i}x \mid \mathbf{b}) - \log S_3(\mathrm{i}x \mid \mathbf{b}) \\ &\sim \frac{\pi}{3\, b_1 b_2 b_3}|x|^3 - \frac{\pi\left(b_{\mathrm{tot}}^2 + b_1 b_2 + b_1 b_3 + b_2 b_3\right)}{6\, b_1 b_2 b_3}|x| \,, \end{aligned} \tag{2.4.2}$$

where we have expanded in the limit $|x| \to \infty$. Here we have assumed that $b_i > 0$ for each $i = 1, 2, 3$, as this is the case of interest – see equation (2.3.13) and the discussion after it. Similarly, for the free energy contribution of the hypermultiplet we obtain

$$G_H(x \mid \mathbf{b}) = \log S_3\left(\tfrac{1}{2} b_{\mathrm{tot}} - \mathrm{i}x \mid \mathbf{b}\right) \sim -\frac{\pi}{6\, b_1 b_2 b_3}|x|^3 - \frac{\pi\left(b_1^2 + b_2^2 + b_3^2\right)}{24\, b_1 b_2 b_3}|x| \,, \tag{2.4.3}$$

in the asymptotic limit $|x| \to \infty$.

Using the Weyl symmetry of $USp(2N)$ we may take $\lambda_i \geq 0$, and we shall furthermore assume that these eigenvalues scale as $\lambda_i = N^\alpha x_i$ to leading order in the large N limit, with $\alpha > 0$. We next introduce the density

$$\rho(x) = \frac{1}{N}\sum_{i=1}^{N} \delta\left(x - x_i\right) \,, \tag{2.4.4}$$

which becomes an $\mathcal{L}^1$ function with

$$\int \rho(x)\mathrm{d}x = 1 \,, \tag{2.4.5}$$

once we take $N \to \infty$. In that limit, the discrete sums in (2.4.1) become Riemann integrals

$$\frac{1}{N}\sum_{i=1}^{N} \longrightarrow \int_0^{x_\star} \rho(x)\mathrm{d}x \,. \tag{2.4.6}$$

Hence taking the large N limit of (2.4.1), we obtain to leading order

$$\begin{aligned} \mathcal{F} \approx\; & N^2 \int_0^{x_\star} \rho(x) \int_0^{x_\star} \rho(y) \Big[G_V(\lambda(x) \pm \lambda(y) \mid \mathbf{b}) + G_H(\lambda(x) \pm \lambda(y) \mid \mathbf{b})\Big] \mathrm{d}y\, \mathrm{d}x \\ & + N \int_0^{x_\star} \rho(x) \Big[G_V(\pm 2\lambda(x) \mid \mathbf{b}) + N_f\, G_H(\pm\lambda(x) \mid \mathbf{b})\Big] \mathrm{d}x \,. \end{aligned} \tag{2.4.7}$$

By assumption we have $\lambda(x) = N^\alpha x$ to leading order in the continuum limit, and hence we may use the above expansions for the vector and hypermultiplet contributions (2.4.2), (2.4.3) respectively.

Then the leading order term in the first line of (2.4.7) scales as $N^{2+\alpha}$, because the cubic terms in the asymptotic expansion of G_H and G_V cancel. The leading order term of the second line in (2.4.7) however does not cancel, and is given by $N^{1+3\alpha}$. In order to obtain a non-trivial saddle point, both terms must contribute and we deduce that $\alpha = 1/2$. Putting everything together we obtain

$$\begin{aligned}\mathcal{F} \quad = \quad & -N^{5/2}\int_0^{x_\star}\rho(x)\int_0^{x_\star}\rho(y)\Bigg[\frac{\pi b_{\text{tot}}^2}{8b_1b_2b_3}\left(|x+y|+|x-y|\right) \\ & -\frac{(8-N_f)\pi}{3\; b_1b_2b_3}|x|^3\Bigg]\text{d}y\;\text{d}x+\mathcal{O}\left(N^{3/2}\right)\,. \end{aligned} \tag{2.4.8}$$

It thus remains to solve a simple variational problem for $\rho(x)$ extremizing the free energy. We add a Lagrange multiplier term to impose the constraint (2.4.5), namely $\mu\left(\int_0^{x_\star}\rho(x)\text{d}x - 1\right)$, and then solve $\frac{\partial F}{\partial \rho} = 0$ for $\rho(x)$. Doing so we find (with $N_f < 8$)

$$\rho(x) \quad = \quad \frac{4(8-N_f)}{b_{\text{tot}}^2}|x|\,, \tag{2.4.9}$$

inside the interval $[0, x_\star]$, with ρ identically zero outside this interval, and where extremizing $\mathcal{F}$ over the end-point $x_\star$ gives

$$x_\star^2 \quad = \quad \frac{b_{\text{tot}}^2}{2(8-N_f)}\,. \tag{2.4.10}$$

We may then evaluate the free energy by substituting these saddle point configurations back into (2.4.7) to obtain

$$\mathcal{F} \quad = \quad -\frac{\sqrt{2}\pi b_{\text{tot}}^3}{15\sqrt{8-N_f}\;b_1b_2b_3}N^{5/2}+\mathcal{O}\left(N^{3/2}\right)\,, \tag{2.4.11}$$

which may be rewritten as (where recall we have assumed that $b_i > 0$ for each $i = 1, 2, 3$)

$$\mathcal{F} \quad = \quad \frac{(b_1+b_2+b_3)^3}{27b_1b_2b_3}\mathcal{F}_{S^5_{\text{round}}}\,, \tag{2.4.12}$$

where $\mathcal{F}_{S^5_{\text{round}}}$ is the large N limit of the free energy on the round five-sphere computed in reference [7]

$$\mathcal{F}_{S^5_{\text{round}}} \quad = \quad -\frac{9\sqrt{2}\pi N^{5/2}}{5\,\sqrt{8-N_f}}+\mathcal{O}\left(N^{3/2}\right)\,. \tag{2.4.13}$$

We note that the above result has a very similar structure to that obtained in three dimensions [34]. Also notice that we get the same result, (2.4.12), for the orbifold theories discussed in [7, 12].

We conclude this chapter by noting that for the $SU(3) \times U(1)$ squashed five-spheres, with the vector $\mathbf{b} = (b_1, b_2, b_3)$ given by (2.3.13), we obtain the large N free energies

$$\mathcal{F} = \begin{cases} \dfrac{1}{27s^2} \dfrac{(3-\sqrt{1-s^2})^3}{1-\sqrt{1-s^2}} \mathcal{F}_{S^5_{\text{round}}} , & 3/4 \text{ BPS} \quad , \\ \mathcal{F}_{S^5_{\text{round}}} , & 1/4 \text{ BPS} \quad . \end{cases} \qquad (2.4.14)$$

Chapter 3

Romans $F(4)$ supergravity

When the $USp(2N)$ superconformal theories discussed in chapter 2 are put on the round S^5, they are conjectured to be dual in the large N limit to the $\mathrm{AdS}_6 \times S^4$ solution of massive type IIA supergravity [10, 11, 12]. In order to find gravity duals to the same superconformal theories put on different background five-manifolds, it is then natural to work in the six-dimenional Romans $F(4)$ supergravity theory [13]. The key here is that, as shown in [14], the Romans theory is a consistent truncation of massive type IIA supergravity on S^4. In the next subsection we shall review this uplift to ten dimensions, and then present the Romans theory in Euclidean signature in section 3.2.

3.1 Uplift to massive type IIA

The Romans theory [13] is a six-dimensional gauged supergravity that admits an AdS_6 vacuum. The bosonic fields consist of the metric, a dilaton ϕ, a two-form potential B, a one-form potential A, together with an $SU(2) \sim SO(3)$ gauge field A^i, $i = 1, 2, 3$. It is convenient to introduce the scalar field $X \equiv \exp(-\phi/2\sqrt{2})$, and we define the field strengths as $H = \mathrm{d}B$, $F = \mathrm{d}A + \frac{2}{3}gB$, $F^i = \mathrm{d}A^i - \frac{1}{2}g\varepsilon_{ijk}A^j \wedge A^k$. Here g denotes the gauge coupling constant. Notice that B appears in the field strength for A.

As shown in [14], this Romans theory is a consistent truncation of massive type IIA supergravity on S^4. This means that any solution to the Romans theory automatically uplifts, via the non-linear

Kaluza-Klein ansatz of [14] presented in (3.1.1) below, to a solution of massive type IIA. Moreover, the $\mathrm{AdS}_6 \times S^4$ solution of the latter is the uplift of the AdS_6 vacuum of the Romans theory.

We shall later need some details of how the six-dimensional solutions uplift to ten dimensions. The gauge coupling constant g is related to the ten-dimensional mass parameter by $m_{\mathrm{IIA}} = \frac{\sqrt{2}}{3}g$, while the remaining fields uplift via

$$
\begin{aligned}
\mathrm{d}s^2_{10} &= (\sin\xi)^{\frac{1}{12}} X^{\frac{1}{8}} \left[\Delta^{\frac{3}{8}} \mathrm{d}s^2_6 + 2g^{-2}\Delta^{\frac{3}{8}} X^2 \mathrm{d}\xi^2 + \tfrac{1}{2} g^{-2} \Delta^{-\frac{5}{8}} X^{-1} \cos^2\xi \sum_{i=1}^{3} (\hat{\tau}^i - gA^i)^2 \right] , \\
F_{(4)} &= -\tfrac{\sqrt{2}}{6} g^{-3} s^{1/3} c^3 \Delta^{-2} U \,\mathrm{d}\xi \wedge \mathrm{vol}_3 - \sqrt{2} g^{-3} s^{4/3} c^4 \Delta^{-2} X^{-3} \,\mathrm{d}X \wedge \mathrm{vol}_3 \\
&\quad + \sqrt{2} g^{-1} s^{1/3} c X^4 * H \wedge \mathrm{d}\xi - \tfrac{1}{\sqrt{2}} s^{4/3} X^{-2} * F + \tfrac{1}{\sqrt{2}} g^{-2} s^{1/3} c\, F^i h^i \wedge \mathrm{d}\xi \\
&\quad - \tfrac{1}{4\sqrt{2}} g^{-2} s^{4/3} c^2 \Delta^{-1} X^{-3} F^i \wedge h^j \wedge h^k \,\varepsilon_{ijk} , \\
F_{(3)} &= s^{2/3} H + g^{-1} s^{-1/3} c\, F \wedge \mathrm{d}\xi , \\
F_{(2)} &= \tfrac{1}{\sqrt{2}} s^{2/3} F , \qquad \mathrm{e}^{\Phi} = s^{-5/6} \Delta^{1/4} X^{-5/4} ,
\end{aligned}
\tag{3.1.1}
$$

where

$$
\begin{aligned}
\Delta &\equiv X \cos^2\xi + X^{-3} \sin^2\xi , \\
U &\equiv X^{-6} s^2 - 3X^2 c^2 + 4X^{-2} c^2 - 6X^{-2} .
\end{aligned}
\tag{3.1.2}
$$

Here $\mathrm{d}s^2_{10}$ is the ten-dimensional metric in Einstein frame, Φ is the ten-dimensional dilaton, $F_{(3)}$ is the NS-NS three-form field strength, while $F_{(4)}$ and $F_{(2)}$ are the RR four-form and two-form field strengths, respectively. The $\hat{\tau}^i$, $i = 1, 2, 3$, are left-invariant one-forms on a copy of $SU(2) \cong S^3$. These are defined precisely as in (2.2.6), except here this S^3 is in the internal space (hence the hats). We have also defined $h^i \equiv \hat{\tau}^i - gA^i$, $\mathrm{vol}_3 \equiv h^1 \wedge h^2 \wedge h^3$, and $s = \sin\xi$ and $c = \cos\xi$. The Hodge duals in (3.1.1) are computed with respect to the six-dimensional metric $\mathrm{d}s^2_6$. This is defined on some six-manifold M_6, and the ten-dimensional metric in (3.1.1) then describes a warped product $M_6 \times S^4$. More precisely, the solution only describes "half" of a four-sphere, where the coordinate $\xi \in (0, \frac{\pi}{2}]$ is a polar coordinate for which constant $\xi \in (0, \frac{\pi}{2})$ slices are three-spheres, parametrized by Euler angles on S^3 as in (2.2.6). The solution is smooth at the north pole $\xi = \frac{\pi}{2}$, where the S^3

slices of S^4 collapse to zero size, but singular on the equator $\xi = 0$. Nevertheless, it is argued in [11, 12] that the supergravity solution (3.1.1) can be trusted away from this singularity.

3.2 Euclidean theory

The equations of motion and action for the Romans theory in Lorentz signature appear in [13, 14]. However, the gravity duals to the large N field theories on the squashed five-sphere of chapter 2 will be constructed in Euclidean signature. The corresponding Wick rotation is not entirely straightforward because the Romans theory contains Chern-Simons-type couplings, that become purely imaginary in Euclidean signature in order that the theory is gauge invariant. The associated factors of i are also crucial for supersymmetry in Euclidean signature. The Euclidean equations of motion for the Romans supergravity fields are

$$\begin{aligned} \mathrm{d}\left(X^4 * H\right) &= \tfrac{\mathrm{i}}{2} F\wedge F + \tfrac{\mathrm{i}}{2} F^i \wedge F^i + \tfrac{2}{3} g X^{-2} * F \ , \\ \mathrm{d}(X^{-2} * F) &= -\mathrm{i} F\wedge H \ , \\ D(X^{-2} * F^i) &= -\mathrm{i} F^i \wedge H \ , \\ \mathrm{d}\left(X^{-1} * \mathrm{d}X\right) &= -g^2\left(\tfrac{1}{6}X^{-6} - \tfrac{2}{3}X^{-2} + \tfrac{1}{2}X^2\right) * 1 \\ &\quad -\tfrac{1}{8}X^{-2}\left(F\wedge *F + F^i\wedge *F^i\right) + \tfrac{1}{4}X^4 H\wedge *H \ . \end{aligned} \tag{3.2.1}$$

Here $D\omega^i = \mathrm{d}\omega^i - g\varepsilon_{ijk}A^j\wedge\omega^k$ is the $SO(3)$ covariant derivative, and our convention for the Hodge duality operator is fixed via

$$\alpha\wedge *\beta = \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p}\beta^{\mu_1\cdots\mu_p} * 1 \ , \tag{3.2.2}$$

where α and β are p-forms.[1] The Einstein equation is

$$\begin{aligned} R_{\mu\nu} &= 4X^{-2}\partial_\mu X\partial_\nu X + g^2\left(\tfrac{1}{18}X^{-6} - \tfrac{2}{3}X^{-2} - \tfrac{1}{2}X^2\right)g_{\mu\nu} + \tfrac{1}{4}X^4\left(H^2_{\mu\nu} - \tfrac{1}{6}H^2 g_{\mu\nu}\right) \\ &\quad + \tfrac{1}{2}X^{-2}\left(F^2_{\mu\nu} - \tfrac{1}{8}F^2 g_{\mu\nu}\right) + \tfrac{1}{2}X^{-2}\left((F^i)^2_{\mu\nu} - \tfrac{1}{8}(F^i)^2 g_{\mu\nu}\right) \ , \end{aligned} \tag{3.2.3}$$

where $F^2_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho$, $H^2_{\mu\nu} = H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}$.

[1] In particular this convention differs from that in [14].

The Euclidean action which gives rise to these field equations is

$$
\begin{aligned}
I_E \;=\; -\frac{1}{16\pi G_N}\int \Big[& R*1 - 4X^{-2}\mathrm{d}X\wedge *\mathrm{d}X - g^2\left(\tfrac{2}{9}X^{-6} - \tfrac{8}{3}X^{-2} - 2X^2\right)*1 \\
& - \tfrac{1}{2}X^{-2}\left(F\wedge *F + F^i\wedge *F^i\right) - \tfrac{1}{2}X^4 H\wedge *H \qquad (3.2.4)\\
& - \mathrm{i}B\wedge\left(\tfrac{1}{2}\mathrm{d}A\wedge\mathrm{d}A + \tfrac{1}{3}B\wedge\mathrm{d}A + \tfrac{2}{27}g^2 B\wedge B + \tfrac{1}{2}F^i\wedge F^i\right)\Big]\,.
\end{aligned}
$$

In particular notice that the final term is a Chern-Simons-type coupling, and is accompanied by a factor of i. This is required for gauge-invariance in the path integral with Euclidean measure $\exp(-I_E)$. It is also implied by supersymmetry. Indeed, a solution to the above equations of motion is supersymmetric provided the following Killing spinor equation and dilatino equation hold:

$$
\begin{aligned}
D_\mu\epsilon_I \;=\;& \frac{\mathrm{i}}{4\sqrt{2}}g(X+\tfrac{1}{3}X^{-3})\Gamma_\mu\Gamma_7\epsilon_I - \frac{\mathrm{i}}{16\sqrt{2}}X^{-1}F_{\nu\rho}(\Gamma_\mu{}^{\nu\rho} - 6\delta_\mu{}^{\nu}\Gamma^\rho)\epsilon_I \qquad (3.2.5)\\
& -\frac{1}{48}X^2 H_{\nu\rho\sigma}\Gamma^{\nu\rho\sigma}\Gamma_\mu\Gamma_7\epsilon_I + \frac{1}{16\sqrt{2}}X^{-1}F^i_{\nu\rho}(\Gamma_\mu{}^{\nu\rho} - 6\delta_\mu{}^{\nu}\Gamma^\rho)\Gamma_7(\sigma^i)_I{}^J\epsilon_J\,,\\
0 \;=\;& -\mathrm{i}X^{-1}\partial_\mu X\Gamma^\mu\epsilon_I + \frac{1}{2\sqrt{2}}g\left(X - X^{-3}\right)\Gamma_7\epsilon_I + \frac{\mathrm{i}}{24}X^2 H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_7\epsilon_I\\
& -\frac{1}{8\sqrt{2}}X^{-1}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon_I - \frac{\mathrm{i}}{8\sqrt{2}}X^{-1}F^i_{\mu\nu}\Gamma^{\mu\nu}\Gamma_7(\sigma^i)_I{}^J\epsilon_J\,. \qquad (3.2.6)
\end{aligned}
$$

Here ϵ_I, $I = 1, 2$, are two Dirac spinors, Γ_μ generate the Clifford algebra Cliff(6, 0) in an orthonormal frame, and we have defined the chirality operator $\Gamma_7 = \mathrm{i}\Gamma_{012345}$, which satisfies $\Gamma_7^2 = 1$. The $SO(3) \sim SU(2)$ gauge field A^i is an R-symmetry gauge field, with the spinor ϵ_I transforming in the two-dimensional representation via the Pauli matrices $(\sigma^i)_I{}^J$. Thus the covariant derivative acting on the spinor is $D_\mu\epsilon_I = \nabla_\mu\epsilon_I + \frac{\mathrm{i}}{2}gA^i_\mu(\sigma^i)_I{}^J\epsilon_J$.

Returning to the equations of motion (3.2.1), notice that the exterior derivative of the first equation (the equation of motion for B) implies the second equation on using the Bianchi identities for F and F^i, where note that $\mathrm{d}F = \frac{2}{3}gH$. This is related to the fact that the theory possesses a gauge invariance $A \to A + \frac{2}{3}g\lambda$, $B \to B - \mathrm{d}\lambda$, where λ is an arbitrary one-form. Using this freedom one can then gauge away $A = 0$, leaving $F = \frac{2}{3}gB$. The kinetic term for F in the action (3.2.4) then becomes a mass term for the B-field; that is, the B-field "eats" the $U(1)$ gauge field A in a Higgs-like mechanism. Notice that there is also a cubic Chern-Simons coupling for B in (3.2.4), making it a somewhat exotic field. We may also make a simple rescaling of the fields via

$g_{\mu\nu} \to \frac{1}{g^2} g_{\mu\nu}$, $B \to \frac{1}{g^2} B$, $A \to \frac{1}{g} A$, $A^i \to \frac{1}{g} A^i$, after which one sees that the coupling constant g only appears in the action as an overall constant $1/g^4$ factor. Thus we may without loss of generality set $g = 1$, which we henceforth will do.

In appendix A we compute the integrability conditions for the Killing spinor equation (3.2.5) and dilatino equation (3.2.6), and show that these are compatible with the equations of motion (3.2.1), (3.2.3).

3.3 Killing vector bilinear

Given a supersymmetric solution to the Euclidean Romans theory, one can verify that the bilinear

$$K_\mu \equiv \varepsilon^{IJ} \epsilon_I^T \mathcal{C} \Gamma_\mu \epsilon_J \ , \tag{3.3.1}$$

is a Killing one-form. Here $\mathcal{C}$ is the charge conjugation matrix, satisfying $\Gamma_\mu^T = \mathcal{C}^{-1} \Gamma_\mu \mathcal{C}$ and in our conventions is antisymmetric satisfying $\mathcal{C}^2 = -1$. If we also impose a symplectic Majorana condition

$$\mathcal{C} \epsilon_I^* = \varepsilon_I{}^J \epsilon_J \ , \tag{3.3.2}$$

then this Killing one-form may be rewritten as

$$K_\mu = \epsilon_I^\dagger \Gamma_\mu \epsilon_I \ , \tag{3.3.3}$$

which is then manifestly real. In particular we will be able to impose this symplectic Majorana condition for the solutions we construct in chapter 4. In this "real" case the Killing spinors ϵ_I define an $SU(2)$ structure on M_6. One could similarly analyse the differential conditions on the corresponding $SU(2)$ structure bilinears, which will be done in chapter 8.

Chapter 4

Supergravity solutions

In this chapter we present supergravity duals to the $SU(3) \times U(1)$ squashed five-sphere backgrounds of chapter 2. Via the consistent truncation to the Romans theory in the previous chapter, this effectively becomes a filling problem in six-dimensional gauged supergravity: one seeks a smooth, asymptotically locally Euclidean AdS_6 supersymmetric supergravity solution, with conformal boundary data given by the squashed five-sphere background in chapter 2. In particular this means the bulk supergravity solution is equipped with an $SU(2)_R$ doublet of Killing spinors ϵ_I, $I = 1, 2$, solving (3.2.5) and (3.2.6), which should suitably approach the boundary Killing spinors in chapter 2.2. We shall indeed find such fillings for both the 3/4 BPS and 1/4 BPS solutions. In the process shall extend the 1/4 BPS solution to a two-parameter family of solutions, containing a one-parameter 1/2 BPS subfamily of new solutions.

4.1 $SU(3) \times U(1)$ invariant ansatz

The squashed five-sphere backgrounds of section 2.2 have $SU(3) \times U(1)$ symmetry, and one expects this symmetry to be preserved by the bulk supergravity filling. Indeed, for asymptotically locally Euclidean AdS solutions of the *vacuum* Einstein equations this is a theorem [35]. This leads to the following ansatz for the Romans supergravity fields

$$\begin{aligned} \mathrm{d}s_6^2 &= \alpha^2(r)\mathrm{d}r^2 + \gamma^2(r)(\mathrm{d}\tau + C)^2 + \beta^2(r)\Big[\mathrm{d}\sigma^2 + \frac{1}{4}\sin^2\sigma(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \\ &\quad + \frac{1}{4}\cos^2\sigma\sin^2\sigma(\mathrm{d}\psi + \cos\theta\mathrm{d}\varphi)^2\Big] , \\ B &= p(r)\mathrm{d}r \wedge (\mathrm{d}\tau + C) + \frac{1}{2}q(r)\mathrm{d}C , \\ A^i &= f^i(r)(\mathrm{d}\tau + C) , \quad i = 1,2,3 , \end{aligned} \tag{4.1.1}$$

together with $X = X(r)$. Recall here that we have used the gauge freedom to set the $U(1)$ gauge field (which is really a Stueckelberg field) to $A = 0$. The additional coordinate r is a radial coordinate, and we shall choose a parametrization in which the conformal boundary is at $r = \infty$. For fixed r, provided $\gamma(r)$ and $\beta(r)$ are non-zero the constant r surfaces in (4.1.1) are squashed five-spheres. We shall seek solutions with the topology of a ball, so that $r \in [r_0, \infty)$ with $r = r_0$ being the origin. At this point the squashed five-spheres must become *round* in order that the metric extends smoothly to the origin of the ball. Similarly, in order for the gauge fields B, A^i in (4.1.1) to be non-singular at the origin they must tend to zero sufficiently quickly at $r = r_0$. In writing the ansatz (4.1.1) we have used the fact that the only $SU(3)\times U(1)$ invariant one-form on the squashed five-sphere is the global angular form $\mathrm{d}\tau + C$ for the Hopf fibration $S^1 \hookrightarrow S^5 \to \mathbb{CP}^2$, while the only invariant two-form is the pull-back $\frac{1}{2}\mathrm{d}C = \omega$ of the Kähler form on $\mathbb{CP}^2$.

Substituting the cohomogeneity one ansatz (4.1.1) into the equations of motion (3.2.1) and Einstein equation (3.2.3) leads to a rather complicated coupled system of ODEs. The equations of motion for the background $SU(2)_R$ gauge field imply $f^i(r) = \kappa_i f(r)$, $i = 1,2,3$. The equations for the other fields then depend only on the $SU(2) \sim SO(3)$ invariant $\kappa_1^2 + \kappa_2^2 + \kappa_3^2$, which we can set to one by rescaling $f(r)$. The equations of motion then result in the coupled ODEs for the functions $\alpha(r)$, $\beta(r)$, $\gamma(r)$, $p(r)$, $q(r)$, $f(r)$, $X(r)$, which can be found in appendix B.1.

Since the solutions we find are continuously connected to Euclidean AdS_6, we first present the latter in these coordinates:

$$\begin{aligned}\alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}}\ , \qquad \beta(r) \;=\; \gamma(r) \;=\; \frac{3\sqrt{6r^2-1}}{\sqrt{2}}\ ,\\ p(r) &= q(r) \;=\; f(r) \;=\; 0\ , \qquad X(r) \;=\; 1\ . \end{aligned} \tag{4.1.2}$$

Here only the metric is non-trivial, and (4.1.2) realizes Euclidean AdS_6 as a hyperbolic ball with radial coordinate $r \in [\frac{1}{\sqrt{6}}, \infty)$, with the conformal boundary at infinity $r = \infty$. Thus the origin is at $r_0 = \frac{1}{\sqrt{6}}$. Notice in particular that the conformal boundary at $r = \infty$ is equipped with a *round* metric on S^5, which is conformally flat. We would like to find families of solutions that generalize (4.1.2) by allowing for a squashed five-sphere boundary, keeping the metric asymptotically locally Euclidean AdS near $r = \infty$. That is, near $r = \infty$ the metric should approach

$$\mathrm{d}s^2_6 \;\simeq\; \frac{9\mathrm{d}r^2}{2r^2} + 27r^2\mathrm{d}s^2_5\ , \tag{4.1.3}$$

where $\mathrm{d}s^2_5$ is the squashed five-sphere (2.2.1). For such solutions we may thus define the squashing parameter by

$$\lim_{r\to\infty}\frac{\gamma(r)}{r} \;=\; 3\sqrt{3}\ \frac{1}{s}\ , \tag{4.1.4}$$

so that $s = 1$ for the round sphere. Even though we did not manage to find supersymmetric solutions in closed form using this approach, solutions may nevertheless be given as expansions around different limits. However, later in this thesis, using the $SU(2)$-structure to be developed, we show that there are analytic solutions to this case. In general notice that we can use reparametrization invariance to set

$$\beta(r) \;=\; \frac{3\sqrt{6r^2-1}}{\sqrt{2}}\ , \tag{4.1.5}$$

which we assume henceforth. In particular this fixes the origin of the ball to be at $r_0 = \frac{1}{\sqrt{6}}$.

In the following we summarize the various families of supersymmetric solutions we have constructed with the ansatz (4.1.1). Details of the computations may be found in appendix B.

4.2 3/4 BPS solutions

There is a one-parameter family of 3/4 BPS solutions parametrized by the squashing parameter s. The solution expanded around the conformal boundary is given by

$$
\begin{aligned}
\alpha(r) &= \frac{3}{\sqrt{2}}\frac{1}{r} + \frac{8+s^2}{36\sqrt{2}s^2}\frac{1}{r^3} + \dots\ , \\
\gamma(r) &= \frac{3\sqrt{3}}{s}r + \frac{-16+7s^2}{12\sqrt{3}s^3}\frac{1}{r} - \frac{-1280+1120s^2+241s^4}{2592\sqrt{3}s^5}\frac{1}{r^3} + \dots\ , \\
X(r) &= 1 + \frac{1-s^2-3\sqrt{1-s^2}}{54s^2}\frac{1}{r^2} + \frac{s^2\sqrt{1-s^2}\kappa}{12\left(1-s^2+\sqrt{1-s^2}\right)}\frac{1}{r^3} + \dots\ , \\
p(r) &= -\frac{\mathrm{i}\sqrt{\frac{2}{3}}\left(s^2+3\sqrt{1-s^2}-1\right)}{s^3}\frac{1}{r^2} + \dots\ , \\
q(r) &= -\frac{3\mathrm{i}\left(\sqrt{6}\sqrt{1-s^2}\right)}{s}r + \frac{\sqrt{\frac{2}{3}}\mathrm{i}\sqrt{1-s^2}\left(5s^2+9\sqrt{1-s^2}-5\right)}{3s^3}\frac{1}{r} + \dots\ , \\
f(r) &= \frac{1-s^2+\sqrt{1-s^2}}{s^2} + \frac{2\left(-2+2s^2-(2+s^2)\sqrt{1-s^2}\right)}{9s^4}\frac{1}{r^2} + \frac{\kappa}{r^3} + \dots\ ,
\end{aligned}
\tag{4.2.1}
$$

where we have computed this expansion up to $\mathcal{O}(1/r^9)$. The extra parameter κ is fixed by requiring regularity at the origin $r = \frac{1}{\sqrt{6}}$ (see (4.2.3) below). Notice that the $SU(2)_R$ gauge field at the conformal boundary agrees with the gauge field (2.2.4) with $Q = 1$. We may also expand the solution around Euclidean AdS_6, which has $s = 1$:

$$
\begin{aligned}
\alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}} \\
&\quad + \frac{\left(-5\sqrt{6}+330\sqrt{6}r^2-3744r^3+1620\sqrt{6}r^4+8640r^5-7560\sqrt{6}r^6+5184\sqrt{6}r^8\right)}{9\sqrt{2}r^2(6r^2-1)^{9/2}}(1-s) + \dots\ , \\
\gamma(r) &= \frac{3\sqrt{6r^2-1}}{\sqrt{2}} \\
&\quad - \frac{\left(55\sqrt{2}-384\sqrt{3}r+1080\sqrt{2}r^2+768\sqrt{3}r^3-5400\sqrt{2}r^4+11232\sqrt{2}r^6-11664\sqrt{2}r^8\right)}{6(6r^2-1)^{7/2}}(1-s) + \dots\ , \\
X(r) &= 1 - \frac{\left(\sqrt{2}\left(1-2\sqrt{6}r+6r^2\right)\right)}{3\left(6r^2-1\right)^2}\sqrt{1-s} + \dots\ , \\
p(r) &= \frac{18\mathrm{i}\sqrt{2}\left(\sqrt{6}-16r+12\sqrt{6}r^2-12\sqrt{6}r^4\right)}{\left(6r^2-1\right)^3}\sqrt{1-s} + \dots\ , \\
q(r) &= -\frac{3\mathrm{i}\sqrt{2}\left(-4+9\sqrt{6}r-24r^2-12\sqrt{6}r^3+36\sqrt{6}r^5\right)}{\left(6r^2-1\right)^2}\sqrt{1-s} + \dots\ , \\
f(r) &= \frac{\sqrt{2}\left(-3+8\sqrt{6}r-36r^2+36r^4\right)}{\left(6r^2-1\right)^2}\sqrt{1-s} + \dots\ .
\end{aligned}
\tag{4.2.2}
$$

In particular one can check that these functions lead to a regular solution at the origin $r = \frac{1}{\sqrt{6}}$, although this is not manifest in the formulas presented above. Indeed, we have computed this expansion up to sixth order, and by comparing the two expansions we find that regularity at the origin fixes the parameter κ in (4.2.1) via

$$\frac{3\sqrt{3}}{4}\kappa = \delta + \frac{\sqrt{2}}{3}\delta^2 + \frac{113}{36}\delta^3 + \frac{25}{9\sqrt{2}}\delta^4 + \frac{1127}{288}\delta^5 + \frac{35}{9\sqrt{2}}\delta^6 + \dots , \tag{4.2.3}$$

where we have introduced

$$\delta^2 \equiv \frac{1}{s} - 1 . \tag{4.2.4}$$

The explicit solution ϵ_I to the Killing spinor (3.2.5) and dilatino equation (3.2.6) for this solution may be found in appendix B. In particular there are three independent constants of integration after imposing the symplectic Majorana condition (3.3.2). Using this solution one can compute the Killing vector bilinear (3.3.1). Requiring that this Killing vector lies in the Lie algebra of the maximal torus $U(1)^3 \subset SU(3) \times U(1)$ fixes the constants of integration, up to an overall irrelevant scaling. In this case we obtain

$$K = b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2} + b_3\partial_{\varphi_3} , \tag{4.2.5}$$

where $b_1 = 1 + \sqrt{1-s^2}$, $b_2 = b_3 = 1 - \sqrt{1-s^2}$ and the coordinates φ_i are related to τ, ψ and φ via (2.3.7).

4.3 1/4 BPS solutions

We also find a two-parameter family of 1/4 BPS solutions, parametrized by the squashing parameter s and the background $SU(2)_R$ field at the conformal boundary, which is parametrized by f_0. The solution expanded around the conformal boundary is given by

$$\begin{aligned}
\alpha(r) &= \frac{3}{\sqrt{2}}\frac{1}{r} - \frac{f_0^2 s^2 + 9\left(-2+s^2\right) - 6f_0\left(-1+s^2\right)}{36\sqrt{2}}\frac{1}{r^3} + \dots \ , \\
\gamma(r) &= \frac{3\sqrt{3}}{s} r + \frac{2f_0^2 s^2 - 12 f_0\left(-1+s^2\right) + 9\left(-3+2s^2\right)}{12\sqrt{3}s}\frac{1}{r} + \dots \ , \\
X(r) &= 1 + \frac{18 - 3f_0 - 18s^2 + 12 f_0 s^2 - 2f_0^2 s^2}{54}\frac{1}{r^2} + \dots \ , \\
p(r) &= \frac{\mathrm{i}\sqrt{\frac{2}{3}}(-3+f_0)\left(3+(-3+f_0)s^2\right)}{s}\frac{1}{r^2} + \dots \ , \\
q(r) &= -\frac{3\mathrm{i}\sqrt{6}\left(3+(-3+f_0)s^2\right)}{s} r \\
&\quad + \frac{\mathrm{i}\left(3+(-3+f_0)s^2\right)\left(f_0^2 s^2 + 9\left(-1+s^2\right) - 6f_0\left(1+s^2\right)\right)}{6\sqrt{6}s}\frac{1}{r} + \frac{\xi_1}{r^2} + \dots \ , \\
f(r) &= f_0 + \frac{2(-3+f_0)f_0}{9}\frac{1}{r^2} + \frac{\xi_2}{r^3} + \dots \ .
\end{aligned} \tag{4.3.1}$$

Again, we have found this solution up to $\mathcal{O}(1/r^9)$. The constants ξ_1 and ξ_2 are again fixed by requiring regularity at the origin.

There are a number of interesting special cases. First, we obtain the one-parameter family of 1/4 BPS squashed five-spheres of section 2.2 by choosing the constant f_0 so as to reproduce (2.2.4) with $Q = -3$. That is, $f_0 = (1 - 3\sqrt{1-s^2})\sqrt{1-s^2}/s^2$. We show explicitly in appendix B that the supergravity Killing spinor matches onto the five-dimensional spinors in section 2.2. Another interesting case is $f_0 = 0$. In this case the $SU(2)_R$ background gauge field is completely switched off, but the solution is still supersymmetric with a squashed five-sphere at the conformal boundary. This solution has enhanced supersymmetry – as we show in appendix B it is 1/2 BPS. On the other hand we may also set $s = 1$, so that the conformal boundary is the *round* five-sphere, but keep the parameter f_0. This shows that one can define non-trivial Killing spinors on the *round* S^5 by turning on other fields.

We may also expand the solution around Euclidean AdS$_6$ with $s=1$:

$$\begin{aligned}
\alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}} + \frac{\sqrt{3}\left(1-54r^2+96\sqrt{6}r^3-324r^4+216r^6\right)}{2r^2\left(6r^2-1\right)^{7/2}}(1-s)+\ldots\ ,\\
\gamma(r) &= \frac{3\sqrt{6r^2-1}}{\sqrt{2}} + \frac{\left(15-48\sqrt{6}r+270r^2-540r^4+648r^6\right)}{\sqrt{2}\left(6r^2-1\right)^{5/2}}(1-s)+\ldots\ ,\\
X(r) &= 1+\frac{\left(1-2\sqrt{6}r+6r^2\right)(4+\omega)}{\left(6r^2-1\right)^2}(1-s)+\ldots\ ,\\
p(r) &= -\frac{18\mathrm{i}\sqrt{2}\left(-\sqrt{3}+8\sqrt{2}r-12\sqrt{3}r^2+12\sqrt{3}r^4\right)(6+\omega)}{\left(6r^2-1\right)^3}(1-s)+\ldots\ ,\\
q(r) &= -\frac{3\mathrm{i}\left(-4+9\sqrt{6}r-24r^2-12\sqrt{6}r^3+36\sqrt{6}r^5\right)(6+\omega)}{\left(6r^2-1\right)^2}(1-s)+\ldots\ ,\\
f(r) &= \frac{\left(-3+8\sqrt{6}r-36r^2+36r^4\right)\omega}{\left(6r^2-1\right)^2}(1-s)+\ldots\ ,
\end{aligned} \tag{4.3.2}$$

where we have introduced the parameter ω via $(1-s)\omega = f_0$. As before it can be checked explicitly that the solution is regular at $r=\frac{1}{\sqrt{6}}$, and we have checked this up to fourth order in the expansion variable

$$\delta \ \equiv\ \frac{1}{s}-1\ . \tag{4.3.3}$$

Comparing this expansion with the expansion around the conformal boundary we deduce

$$\begin{aligned}
\xi_1 &= 2\mathrm{i}(6+\omega)\delta - \frac{\mathrm{i}\left(144+98\omega+13\omega^2\right)}{5}\delta^2\\
&+ \frac{\mathrm{i}\left(307719+209547\omega+41094\omega^2+1282\omega^3\right)}{9450}\delta^3\\
&- \frac{\mathrm{i}\left(26693550+21683700\omega+6126111\omega^2+771474\omega^3+51568\omega^4\right)}{623700}\delta^4+\ldots\ ,\\
\xi_2 &= \frac{2}{3}\sqrt{\frac{2}{3}}\omega\delta - \frac{2\left(-\sqrt{6}\omega+2\sqrt{6}\omega^2\right)}{45}\delta^2 + \frac{\left(-999\sqrt{6}\omega-594\sqrt{6}\omega^2+244\sqrt{6}\omega^3\right)}{42525}\delta^3\\
&+ \frac{\left(32724\sqrt{6}\omega+26082\sqrt{6}\omega^2+6105\sqrt{6}\omega^3+935\sqrt{6}\omega^4\right)}{1403325}\delta^4+\ldots\ .
\end{aligned} \tag{4.3.4}$$

The explicit solution ϵ_I to the dilatino and Killing spinor equation (3.2.6), (3.2.5) for this solution may also be found in appendix B. In this case there is a single integration constant (for generic f_0, or equivalently ω). The Killing vector automatically lies in the Lie algebra of the torus $U(1)^3 \subset SU(3)\times U(1)$, and with an appropriate scaling we obtain

$$K \ =\ \partial_\tau \ =\ b_1\partial_{\varphi_1}+b_2\partial_{\varphi_2}+b_3\partial_{\varphi_3}\ , \tag{4.3.5}$$

where $b_1=b_2=b_3=1$ and the coordinates φ_i are related to τ, ψ and φ via (2.3.7).

Chapter 5

Holographic free energy

In this section we describe how the on-shell action for the Euclidean Romans theory detailed in chapter 3 can be computed, and for asymptotically locally Euclidean AdS solutions holographically renormalized by adding boundary counterterms [36, 37, 38]. For the supersymmetric solutions presented in chapter 4 we evaluate the renormalized on-shell action and determine the holographic free energies.

5.1 Renormalization theory

The AdS/CFT correspondence conjectures an equivalence between gravity theory in a d dimensional AdS space and a conformal field theory living in its $d-1$ dimensional conformal boundary [39]. The correspondence equates generating functions

$$Z_{AdS}(\phi_{0,i}) = Z_{CFT}(\phi_{0,i}), \tag{5.1.1}$$

where $\phi_{0,i}$ have different meanings in each theory. In the supergravity side, $\phi_{0,i}$ are the boundary values of the bulk fields, while in the field theory, $\phi_{0,i}$ are the external source currents coupled to operators, so that AdS fields are dual to CFT operators [36]. We shall mainly be concerned with the partition function, as seen in chapter 2, where we use a saddle point approximation. The duality then can more precisely by written as

$$e^{-I_{AdS(\phi_i)}} = \left\langle e^{\int \phi_{0,i}\mathcal{O}^i} \right\rangle_{CFT} , \tag{5.1.2}$$

where $I_{AdS}(\phi)$ is the on-shell gravitational action and $\mathcal{O}^i$ are the CFT dual operators.

One difficulty of this equality is in the fact that the gravity action diverges. As we expect the dual CFT to have a finite partition function after appropriate regularization, we must be able to remove the divergences in the supergravity theory. We do this by defining local counterterms on the boundary.

Consider the Euclidean action given by

$$I = -\frac{1}{16\pi R}\int_{\mathcal{M}} d^{n+1}x\sqrt{g}\left(R + \frac{n(n-1)}{l^2}\right) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^n x\sqrt{h}\mathcal{K}, \tag{5.1.3}$$

where the first term is the usual Einstein-Hilbert action, evaluated in the bulk, and the second term is the Gibbons-Hawking boundary term. The Gibbons-Hawking term is necessary so that upon variation with the metric fixed at the boundary, the action yields the Einstein equations. This involves the trace $\mathcal{K}$ of the extrinsic curvature of the boundary, and where h is the induced boundary metric, and also leads to divergences.

In the AdS context, both these terms are divergent, as the first is the volume of $\mathcal{M}$, which is infinite, and the second is a derivative of the volume in the r direction, which also diverges. But the divergences arising from this action are all proportional to local integrals of the boundary metric.

We write the gravity metric in the asymptotically locally AdS form

$$\mathrm{d}s^2 = \frac{\ell^2}{z^2}\mathrm{d}z^2 + \frac{1}{z^2}\gamma_{mn}(z,x)\mathrm{d}x^m\mathrm{d}x^n , \tag{5.1.4}$$

where l is the AdS radius, meaning that the curvature scale is essential in defining the counterterms.

The action is then modified to include the subtraction of a finite set of boundary integrals involving curvature scalars constructed from the background metric γ_{ij}.

The action then becomes

$$I_{\text{renormalized}} = I_{\text{bulk}} + S_{\text{boundary}} + I_{\text{GH}} + I_{\text{counterterms}} \tag{5.1.5}$$

where the counterterms will have the form

$$I_{\text{counterterms}} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^n x\sqrt{h}\; F(l, R, \nabla R). \tag{5.1.6}$$

One may encounter two possible problems when doing this holographic renormalization. For instance, in odd dimensional supergravities, one finds logarithmic divergences that do not cancel out. This however can be regarded as a theory check, as they agree with the manifestation of Weyl anomalies in the CFT. We will discuss this further shortly. It is also interesting to notice that if the boundary metric becomes degenerate (i.e., the metric is non compact and/or singular), one can no longer, in general, remove the divergences via counterterm regularization. This is a manifestation of that the dual CFT does not have a finite partition function in the degenerate limit.

By writing the counterterm action not as $I_{ct}(\gamma_{ij})$, but as $I_{ct}(h_{ij})$, where $h_{ij} = g_{ij} - n_i n_j$ is the induced metric on the cut-off boundary, one automatically picks up the appropriate divergences, as it becomes much easier to analyse the terms order by order, and the counterterms action can in fact be expressed as an expansion in powers of the boundary curvature and its derivatives. The number of terms that contribute grows with the dimension of the space-time (and the number of fields that are turned on in the theory).

The counterterm action will then take the form of

$$I_{ct} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^n x\sqrt{h}\;\left[\frac{n-1}{l} + \frac{l}{2(n-2)}R + \frac{l^3}{2(n-4)(n-2)^2}\left(R_{ab}R^{ab} - \frac{n}{4(n-1)}R^2\right) + \dots\right]. \tag{5.1.7}$$

These terms are sufficient to cancel out the divergences of the bulk action in dimensions up to $n \leq 6$.

Here we deal with a supergravity in an even dimension, that has no logarithmic contributions to the renormalized action. Nevertheless, it is interesting to make a comment on what happens in the odd dimensinal supergravity. The logarithmically divergent contributions in the action do not get cancelled by the counterterms, in fact, there is no way they could, as the counterterms are given by a local integral of a polynomial over the boundary of a curvature invariant.

At first, one may think that this presents a limitation for the counterterms subtraction technique for asymptotically locally AdS solutions, but instead, it turns out to be a fantastic consistency check of the AdS/CFT conjecture. The relation between these divergences in the gravity side and the conformal anomalies in the dual field theory was first spotted by Witten in 1999 [41]. It is interesting to see that these conformal anomalies in the classically conformally invariant theory are due to logarithmic UV divergences at the one loop level in the quantum field theory. In fact, this is a UV/IR relation of the AdS/CFT correspondence, meaning that an infinite volume singularity in AdS is a reflection of the existence of a UV divergence in the CFT.

5.2 On-shell action

We will now apply this general method to our case. The formalism presented in the previous section is done for pure gravity, but in principle, the holographic renormalization should work in the same way for supergravities. Indeed, this technique has been applied to supergravities before (see [37],[38],[40]), but not to Romans theory, due to its exotic structure.

Romans theory contains a scalar matter field, an Abelian gauge-field A, a $SU(2)$ gauge-field A^i and a B-field. Here we will work in the gauge $A=0$. Starting from the Euclidean action (3.2.4) and using the equations of motion (3.2.1) together with the Einstein equation (3.2.3) and its trace, we can write the on-shell action defined on a manifold M_6 with boundary ∂M_6 to be

$$I_{\text{on-shell}} = I_{\text{bulk}} + I_{\text{boundary}} \, , \tag{5.2.1}$$

previously stated. Here

$$I_{\text{bulk}} = \frac{1}{16\pi G_N} \int_{M_6} \tfrac{4}{9} X^{-2} \left(2 + 3X^4\right) * 1 + \tfrac{1}{3} X^{-2} F^i \wedge *F^i + \tfrac{\mathrm{i}}{3} B \wedge F^i \wedge F^i \, , \tag{5.2.2}$$

$$I_{\text{boundary}} = \frac{1}{16\pi G_N} \int_{\partial M_6} \tfrac{2}{3} \left(X^{-1} * \mathrm{d}X\right) + \tfrac{1}{3}(B \wedge X^4 * H) \, . \tag{5.2.3}$$

Here we have used Stokes' theorem to write a total derivative as a boundary integral. In particular this assumes that the potentials B and A^i are globally defined, which is the case for our supergravity solutions. The Hodge duals in (5.2.3) are defined on M_6, and then restricted to the boundary. The on-shell action is divergent due to the infinite volume of M_6 and ∂M_6, and from divergences in the supergravity fields as the conformal boundary $r \to \infty$ is approached. Consequently, I_{bulk} should be understood as integrated up to a finite cut-off which is then sent to infinity only after adding counterterms which regularize the divergences. In addition, as explained earlier, due to the presence of boundary terms in the on-shell action, one should add a Gibbons-Hawking term [42]. Hence the finite on-shell action is

$$I_{\text{renormalized}} = I_{\text{on-shell}} + I_{\text{GH}} + I_{\text{counterterms}} \,. \tag{5.2.4}$$

In the next section we determine the precise form of the counterterms.

5.3 Boundary counterterms

The counterterms needed to regularize the action of the Euclidean Romans $F(4)$ theory were stated without derivation in [28]. Here we provide a full account of their construction. We assume a general expansion of the fields for an asymptotically locally Euclidean AdS$_6$ solution. In particular, we take the metric to be given in Fefferman-Graham form [43]

$$\mathrm{d}s_6^2 = \frac{\ell^2}{z^2}\mathrm{d}z^2 + \frac{1}{z^2}\gamma_{mn}(z,x)\mathrm{d}x^m\mathrm{d}x^n \,, \tag{5.3.1}$$

where $\ell = 3/\sqrt{2}$ is the AdS$_6$ radius, and in turn

$$\gamma_{mn}(z,x) = \gamma^0_{mn} + z^2\gamma^2_{mn} + z^4\gamma^4_{mn} + \mathcal{O}(z^5) \,. \tag{5.3.2}$$

Here $\gamma^0_{mn}(x)$ is the metric induced on the conformal boundary which, due to the radial coordinate transformation $r \to \frac{1}{z}$, is now at $z = 0$. The Gibbons-Hawking term is then

$$I_{\text{GH}} = \frac{1}{8\pi G_N}\int_{\partial M_6} \frac{z}{\ell}\partial_z\sqrt{\det h}\ \mathrm{d}^5x \,, \tag{5.3.3}$$

and $h_{mn} = \frac{1}{z^2}\gamma_{mn}$ is the induced metric on the boundary.

The Ricci tensor of the six-dimensional metric (5.3.1) is

$$\begin{aligned}
R_{zz} &= -\frac{5}{z^2} - \frac{1}{2}\left[\mathrm{Tr}\left(\gamma^{-1}\partial_z^2\gamma\right) - \frac{1}{z}\mathrm{Tr}\left(\gamma^{-1}\partial_z\gamma\right) - \frac{1}{2}\mathrm{Tr}\left(\gamma^{-1}\partial_z\gamma\right)^2\right] , \\
R_{mn} &= -\frac{5}{\ell^2 z^2}\gamma_{mn} - \frac{1}{\ell^2}\Big[\frac{1}{2}\partial_z^2\gamma - \frac{2}{z}\partial_z\gamma - \frac{1}{2}(\partial_z\gamma)\gamma^{-1}(\partial_z\gamma) + \frac{1}{4}(\partial_z\gamma)\mathrm{Tr}\left(\gamma^{-1}\partial_z\gamma\right) \\
&\quad -\ell^2 R(\gamma) - \frac{1}{2z}\gamma\mathrm{Tr}\left(\gamma^{-1}\partial_z\gamma\right)\Big]_{mn} , \\
R_{zm} &= \tfrac{1}{2}(\gamma^{-1})^{np}\left[\nabla_m\gamma_{np,z} - \nabla_p\gamma_{mn,z}\right] ,
\end{aligned} \tag{5.3.4}$$

with ∇ being the covariant derivative for $\gamma(z,x)$. We also assume an asymptotic expansion for bulk scalar and gauge fields, namely

$$\begin{aligned}
X &= 1 + zX_1 + z^2X_2 + \cdots , \\
B &= \frac{1}{z}b + \mathrm{d}z\wedge A_0 + B_0 + z\mathrm{d}z\wedge A_1 + zB_1 + \cdots , \\
H = \mathrm{d}B &= -\frac{1}{z^2}\mathrm{d}z\wedge b + \frac{1}{z}\mathrm{d}b - \mathrm{d}z\wedge\mathrm{d}A_0 + \mathrm{d}B_0 + \mathrm{d}z\wedge B_1 - z\mathrm{d}z\wedge\mathrm{d}A_1\cdots , \\
F^i &= f^i + \mathrm{d}z\wedge A_0^i + z\mathrm{d}z\wedge A_1^i + zF_1^i + \cdots .
\end{aligned} \tag{5.3.5}$$

The $1/z$ term appearing in the B-field expansion is non-standard but is justified by being compatible with the equations of motion as we will see below.

It is useful to establish some formulas. We write (in general)

$$\alpha\wedge *\alpha = \|\alpha\|^2\mathrm{vol} , \tag{5.3.6}$$

to define the norm $\|\cdot\|$ of a p-form. The inner product of two p-forms α,β is denoted $\langle\alpha,\beta\rangle$. First we compute

$$\begin{aligned}
*\alpha_p &= -\ell z^{2p-6}\left(*_\gamma\alpha_p\right)\wedge\mathrm{d}z , \\
*(\mathrm{d}z\wedge\alpha_{p-1}) &= \frac{1}{\ell}z^{2p-6} *_\gamma \alpha_{p-1} ,
\end{aligned} \tag{5.3.7}$$

where α_p represents a general p-form that is orthogonal to ∂_z. Here the volume forms are related as

$$\mathrm{vol}_6 = \frac{\ell}{z^6}\mathrm{d}z\wedge\mathrm{vol}_\gamma = \frac{\ell}{z^6}\mathrm{d}z\wedge\sqrt{\det\gamma}\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^5 . \tag{5.3.8}$$

We will need the expansion of the determinant and Hodge dual for γ_{mn}. The former is

$$\sqrt{\det\gamma} = \sqrt{\det\gamma^0}\Big[1 + \tfrac{z^2}{2}\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right] + \tfrac{z^4}{2}\mathrm{Tr}\left[\gamma^4(\gamma^0)^{-1}\right] - \tfrac{z^4}{4}\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]^2 + \tfrac{z^4}{8}\left(\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]\right)^2 + \mathcal{O}(z^5)\Big] , \qquad (5.3.9)$$

whilst the latter may be computed similarly as

$$*_\gamma\alpha_p \quad = \quad *_{\gamma^0}\alpha_p + z^2\left[\tfrac{1}{2}\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right] *_{\gamma^0}\alpha_p - p *_{\gamma^0}(\gamma^2\circ\alpha_p)\right] + \mathcal{O}(z^4) \ . \qquad (5.3.10)$$

Here we have defined the p-form

$$(\gamma^2\circ\alpha_p)_{m_1\cdots m_p} \quad \equiv \quad (\gamma^2)_{[m_1}{}^n(\alpha_p)_{|n|m_2\cdots m_p]} \ , \qquad (5.3.11)$$

and indices are always raised with γ^0, so $(\gamma^2)_m{}^n \equiv (\gamma^2)_{mp}(\gamma^0)^{pn}$.

The idea now is to substitute these expansions into the Romans field equations and then on-shell action. We first look at the lowest order term in z in each of the X, B and Einstein equations. The leading order term in the X equation of motion dictates

$$X_1 \quad = \quad 0 \ . \qquad (5.3.12)$$

Specifically, the term $\frac{1}{z^5}\mathrm{d}z\wedge\mathrm{vol}_{\gamma^0}$ has a coefficient proportional to X_1 times a non-zero number, thus forcing $X_1 = 0$. Next one finds that the leading order term in the B equation of motion, which is proportional to $\frac{1}{z^3}\mathrm{d}z\wedge *_{\gamma^0}b$, has a coefficient that is zero if and only if $\ell^2 = 9/2$. Similarly, the leading order term in the mn component of the Einstein equation, which is $\mathcal{O}(1/z^2)$, is satisfied if and only if $\ell^2 = 9/2$. We will substitute $\ell = 3/\sqrt{2}$ from now on.

The first divergence we encounter, which is at order $\mathcal{O}(1/\epsilon^5)$ where $z = \epsilon$ is the finite cut-off, comes from expanding the $\frac{4}{9}X^{-2}\left(2+3X^4\right)*1$ integrand in I_{bulk} and the Gibbons-Hawking term. It is

$$I^{\mathrm{div}}_{\mathcal{O}(1/\epsilon^5)} \quad = \quad \frac{1}{8\pi G_N}\frac{1}{\epsilon^5}\int_{\partial M_6} -\frac{4\sqrt{2}}{3}\sqrt{\det\gamma^0}\ \mathrm{d}^5x\,, \qquad (5.3.13)$$

and is simply cancelled by adding the counterterm

$$I_5^{\mathrm{counterterm}} \quad = \quad \frac{1}{8\pi G_N}\cdot\frac{4\sqrt{2}}{3}\int_{\partial M_6}\sqrt{\det h}\ \mathrm{d}^5x\ . \qquad (5.3.14)$$

We write the counterterm action in terms of the induced boundary metric h_{mn} as the divergences most naturally appear in this form [44]. There is no divergence at $\mathcal{O}(1/\epsilon^4)$ as a consequence of $X_1 = 0$. The divergence at $\mathcal{O}(1/\epsilon^3)$ has contributions from each of I_{bulk}, I_{boundary}, I_{GH} and the expansion of $I_5^{\text{counterterm}}$, and is

$$I^{\text{div}}_{\mathcal{O}(1/\epsilon^3)} \;=\; \frac{1}{8\pi G_N}\frac{1}{\epsilon^3}\int_{\partial M_6}\left[\frac{4\sqrt{2}}{9}\text{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]+\frac{1}{9\sqrt{2}}\|b\|^2_{\gamma^0}\right]\sqrt{\det\gamma^0}\,\mathrm{d}^5x\,. \tag{5.3.15}$$

Clearly we will need some control on γ^2, and this comes from the $\mathcal{O}(1)$ term in the mn direction of the Einstein equation. Carefully expanding we find this fixes

$$\gamma^2_{mn} \;=\; -\frac{3}{2}\left[R(\gamma^0)_{mn}-\frac{1}{8}R(\gamma^0)\gamma^0_{mn}\right]+\frac{1}{2}b^2_{mn}-\frac{3}{16}\|b\|^2_{\gamma^0}\gamma^0_{mn}\,. \tag{5.3.16}$$

Here $R(g)_{mn} = \text{Ric}(g)_{mn}$ denotes the Ricci tensor of a metric g_{mn}, with $R(g)$ the Ricci scalar. The curvature terms in γ^2_{mn} are standard [36], while the terms involving b are specific to the Romans theory and boundary conditions we are considering. Taking the trace of (5.3.16), or alternatively examining the zz component of the Einstein equation at order $\mathcal{O}(1)$, gives

$$\text{Tr}\left[\gamma^2(\gamma^0)^{-1}\right] \;=\; -\frac{9}{16}R(\gamma^0)+\frac{1}{16}\|b\|^2_{\gamma^0}\,. \tag{5.3.17}$$

This expression will need to be used extensively due to its appearance in the Hodge dual and metric determinant. Substituting $\text{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]$ into the right hand side of $I^{\text{div}}_{\mathcal{O}(1/\epsilon^3)}$ leads to

$$I^{\text{div}}_{\mathcal{O}(1/\epsilon^3)} \;=\; \frac{1}{8\pi G_N}\frac{1}{\epsilon^3}\int_{\partial M_6}\left[-\frac{1}{2\sqrt{2}}R(\gamma^0)+\frac{1}{6\sqrt{2}}\|b\|^2_{\gamma^0}\right]\sqrt{\det\gamma^0}\,\mathrm{d}^5x\,, \tag{5.3.18}$$

and the appropriate counterterm is therefore

$$I_3^{\text{counterterm}} \;=\; \frac{1}{8\pi G_N}\int_{\partial M_6}\left[\frac{1}{2\sqrt{2}}R(h)-\frac{1}{6\sqrt{2}}\|B\|^2_h\right]\sqrt{\det h}\,\mathrm{d}^5x\,. \tag{5.3.19}$$

A priori there is also an $\mathcal{O}(1/\epsilon^2)$ divergence, but one easily sees from the various expansions that only the scalar field contributes to it. This term (temporarily reinstating the AdS length scale) is

$$I^{\text{div}}_{\mathcal{O}(1/\epsilon^2)} \;=\; \frac{1}{8\pi G_N}\left(\frac{4\ell}{9}\cdot\frac{1}{2}-\frac{1}{\ell}\right)\frac{1}{\epsilon^2}\int_{\partial M_6}X_3\sqrt{\det\gamma^0}\,\mathrm{d}^5x \;=\; 0\,, \tag{5.3.20}$$

where the first term comes from expanding the bulk integral (5.2.2), while the second (which cancels it) comes from the boundary $X^{-1} * \mathrm{d}X$ term in (5.2.3). Thus this potential divergence is zero, without needing a counterterm or indeed even needing to use any of the equations of motion.

Continuing we find there are many terms that contribute at $\mathcal{O}(1/\epsilon)$ including A_1 and B_0 from the asymptotic expansion of the B-field. It is prudent to look at higher orders of z in the equations of motion for simplifications along the lines of $X_1 = 0$. Indeed by looking at the $z^{-2}\mathrm{d}z \wedge \alpha_3$ coefficient of the B-field equation of motion we find

$$B_0 \;=\; 0\,. \tag{5.3.21}$$

The $z^{-1}\alpha_4$ coefficient similarly implies

$$A_1 \;=\; 0\,. \tag{5.3.22}$$

With these simplifications the $\mathcal{O}(1/\epsilon)$ divergence becomes

$$\begin{aligned} I^{\mathrm{div}}_{\mathcal{O}(1/\epsilon)} \;=\; & \frac{1}{8\pi G_N}\frac{1}{\epsilon}\int_{\partial M_6}\Big[\frac{29\sqrt{2}}{9}(X_2)^2 + \frac{2\sqrt{2}}{9}X_4 + \frac{2\sqrt{2}}{9}X_2 \mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right] + \frac{\sqrt{2}}{4}\|f^i\|^2_{\gamma^0} \\ & -\frac{\sqrt{2}}{72}\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]\|b\|^2_{\gamma^0} + \frac{\sqrt{2}}{18}\langle b, \gamma^2 \circ b\rangle + \frac{2\sqrt{2}}{9}X_2\,\|b\|^2_{\gamma^0} + \frac{\sqrt{2}}{18}\langle b, \mathrm{d}A_0\rangle \\ & +\frac{4\sqrt{2}}{3}\mathrm{Tr}\left[\gamma^4(\gamma^0)^{-1}\right] - \frac{2\sqrt{2}}{3}\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]^2 + \frac{\sqrt{2}}{3}\left(\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]\right)^2 \\ & -\frac{\sqrt{2}}{4}R(\gamma^0)_{ij}(\gamma^2)^{ij} + \frac{\sqrt{2}}{8}R(\gamma^0)\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]\Big]\sqrt{\det\gamma^0}\;\mathrm{d}^5x\,. \end{aligned} \tag{5.3.23}$$

We now seek to determine A_0, X_4 and γ^4 in terms of lower order boundary quantities such as b. Examination of the $z^{-2}\alpha_4$ coefficient of the B-field equation of motion gives

$$\mathrm{d} *_{\gamma^0} b \;=\; -\frac{\mathrm{i}\sqrt{2}}{3}b\wedge b - \frac{4}{9} *_{\gamma^0} A_0\ , \tag{5.3.24}$$

which we should regard as fixing A_0 in terms of the boundary field b. Specifically, since $*_5^2 = 1$ on any form, we solve this as

$$A_0 \;=\; -\frac{9}{4} *_{\gamma^0}\left(\mathrm{d} *_{\gamma^0} b + \frac{\mathrm{i}\sqrt{2}}{3}b\wedge b\right)\ . \tag{5.3.25}$$

Note we may also write $*_{\gamma^0}\mathrm{d} *_{\gamma^0} b = \delta_{\gamma^0} b$ in terms of the adjoint δ_{γ^0} of d with respect to γ^0. The

$z^{-1}\mathrm{d}z \wedge \alpha_3$ coefficient determines B_1 to be

$$B_1 = *_{\gamma^0}\left(\frac{9}{4}\mathrm{d} *_{\gamma^0} \mathrm{d}b - \frac{\mathrm{i}\sqrt{2}}{3} b \wedge A_0\right) + 2bX_2 - \frac{1}{2}\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right] b + 2\gamma^2 \circ b \ , \qquad (5.3.26)$$

which may be rewritten as

$$\begin{aligned} B_1 &= \frac{9}{4} *_{\gamma^0}\left[\mathrm{d} *_{\gamma^0} \mathrm{d}b + \frac{\mathrm{i}\sqrt{2}}{3} b \wedge \delta_{\gamma^0} b - \frac{2}{9} b \wedge *_{\gamma^0}(b \wedge b)\right] + 2bX_2 \\ &\quad - \frac{1}{2}\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right] b + 2\gamma^2 \circ b \ . \qquad (5.3.27) \end{aligned}$$

The next coefficient we need is X_4, the coefficient of z^4 in the expansion of $X(z, x^m)$ and is found from the $z^{-2}\mathrm{d}z \wedge \mathrm{vol}_{\gamma^0}$ terms in the X field equation

$$\begin{aligned} X_4 &= -\frac{9}{4}\Delta_{\gamma^0} X_2 - X_2 \mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right] - \frac{11}{2}(X_2)^2 + \frac{3}{4} X_2 \|b\|^2_{\gamma^0} \\ &\quad + \frac{9}{16}\|\mathrm{d}b\|^2_{\gamma^0} - \frac{1}{36}\|A_0\|^2_{\gamma^0} - \frac{1}{2}\langle B_1, b\rangle + \frac{1}{4}\langle b, \mathrm{d}A_0\rangle - \frac{9}{32}\|f^i\|^2_{\gamma^0} \ . \qquad (5.3.28) \end{aligned}$$

Here $\Delta_{\gamma^0} = \delta_{\gamma^0}\mathrm{d}$ acting on functions but will not contribute for a compact boundary (after integrating by parts).

We also need γ^4_{mn}, which comes from expanding the zz component of the Einstein equation at $\mathcal{O}(z^2)$:

$$\begin{aligned} \mathrm{Tr}\left[\gamma^4(\gamma^0)^{-1}\right] &= +\frac{1}{4}\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]^2 - \frac{5}{2}(X_2)^2 - \frac{1}{24}\|A_0\|^2_{\gamma^0} + \frac{9}{32}\|\mathrm{d}b\|^2_{\gamma^0} - \frac{3}{8}X_2\|b\|^2_{\gamma^0} \\ &\quad + \frac{1}{4}\langle b, B_1\rangle - \frac{1}{8}\langle b, \mathrm{d}A_0\rangle + \frac{9}{64}\|f^i\|^2_{\gamma^0} \ . \qquad (5.3.29) \end{aligned}$$

Next we record some intermediate formulae which follow from the expression for γ^2_{mn} in (5.3.16):

$$\begin{aligned} \mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]^2 &= \frac{9}{4}\left[R(\gamma^0)_{mn}R(\gamma^0)^{mn} - \frac{11}{64}R(\gamma^0)^2\right] + \frac{1}{4}\mathrm{Tr}_{\gamma^0} b^4 \qquad (5.3.30) \\ &\quad - 3\langle \mathrm{Ric}(\gamma^0) \circ b, b\rangle_{\gamma^0} + \frac{75}{128}R(\gamma^0)\|b\|^2_{\gamma^0} - \frac{51}{256}\|b\|^4_{\gamma^0} \ , \\ R(\gamma^0)_{mn}(\gamma^2)^{mn} &= -\frac{3}{2}R(\gamma^0)_{mn}R(\gamma^0)^{mn} + \frac{3}{16}R(\gamma^0)^2 + \langle \mathrm{Ric}(\gamma^0) \circ b, b\rangle_{\gamma^0} \\ &\quad - \frac{3}{16}R(\gamma^0)\|b\|^2_{\gamma^0} \ , \\ \langle \gamma^2 \circ b, b\rangle &= -\frac{3}{2}\langle \mathrm{Ric}(\gamma^0) \circ b, b\rangle_{\gamma^0} + \frac{1}{4}\mathrm{Tr}_{\gamma^0} b^4 + \frac{3}{16}R(\gamma^0)\|b\|^2_{\gamma^0} - \frac{3}{16}\|b\|^4_{\gamma^0} \ . \end{aligned}$$

Here we have defined $\mathrm{Tr}_{\gamma^0} b^4 \equiv b_m{}^n b_n{}^p b_p{}^q b_q{}^m$. Notice that $\mathrm{Tr}_{\gamma^0} b^2 = -2\|b\|^2_{\gamma^0}$, with this notation.

We now have all that we need to compute the $\mathcal{O}(1/\epsilon)$ counterterm. Inserting all our intermediate results along with the newfound expressions for X_4 *etc* into $I^{\text{div}}_{\mathcal{O}(1/\epsilon)}$ in (5.3.23) leads to

$$\begin{aligned}
I^{\text{div}}_{\mathcal{O}(1/\epsilon)} \quad = \quad & \frac{1}{8\pi G_N}\frac{1}{\epsilon}\int_{\partial M_6}\Bigg\{\Bigg[-\frac{3}{4\sqrt{2}}R(\gamma^0)_{mn}R(\gamma^0)^{mn}+\frac{15}{64\sqrt{2}}R(\gamma^0)^2\\
&+\frac{3}{4\sqrt{2}}\|f^i\|^2_{\gamma^0}-\frac{1}{12\sqrt{2}}\text{Tr}_{\gamma^0}b^4+\frac{13}{192\sqrt{2}}\|b\|^4_{\gamma^0}+\frac{1}{\sqrt{2}}\|\text{d}b\|^2_{\gamma^0}\\
&-\frac{5}{8\sqrt{2}}\|\text{d}*_{\gamma^0}b+\tfrac{\text{i}\sqrt{2}}{3}b\wedge b\|^2_{\gamma^0}+\frac{1}{4\sqrt{2}}\langle b,\text{d}\delta_{\gamma^0}b+\tfrac{\text{i}\sqrt{2}}{3}\text{d}[*_{\gamma^0}b\wedge b]\rangle\\
&-\frac{4\sqrt{2}}{3}(X_2)^2+\frac{1}{\sqrt{2}}\langle\text{Ric}(\gamma^0)\circ b,b\rangle_{\gamma^0}-\frac{9}{32\sqrt{2}}R(\gamma^0)\|b\|^2_{\gamma^0}\Bigg]\sqrt{\det\gamma^0}\;\text{d}^5x\\
&+\frac{1}{4\sqrt{2}}\langle b,*_{\gamma^0}\left[\text{d}*_{\gamma^0}\text{d}b+\tfrac{\text{i}\sqrt{2}}{3}b\wedge\delta b-\tfrac{2}{9}b\wedge*_{\gamma^0}(b\wedge b)\right]\rangle\Bigg\}\,. \qquad (5.3.31)
\end{aligned}$$

The corresponding counterterm is hence

$$\begin{aligned}
I^{\text{counterterm}}_1 \quad = \quad & \frac{1}{8\pi G_N}\int_{\partial M_6}\Bigg\{\Bigg[\frac{3}{4\sqrt{2}}R(h)_{mn}R(h)^{mn}-\frac{15}{64\sqrt{2}}R(h)^2\\
&-\frac{3}{4\sqrt{2}}\|F^i\|^2_h+\frac{1}{12\sqrt{2}}\text{Tr}_hB^4-\frac{13}{192\sqrt{2}}\|B\|^4_h-\frac{1}{\sqrt{2}}\|\text{d}B\|^2_h\\
&+\frac{5}{8\sqrt{2}}\|\text{d}*_hB+\tfrac{\text{i}\sqrt{2}}{3}B\wedge B\|^2_h-\frac{1}{4\sqrt{2}}\langle B,\text{d}\delta_hB+\tfrac{\text{i}\sqrt{2}}{3}\text{d}*_hB\wedge B\rangle_h\\
&+\frac{4\sqrt{2}}{3}(1-X)^2-\frac{1}{\sqrt{2}}\langle\text{Ric}(h)\circ B,B\rangle_h+\frac{9}{32\sqrt{2}}R(h)\|B\|^2_h\Bigg]\sqrt{\det h}\,\text{d}^5x\\
&-\frac{1}{4\sqrt{2}}B\wedge\left[\text{d}*_h\text{d}B+\frac{\sqrt{2}\text{i}}{3}B\wedge\delta_hB-\frac{2}{9}B\wedge*_h(B\wedge B)\right]\Bigg\}\,. \qquad (5.3.32)
\end{aligned}$$

Once again the pure gravity terms found in the first line agree with the literature [36].

A priori the bulk integral in (5.2.2) is logarithmically divergent. Of course a log divergence should not appear, as the boundary is odd-dimensional and on general grounds one does not expect local anomalies. In keeping with this argument the equations of motion at even higher order in z constrain the fields such that the potential log divergence cancels without the need for a counterterm.

Collating all the expressions for the counterterms we finally arrive at [28]

$$
\begin{aligned}
I_{\text{counterterms}} \;=\; & \frac{1}{8\pi G_N}\int_{\partial M_6}\Bigg\{\Bigg[\frac{4\sqrt{2}}{3}+\frac{1}{2\sqrt{2}}R(h)-\frac{1}{6\sqrt{2}}\|B\|_h^2 \\
& +\frac{3}{4\sqrt{2}}R(h)_{mn}R(h)^{mn}-\frac{15}{64\sqrt{2}}R(h)^2 \\
& -\frac{3}{4\sqrt{2}}\|F^i\|_h^2+\frac{1}{12\sqrt{2}}\mathrm{Tr}_h B^4-\frac{13}{192\sqrt{2}}\|B\|_h^4-\frac{1}{\sqrt{2}}\|\mathrm{d}B\|_h^2 \\
& +\frac{5}{8\sqrt{2}}\|\mathrm{d}*_h B+\tfrac{\mathrm{i}\sqrt{2}}{3}B\wedge B\|_h^2-\frac{1}{4\sqrt{2}}\langle B,\mathrm{d}\delta_h B+\tfrac{\mathrm{i}\sqrt{2}}{3}\mathrm{d}*_h B\wedge B\rangle_h \\
& +\frac{4\sqrt{2}}{3}(1-X)^2-\frac{1}{\sqrt{2}}\langle \mathrm{Ric}(h)\circ B,B\rangle_h+\frac{9}{32\sqrt{2}}R(h)\|B\|_h^2\Bigg]\sqrt{\det h}\,\mathrm{d}^5x \\
& -\frac{1}{4\sqrt{2}}B\wedge\left[\mathrm{d}*_h\mathrm{d}B+\frac{\sqrt{2}\mathrm{i}}{3}B\wedge\delta_h B-\frac{2}{9}B\wedge *_h(B\wedge B)\right]\Bigg\}\,. \qquad (5.3.33)
\end{aligned}
$$

5.4 Free energy of the solutions

The renormalized on-shell action determined in the previous section holds for all Romans supergravity solutions which are asymptotically locally AdS. In particular we may use these results to compute the holographic free energy for the supersymmetric solutions of chapter 4. In order to present the results, we first split the renormalized action as

$$I_{\text{renormalized}} \;=\; I_{\text{bulk}}+I_{\text{non-bulk}}\;, \qquad (5.4.1)$$

where I_{bulk} is the bulk integral given by (5.2.2), while

$$I_{\text{non-bulk}} \;=\; I_{\text{boundary}}+I_{\text{GH}}+I_{\text{counterterms}}\;, \qquad (5.4.2)$$

where I_{boundary} is the boundary contribution to the on-shell action (5.2.3), I_{GH} is the Gibbons-Hawking term, while $I_{\text{counterterms}}$ is the full counterterm (5.3.33). For our $SU(3)\times U(1)$ ansatz (4.1.1), with $f^1(r)\equiv f^2(r)\equiv 0$ and $f^3(r)=f(r)$, we have in particular

$$
\begin{aligned}
I_{\text{bulk}} \;=\; & \frac{\pi^2}{36G_N}\int_{r=\frac{1}{\sqrt{6}}}^{\Lambda}\Bigg[3X^2(r)\alpha(r)\beta^4(r)\gamma(r)+6\mathrm{i}f(r)\left[f(r)p(r)+q(r)f'(r)\right] \\
& +\frac{24f^2(r)\alpha^2(r)\gamma^2(r)+8\alpha^2(r)\beta^4(r)\gamma^2(r)+3\beta^4(r)(f'(r))^2}{4X^2(r)\alpha(r)\gamma(r)}\Bigg]\mathrm{d}r\;, \qquad (5.4.3)
\end{aligned}
$$

where Λ is the cut-off for the r coordinate.

3/4 **BPS solution**

For the one-parameter family of 3/4 BPS solutions in section 4.2 we obtain

$$
\begin{aligned}
I_{\text{bulk}} \;=\;& \frac{\pi^2}{36G_N}\Bigg[\frac{6561\sqrt{\frac{3}{2}}}{s}\Lambda^5 - \frac{243\sqrt{\frac{3}{2}}\left(3+12s^2+\sqrt{1-s^2}\right)}{s^3}\Lambda^3 \\
& -\frac{2187\sqrt{6}\kappa\left(-1+\sqrt{1-s^2}\right)}{8s}\Lambda^2 \\
& +\frac{27\left[\sqrt{\frac{3}{2}}\left(74+66s^4-14\sqrt{1-s^2}-s^2\left(5+4\sqrt{1-s^2}\right)\right)\right]}{4s^5}\Lambda \\
& -243+\frac{81\delta}{2\sqrt{2}}-1377\delta^2-\frac{1467\delta^3}{8\sqrt{2}}-\frac{6693\delta^4}{2}-\frac{44073\delta^5}{64\sqrt{2}}-4482\delta^6+\mathcal{O}(\delta^7)\Bigg] \\
& +\mathcal{O}\left(\frac{1}{\Lambda}\right) ,
\end{aligned} \tag{5.4.4}
$$

together with

$$
\begin{aligned}
I_{\text{non-bulk}} \;=\;& \frac{\pi^2}{36G_N}\Bigg[-\frac{6561\sqrt{\frac{3}{2}}}{s}\Lambda^5 + \frac{243\sqrt{\frac{3}{2}}\left(3+12s^2+\sqrt{1-s^2}\right)}{s^3}\Lambda^3 \\
& +\frac{2187\sqrt{6}\kappa\left(-1+\sqrt{1-s^2}\right)}{8s}\Lambda^2 \\
& -\frac{27\left[\sqrt{\frac{3}{2}}\left(74+66s^4-14\sqrt{1-s^2}-s^2\left(5+4\sqrt{1-s^2}\right)\right)\right]}{4s^5}\Lambda \\
& +\frac{81\sqrt{\frac{3}{2}}\left(-16+16\sqrt{1-s^2}+13s^2\left(1+3\sqrt{1-s^2}\right)\right)\kappa}{8s^3}\Bigg]+\mathcal{O}\left(\frac{1}{\Lambda}\right) ,
\end{aligned} \tag{5.4.5}
$$

where recall that κ is given as a series in δ in (4.2.3). Adding the two contributions and taking the cut-off $\Lambda \to \infty$, the divergences cancel and we are left with the following finite result

$$
I_{\text{renormalized}} \;=\; -\frac{27\pi^2}{4G_N}\left(1+\frac{8}{3}\delta^2+\frac{16\sqrt{2}}{27}\delta^3+\frac{68}{27}\delta^4+\frac{28\sqrt{2}}{27}\delta^5+\frac{32}{27}\delta^6+\dots\right) , \tag{5.4.6}
$$

where the six-dimensional Newton constant is given by[1]

$$
G_N \;=\; \frac{15\pi\sqrt{8-N_f}}{4\sqrt{2}N^{5/2}}. \tag{5.4.7}
$$

The holographic free energy is identified with $I_{\text{renormalized}}$ and agrees precisely with the series expansion of the large N field theory result (2.4.14)

$$
\mathcal{F} \;=\; \frac{1}{27s^2}\frac{(3-\sqrt{1-s^2})^3}{1-\sqrt{1-s^2}}\mathcal{F}_{\text{round}\,S^5} , \tag{5.4.8}
$$

[1]This was effectively calculated in [7] by identifying the holographic free energy of Euclidean AdS$_6$ with an entanglement entropy. The $N^{5/2}$ scaling of the free energy had previously been predicted in [11].

where recall that $s = 1/(1+\delta^2)$.

1/4 **BPS Solution**

We may similarly compute the holographic free energy of the two-parameter family of 1/4 BPS solutions in section 4.3. Again we obtain two divergent contributions whose divergences cancel. The finite piece may be computed as an expansion in $\delta = \frac{1}{s} - 1$ using the series expansions of the parameters ξ_1, ξ_2 in (4.3.4). Putting everything together we obtain

$$I_{\text{renormalized}} = -\frac{27\pi^2}{4G_N}\left(1 + \mathcal{O}(\delta^5)\right) \ . \tag{5.4.9}$$

This again agrees with large N field theory result (2.4.14). Of course the latter field theory result was computed for a one-parameter subfamily of boundary conditions in chapter 2, while here we have a more general two-parameter family.

Chapter 6

Wilson loops

In this chapter we compute the expectation values of certain BPS Wilson loops, both in the large N matrix model of section 2.4 and also in the supergravity dual solutions of chapter 4. More precisely it will be important to uplift these solutions to massive type IIA supergravity, where the Wilson loop in the fundamental representation is dual to a fundamental string. Minus the action of this string precisely matches the logarithm of the Wilson loop VEV in the large N limit, as a function of the parameters of the solutions.

6.1 Large N field theory

An interesting observable to consider is the VEV of the Wilson loop in a representation $\mathbf{R}$ of the gauge group G:

$$\langle W_{\mathbf{R}} \rangle \quad = \quad \frac{1}{\dim \mathbf{R}} \left\langle \mathrm{Tr}_{\mathbf{R}} \, \mathcal{P} \exp \int (\mathscr{A}_m \dot{x}^m + \sigma |\dot{x}|) \, \mathrm{d}t \right\rangle \, . \tag{6.1.1}$$

Here $\mathscr{A}$ denotes the dynamical gauge field for the gauge group G, σ is the scalar in the corresponding vector multipet, and the worldline is parametrized by $x^m(t)$. It is straightforward to see that (6.1.1) is invariant under the supersymmetry transformations for the squashed five-sphere (2.3.4) appearing in section 3.3 of [8] provided the Wilson loop wraps an orbit of the Killing vector bilinear[1]

$$K_m \;=\; \varepsilon^{IJ}\chi_I{}^T\mathcal{C}_{(5)}\gamma_m\chi_J \ . \tag{6.1.2}$$

That is, we take $x^m(t)$ to be an integral curve of K. The supersymmetry variations of the two terms in (6.1.1) then cancel each other.

The large N limit of (6.1.1) for the $USp(2N)$ gauge theories described in section 2.4 was computed for the round five-sphere in [45]. It is straightforward to extend this to the more general squashed sphere matrix model in section 2.4. The key point is that the insertion of the Wilson loop into the path integral does not affect the leading order saddle point configuration because its logarithm scales as $N^{1/2}$, while the free energy instead scales as $N^{5/2}$. The dynamical gauge field $\mathscr{A}$ localizes to zero, so only the constant scalar σ contributes to the Wilson loop (6.1.1) in the localization computation. Thus the VEV (6.1.1), for the fundamental representation of $USp(2N)$, is effectively computed in the large N matrix model as

$$\langle\, W_{\rm fund}\,\rangle \;=\; \int_0^{x_\star} \mathrm{e}^{2\pi\mathscr{L}\lambda(x)}\,\rho(x)\mathrm{d}x \ , \tag{6.1.3}$$

where $\rho(x)$ is the saddle point eigenvalue density (2.4.9), with the eigenvalues supported on $[0, x_\star]$ with $x_\star$ given by (2.4.10). We have also denoted by $2\pi\mathscr{L} = \int |\dot{x}|\, \mathrm{d}t$ the length of the integral curve of K that is wrapped by the Wilson loop, and recall that $\lambda(x) = N^{1/2}x$ to leading order. Thus we find the large N result

$$\log\langle\, W_{\rm fund}\,\rangle \;=\; \frac{(b_1+b_2+b_3)\sqrt{2}\pi\mathscr{L}}{\sqrt{8-N_f}}N^{1/2} + o(N^{1/2}) \ . \tag{6.1.4}$$

Relative to the round sphere result we thus have

$$\log\langle\, W_{\rm fund}\,\rangle \;=\; \frac{(b_1+b_2+b_3)\mathscr{L}}{3}\log\langle\, W_{\rm fund}\,\rangle_{\rm round} \ . \tag{6.1.5}$$

Indeed, recalling that

$$K \;=\; b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2} + b_3\partial_{\varphi_3} \ , \tag{6.1.6}$$

[1]Of course we have similarly defined a Killing vector K in the six-dimensional bulk as (3.3.1). The latter restricts to (6.1.2) on the conformal boundary, so this is only a slight abuse of notation.

in terms of the standard $U(1)^3$ action on $S^5 \subset \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$, then the orbits of K are always closed circles at the origins of any two copies of $\mathbb{R}^2$. If we call these $U(1)^3$ invariant circles S^1_i, $i = 1, 2, 3$, then $\mathscr{L} = 1/b_i$ and we may write

$$\log \langle W_{\mathrm{fund}, S^1_i} \rangle \quad = \quad \frac{(b_1 + b_2 + b_3)}{3b_i} \log \langle W_{\mathrm{fund}} \rangle_{\mathrm{round}} \ . \tag{6.1.7}$$

Notice that this formula is invariant under a constant rescaling $K \to c \cdot K$. We now explain how to reproduce this large N result from the dual supergravity solutions.

6.2 Dual fundamental strings

The supergravity dual of the Wilson loop W_{fund} was studied in [45] for the round five-sphere. The supergravity background is in this case the massive type IIA uplift $\mathrm{AdS}_6 \times S^4$ of the AdS_6 vacuum of the Romans theory of section 3. The Wilson loop maps to a fundamental string sitting at the north pole $\xi = \frac{\pi}{2}$ of the internal S^4, in the notation of section 3.1. The string then wraps a copy of $\mathbb{R}^2 \subset \mathrm{AdS}_6$ parametrized by the radial direction r in AdS together with the Wilson loop curve $S^1 \subset S^5$.

We now generalize this to our supergravity backgrounds in chapter 4. Here the type IIA background is a warped and fibred product $M_6 \times S^4$, together with various non-trivial background fluxes. However, M_6 still has the topology of a ball, with a natural radial direction r. Thus the candidate dual of the Wilson loops computed in the previous section is a fundamental string sitting at $\xi = \frac{\pi}{2}$ in the internal S^4 of (3.1.1), together with the Wilson loop curve $S^1 \subset S^5_{\mathrm{squashed}}$ and the radial direction r. This is then a copy of $\Sigma_2 \cong \mathbb{R}^2 \subset M_6$, and we would like to compute the regularized action of a fundamental string wrapping this submanifold.

In order to compute the string action we must first convert to the string frame metric in (3.1.1), which introduces a factor of $\mathrm{e}^{\Phi/2}$, where Φ is the ten-dimensional dilaton. The induced string frame metric on M_6 at the north pole $\xi = \frac{\pi}{2}$ of S^4 is then

$$\mathrm{d}s^2_{M_6} \mid_{\xi = \frac{\pi}{2}, \mathrm{string}} \quad = \quad X^{-2} \mathrm{d}s^2_6 \ , \tag{6.2.1}$$

where $\mathrm{d}s_6^2$ is the Romans supergravity metric. The B-field then uplifts to the type IIA B-field with curvature $F_{(3)} = H = \mathrm{d}B$ via (3.1.1) at the north pole $\xi = \frac{\pi}{2}$. In section 3 we have set most of the physical scaling parameters to specific numerical values – for example the Romans mass is set to $m_{\mathrm{IIA}} = \frac{\sqrt{2}}{3}$, while the correctly normalized value for the supergravity dual to the $USp(2N)$ gauge theories is $(8-N_f)/(2\pi\ell_s)$ where ℓ_s is the string length. In particular restoring the AdS radius to its physical value

$$L^4 \;=\; \frac{8\pi^2 N}{9(8-N_f)}\ell_s^4 \;, \qquad (6.2.2)$$

(as in [45]) the string frame action is

$$S \;=\; \frac{N^{1/2}\sqrt{2}}{3\sqrt{(8-N_f)}}\int_{\Sigma_2} X^{-2}\sqrt{\det\gamma}\,\mathrm{d}^2x + \mathrm{i}B \;, \qquad (6.2.3)$$

where γ_{ab} is the metric induced on Σ_2 via its embedding into the Romans metric $\mathrm{d}s_6^2$ on M_6, and we have included the usual Wess-Zumino coupling to the ten-dimensional B-field. More precisely, (6.2.3) is divergent, and as usual one may regularize it by cutting off the r integral at some $r = \Lambda$, and including a boundary counterterm given by the length of the boundary $S^1 \subset S^5$ at $r = \Lambda$. Thus the regularized action reads

$$S_{\text{string}} \;=\; \frac{N^{1/2}\sqrt{2}}{3\sqrt{(8-N_f)}}\left[\int_{\Sigma_2}\left(X^{-2}\sqrt{\det\gamma}\,\mathrm{d}^2x + \mathrm{i}B\right) - \frac{3}{\sqrt{2}}\mathrm{length}(\partial\Sigma_2)\right] \;, \qquad (6.2.4)$$

where this is understood to mean the limit as one takes the cut-off $\Lambda \to \infty$. We now compute this for our various solutions.

1/4 BPS background

We begin with the 1/4 BPS background, as in this case the supersymmetric Killing vector bilinear is simply $K = \partial_\tau$ (up to an irrelevant constant rescaling). Via the $SU(3)$ symmetry of the background all orbits of K are equivalent, and thus there is effectively only one Wilson loop to compute. This wraps the τ and r directions at, say, $\sigma = 0$ (which is a point on the base $\mathbb{CP}^2$ of $S^1_{\text{Hopf}} \hookrightarrow S^5 \to \mathbb{CP}^2$, all points being equivalent under $SU(3)$). The regularized string action (6.2.4) is

$$S_{\text{string}} \;=\; \lim_{\Lambda\to\infty}\frac{N^{1/2}2\sqrt{2}\pi}{3\sqrt{(8-N_f)}}\left[\int_{r=\frac{1}{\sqrt{6}}}^{\Lambda}\left[X^{-2}(r)\alpha(r)\gamma(r) + \mathrm{i}\,p(r)\right]\mathrm{d}r - \frac{3}{\sqrt{2}}\gamma(\Lambda)\right] \;, \qquad (6.2.5)$$

where we have used that τ has period 2π. Evaluating this for the two-parameter family of 1/4 BPS solutions, as a series in the parameter δ, we find

$$-S_{\text{string}} = \frac{3\sqrt{2}\pi}{\sqrt{8-N_f}} N^{1/2} + \mathcal{O}(\delta^5) \ , \tag{6.2.6}$$

which agrees precisely with the large N field theory result (6.1.4) since $K = \partial_\tau = \partial_{\varphi_1} + \partial_{\varphi_2} + \partial_{\varphi_3}$ so that $b_1 = b_2 = b_3 = 1$.

3/4 BPS background

For the 3/4 BPS solution recall that the supersymmetric Killing vector K has $b_1 = 1 + \sqrt{1-s^2}$, $b_2 = b_3 = 1 - \sqrt{1-s^2}$. For generic values of the squashing parameter s the generic orbit of K will be open. However, the orbits always close over the circles S^1_i defined in section 6.1, which have lengths $\mathscr{L} = 2\pi/b_i$. Since $b_2 = b_3$ these circles give rise to two distinct Wilson loop VEVs:

$$\frac{\log \langle W_{\text{fund}, S^1_i} \rangle}{\log \langle W_{\text{fund}} \rangle_{\text{round}}} = \begin{cases} \dfrac{3-\sqrt{1-s^2}}{3(1+\sqrt{1-s^2})} \ , & i = 1 \ , \\[2ex] \dfrac{3-\sqrt{1-s^2}}{3(1-\sqrt{1-s^2})} \ , & i = 2, 3 \ . \end{cases} \tag{6.2.7}$$

We may then compare these results to the regularized string action (6.2.4), where for S^1_i the fundamental string wraps the circle φ_i together with the r direction. More precisely, S^1_1 is located at $\sigma = 0$ in the coordinates (2.2.1), while S^1_2 is located at $\{\sigma = \frac{\pi}{2}, \theta = 0\}$, as one sees from (2.3.6). The result for S^1_3 is the same as that for S^1_2 due to the $SU(2) \subset SU(3)$ symmetry preserved by the bosonic solution and supersymmetric Killing vector. On the other hand, due to the signs in (2.3.7) the relevant string actions to compute are then

$$\frac{N^{1/2} 2\sqrt{2}\pi}{3\sqrt{(8-N_f)}} \left[\int_{r=\frac{1}{\sqrt{6}}}^{\Lambda} \left[X^{-2}(r)\alpha(r)\gamma(r) \pm \mathrm{i}\, p(r) \right] \mathrm{d}r - \frac{3}{\sqrt{2}} \gamma(\Lambda) \right] \ , \tag{6.2.8}$$

respectively. Evaluating this for the one-parameter family of 3/4 BPS solutions, as a series in the parameter δ up to sixth order where $\delta^2 = \frac{1}{s} - 1$, we find

$$\frac{S_{\text{string}, S^1_1}}{S_{\text{string}} \,|_{\delta=0}} = 1 - \frac{4\sqrt{2}}{3}\delta + \frac{8}{3}\delta^2 - \frac{5\sqrt{2}}{3}\delta^3 + \frac{4}{3}\delta^4 - \frac{7}{12\sqrt{2}}\delta^5 + 0 \cdot \delta^6 + \ldots \ , \tag{6.2.9}$$

while

$$\frac{S_{\text{string},S^1_2}}{S_{\text{string}}\,|_{\delta=0}} \;=\; 1+\frac{2\sqrt{2}}{3}\delta+\frac{4}{3}\delta^2+\frac{5}{3\sqrt{2}}\delta^3+\frac{2}{3}\delta^4+\frac{7}{24\sqrt{2}}\delta^5+0\cdot\delta^6+\dots\;. \tag{6.2.10}$$

These agree precisely with the series expansions of (6.2.7) computed in field theory.

Chapter 7

Boundary supersymmetry conditions

In this chapter we determine the form of the Euclidean Romans supersymmetry conditions, given in section 3, near the five-dimensional conformal boundary. Closely related work has appeared in [46]. Our conventions are the following: we use $x^\mu = (r, x^m)$ to denote six-dimensional coordinates, so that the indices $\mu, \nu, \ldots \in \{0,1,2,3,4,5\}$. Six-dimensional frame indices are indexed by $A, B, \ldots \in \{0,1,2,3,4,5\}$ and five-dimensional frame indices by early Roman letters a, b *etc.*

We continue to use the Fefferman-Graham coordinates outlined in section 5.3, although compared to that section we change coordinates $z \to 1/r$ so that the conformal boundary is now at $r = \infty$. We can then scale the r coordinate $r \to \lambda r$ without changing the position of the conformal boundary or modifying the five-dimensional boundary metric γ^0. After this scaling the asymptotic six-dimensional metric is now

$$\mathrm{d}s_6^2 \;=\; \frac{\ell^2}{r^2}\mathrm{d}r^2 + \lambda^2 r^2 \gamma_{mn}\mathrm{d}x^m \mathrm{d}x^n \ , \tag{7.0.1}$$

where

$$\gamma_{mn} \;=\; \gamma^0_{mn} + \frac{1}{\lambda^2 r^2}\gamma^2_{mn} + \frac{1}{\lambda^4 r^4}\gamma^4_{mn} + \frac{1}{\lambda^5 r^5}\gamma^5_{mn} + \mathcal{O}\left(\frac{1}{r^6}\right) \ . \tag{7.0.2}$$

We introduce a six-dimensional vielbein e^A such that

$$\mathrm{d}s_6^2 \;=\; e^A e^A \;=\; e^0 e^0 + e^a e^a \, . \tag{7.0.3}$$

If we denote by $e^a_{(5)}$ the vielbein for γ^0, then the six-dimensional frame components may be written as

$$e^0_r \;=\; \frac{\ell}{r}\,, \qquad e^0_m \;=\; 0\,, \qquad e^a_r \;=\; 0\,, \qquad e^a_m(r,x) \;=\; \lambda r e^a_{(5)m}(x) + \cdots\,, \tag{7.0.4}$$

where the ellipsis denotes subleading powers of r which will not play a part in what follows. The inverse frame is

$$e^r_0 \;=\; \frac{r}{\ell}\,, \qquad e^m_0 \;=\; 0\,, \qquad e^r_a \;=\; 0\,, \qquad (e^a_m)^{-1} \;=\; e^m_a \;=\; \frac{1}{\lambda r} e^m_{(5)a} + \ldots\,. \tag{7.0.5}$$

The six-dimensional spin connection is given by $\omega_\mu{}^{AB} = e^{\nu[A}\partial_\mu e_\nu{}^{B]} - e^{\nu[A}\partial_\nu e_\mu{}^{B]} - e^{[A}_\nu e^{B]}_\sigma e^C_\mu \partial^\nu e^\sigma_C$ and from this expression it is easy to show that

$$\omega_r{}^{bc} \;=0\;=\; \omega_r{}^{0b}\,, \qquad \omega_a{}^{0c} \;=\; -\frac{1}{\ell}\delta^c_a + \ldots\,, \qquad \omega_a{}^{bc} \;=\; \frac{1}{\lambda r}\omega^{(5d)bc}_a + \ldots\,, \tag{7.0.6}$$

where $\omega^{(5d)bc}_a$ is the spin connection associated with the 5d boundary metric γ^0.

Incorporating some of the results from the holographic renormalization in section 5.3, the asymptotic bulk field expansions in the local six-dimensional coordinates are[1]

$$\begin{aligned}
X &= 1 + \frac{1}{r^2}X_2 + \cdots\,, \\
F = \tfrac{2}{3}B &= \frac{2r}{3}b - \frac{2}{3r^2}\mathrm{d}r\wedge A_0 + \cdots\,, \\
H = \mathrm{d}B &= \mathrm{d}r\wedge b + r\mathrm{d}b + \cdots\,, \\
F^i &= f^i + \cdots\,, \\
\mathcal{A}^i &= a^i + \cdots\,.
\end{aligned} \tag{7.0.7}$$

Note that not all the fields appearing on the right hand side are independent. For example $f^i = \mathrm{d}a^i - \frac{1}{2}\varepsilon^{ijk}a^j\wedge a^k$ and A_0 was found in section 5.3 to be given by

$$A_0 \;=\; -\frac{9}{4}*_{\gamma^0}\left(\mathrm{d}*_{\gamma^0} b + \frac{\mathrm{i}\sqrt{2}}{3} b\wedge b\right)\,. \tag{7.0.8}$$

[1] In this section we use a calligraphic font $\mathcal{A}^i$ to denote the $SU(2)$ gauge field so that there is no confusion with other notation.

However, for simplicity we keep A_0 and substitute in terms of b only at the end of our computation. Converting the bulk field expansions first into the six-dimensional frame and then into the 5d frame using (7.0.5) we can read off the following components for the asymptotic fields

$$\begin{aligned}
H_{0ab} &= \frac{r}{\ell(\lambda r)^2} b_{ab} + \mathcal{O}\left(\frac{1}{r^3}\right), \qquad H_{abc} = \frac{r}{(\lambda r)^3}(\mathrm{d}b)_{abc} + \mathcal{O}\left(\frac{1}{r^4}\right), \\
F_{0a} &= -\frac{2}{3\ell\lambda r^2}(A_0)_a + \mathcal{O}\left(\frac{1}{r^3}\right), \qquad F_{ab} = \frac{2r}{3(\lambda r)^2} b_{ab} + \mathcal{O}\left(\frac{1}{r^3}\right), \\
X + \tfrac{1}{3}X^{-3} &= \frac{4}{3} + \mathcal{O}\left(\frac{1}{r^4}\right), \qquad X - X^{-3} = \frac{4}{r^2}X_2 + \mathcal{O}\left(\frac{1}{r^3}\right), \\
X^{-1}\partial_0 X &= -\frac{2}{\ell r^2}X_2 + \mathcal{O}\left(\frac{1}{r^3}\right), \qquad X^{-1}\partial_a X = \mathcal{O}\left(\frac{1}{r^3}\right), \\
F^i_{ab} &= \frac{1}{(\lambda r)^2} f^i_{ab} + \mathcal{O}\left(\frac{1}{r^3}\right), \qquad F^i_{0a} = \mathcal{O}\left(\frac{1}{r^3}\right), \\
\mathcal{A}^i_a &= \frac{1}{\lambda r} a^i_a + \mathcal{O}\left(\frac{1}{r^3}\right), \qquad \mathcal{A}^i_0 = \mathcal{O}\left(\frac{1}{r^3}\right).
\end{aligned} \tag{7.0.9}$$

The full six-dimensional Killing spinor equation for the Euclidean Romans theory, where all indices are orthonormal frame indices, is

$$\begin{aligned}
D_A\epsilon_I &= \frac{\mathrm{i}}{4\sqrt{2}}(X + \tfrac{1}{3}X^{-3})\Gamma_A\Gamma_7\epsilon_I - \frac{1}{48}X^2 H_{BCD}\Gamma^{BCD}\Gamma_A\Gamma_7\epsilon_I \\
&\quad - \frac{\mathrm{i}}{16\sqrt{2}}X^{-1}F_{BC}(\Gamma_A{}^{BC} - 6\delta_A{}^B\Gamma^C)\epsilon_I \\
&\quad + \frac{1}{16\sqrt{2}}X^{-1}F^i_{BC}(\Gamma_A{}^{BC} - 6\delta_A{}^B\Gamma^C)\Gamma_7(\sigma^i)_I{}^J\epsilon_J,
\end{aligned} \tag{7.0.10}$$

where $D_A\epsilon_I = \partial_A\epsilon_I + \frac{1}{4}\omega_A{}^{BC}\Gamma_{BC}\epsilon_I + \frac{\mathrm{i}}{2}\mathcal{A}^i_A(\sigma^i)_I{}^J\epsilon_J$. Taking the free index to be $A = 0$ and substituting the field components (7.0.9) leads to

$$\partial_r\epsilon_I = +\frac{\mathrm{i}}{2r}\Gamma_0\Gamma_7\epsilon_I + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{7.0.11}$$

Similarly, if we take the free index in the Killing spinor equation to be $A = a$ then we find

$$\begin{aligned}
\nabla_a\epsilon_I &= \frac{\lambda}{3\sqrt{2}}r\Gamma_a(\mathrm{i}\Gamma_7 - \Gamma_0)\epsilon_I - \frac{\mathrm{i}}{2}a^i_a(\sigma^i)_I{}^J\epsilon_J \\
&\quad - \frac{\mathrm{i}}{24\lambda\sqrt{2}}b_{bc}\Gamma_a{}^{bc}(1 + \mathrm{i}\Gamma_0\Gamma_7)\epsilon_I + \frac{\mathrm{i}}{4\lambda\sqrt{2}}b_{ab}\Gamma^b\left(1 + \tfrac{\mathrm{i}}{3}\Gamma_0\Gamma_7\right)\epsilon_I + \mathcal{O}\left(\frac{1}{r}\right),
\end{aligned} \tag{7.0.12}$$

with ∇_a being the covariant derivative with respect to the 5d spin connection.

Now we decompose the six-dimensional gamma matrices and spinors. We take our coordinate independent Cliff$(6,0)$ gamma matrices to be

$$\Gamma_0 = \begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix} , \qquad \Gamma_a = \begin{pmatrix} 0 & \mathrm{i}\gamma_a \\ -\mathrm{i}\gamma_a & 0 \end{pmatrix} , \qquad \Gamma_7 = \begin{pmatrix} -1_4 & 0 \\ 0 & 1_4 \end{pmatrix} , \tag{7.0.13}$$

where γ_a are a Hermitian basis of Cliff$(5,0)$. The six-dimensional spinor ϵ_I is decomposed as

$$\epsilon_I = \begin{pmatrix} \epsilon_I^+ \\ \epsilon_I^- \end{pmatrix} , \tag{7.0.14}$$

where $\epsilon_I^{\pm}$ are 4-component spinors.

With this basis of gamma matrices and splitting of the spinors, the r direction of the Killing spinor equation (G.0.6), to lowest order in r, is

$$\begin{pmatrix} \partial_r \epsilon_I^+ \\ \partial_r \epsilon_I^- \end{pmatrix} = \frac{\mathrm{i}}{2r} \begin{pmatrix} \epsilon_I^- \\ -\epsilon_I^+ \end{pmatrix} . \tag{7.0.15}$$

The general solution determines the asymptotic dependence on r:

$$\epsilon_I = \begin{pmatrix} \epsilon_I^+ \\ \epsilon_I^- \end{pmatrix} = \sqrt{r} \begin{pmatrix} \chi_I \\ -\mathrm{i}\chi_I \end{pmatrix} + \frac{1}{\sqrt{r}} \begin{pmatrix} \varphi_I \\ \mathrm{i}\varphi_I \end{pmatrix} + \cdots , \tag{7.0.16}$$

where χ_I, φ_I depend only on the boundary coordinates x^m. Having found the asymptotic dependence on r for the spinors ϵ_I we can then substitute into the remaining components of the Killing spinor equation (7.0.12). Taking only the lowest terms in r gives two copies of

$$\nabla_a \chi_I = -\frac{\lambda\sqrt{2}\mathrm{i}}{3}\gamma_a \varphi_I - \frac{\mathrm{i}}{2} a_a^i (\sigma^i)_I{}^J \chi_J - \frac{\mathrm{i}}{12\lambda\sqrt{2}} b_{bc} \gamma_a{}^{bc} \chi_I + \frac{\mathrm{i}}{3\lambda\sqrt{2}} b_{ab} \gamma^b \chi_I \, . \tag{7.0.17}$$

This is the five-dimensional boundary Killing spinor equation.

Now recall that the six-dimensional dilatino condition in the frame reads

$$\begin{aligned} 0 = & -\mathrm{i}X^{-1}\partial_A X \Gamma^A \epsilon_I + \frac{1}{2\sqrt{2}}\left(X - X^{-3}\right)\Gamma_7 \epsilon_I + \frac{\mathrm{i}}{24} X^2 H_{ABC}\Gamma^{ABC}\Gamma_7 \epsilon_I \\ & - \frac{1}{8\sqrt{2}} X^{-1} F_{AB} \Gamma^{AB} \epsilon_I - \frac{\mathrm{i}}{8\sqrt{2}} X^{-1} F^i_{AB} \Gamma^{AB} \Gamma_7 (\sigma^i)_I{}^J \epsilon_J \, . \end{aligned} \tag{7.0.18}$$

We may follow precisely the same steps as for the Killing spinor equation to determine the asymptotic form of the dilatino equation. Doing so we find the five-dimensional constraint

$$\begin{aligned} 0 &= -\frac{1}{6\sqrt{2}} b_{ab}\gamma^{ab}\varphi_I - \frac{\sqrt{2}}{3}\lambda^2 X_2\chi_I + \frac{\mathrm{i}}{24\lambda}(\mathrm{d}b)_{abc}\gamma^{abc}\chi_I + \frac{\lambda\mathrm{i}}{8}\nabla^b b_{ab}\gamma^a\chi_I \\ &\quad + \frac{\lambda}{48\sqrt{2}} b_{ab}b_{cd}\gamma^{abcd}\chi_I + \frac{\mathrm{i}}{8\sqrt{2}} f^i_{ab}\gamma^{ab}(\sigma^i)_I{}^J\chi_J\,. \end{aligned} \tag{7.0.19}$$

We would prefer to have five-dimensional supersymmetry conditions which are homogeneous in the spinor χ_I instead of the current dependence on both χ_I and φ_I. To remove φ_I we contract (7.0.17) with γ^a. This gives

$$\begin{aligned} \varphi_I &= \frac{\mathrm{i}}{5}\frac{3}{\lambda\sqrt{2}}\left[\gamma^a\left(\delta_I^J\nabla_a + \frac{\mathrm{i}}{2}a^i_a(\sigma^i)_I{}^J + \frac{\mathrm{i}}{12\lambda\sqrt{2}}b_{bc}(\gamma_a{}^{bc} - 4\delta^b_a\gamma^c)\delta_I^J\right)\right]\chi_J \\ &\equiv \frac{\mathrm{i}}{5}\frac{3}{\lambda\sqrt{2}}D_I{}^J\chi_J\,. \end{aligned} \tag{7.0.20}$$

We may then write the boundary Killing spinor equation in the form

$$\left(\tilde{\nabla}_I{}^J{}_a - \frac{1}{5}\gamma_a D_I{}^J\right)\chi_J = 0\,, \tag{7.0.21}$$

where $\tilde{\nabla}_I{}^J{}_a = \delta_I^J\nabla_a + \frac{\mathrm{i}}{2}a^i_a(\sigma^i)_I{}^J + \frac{\mathrm{i}}{12\lambda\sqrt{2}}b_{bc}(\gamma_a{}^{bc} - 4\delta^b_a\gamma^c)\delta_I^J$. The boundary dilatino constraint reads

$$\begin{aligned} 0 &= -\frac{\mathrm{i}}{20\lambda}b_{ab}\gamma^{ab}D_I{}^J\chi_J - \frac{\sqrt{2}}{3}\lambda^2 X_2\chi_I + \frac{\mathrm{i}}{24\lambda}(\mathrm{d}b)_{abc}\gamma^{abc}\chi_I + \frac{\lambda\mathrm{i}}{8}\nabla^b b_{ab}\gamma^a\chi_I \\ &\quad + \frac{\lambda}{48\sqrt{2}}b_{ab}b_{cd}\gamma^{abcd}\chi_I + \frac{\mathrm{i}}{8\sqrt{2}}f^i_{ab}\gamma^{ab}(\sigma^i)_I{}^J\chi_J\,. \end{aligned} \tag{7.0.22}$$

For vanishing b-field, solutions of (7.0.21) are known as charged conformal Killing spinors (CCKS), or twistor spinors. Within the current context of gauge/gravity duality, CCKS have been classified for 3-manifolds and 4-manifolds in both Euclidean and Lorentzian signature in [47, 48, 49, 50]. More recently, solutions in five dimensions (with arbitrary signature) have been studied in [51]. To our knowledge the more general charged conformal Killing spinor equation, where the charge is with respect to both the triplet of one-forms a^i and the two-form b, has not been studied in the literature. It would be interesting to understand the relationship between the five-dimensional conditions found here from the Romans supergravity theory and the rigid limit of five-dimensional $\mathcal{N} = 1$ Poincaré supergravity [52, 53] studied in [19, 20].

This equation was studied in [21] and reformulated geometrically. We are able to state the precise relation between the spinors φ_I and χ_I for our supersymmetric solutions (for which $\lambda = 3\sqrt{3}$). For the 3/4 BPS solution we find

$$\varphi_I \;=\; (-1)^I \frac{3-\sqrt{1-s^2}}{6\sqrt{6}s}\chi_I - (-1)^I \frac{4\sqrt{1-s^2}}{6\sqrt{6}s}\gamma_1\chi_I \,, \tag{7.0.23}$$

and for the two-parameter family of 1/4 BPS solutions

$$\varphi_I \;=\; \frac{(f_0-3)s}{6\sqrt{6}}\chi_I \,. \tag{7.0.24}$$

In appendix B we give further details of the explicit six-dimensional Killing spinors and their relation to the five-dimensional spinors of chapter 2.

Chapter 8

$SU(2)$ structure and conditions for supersymmetry

We analyse Euclidean Romans supergravity, reconstructing its supersymmetry conditions in terms of a canonical local $SU(2)$ structure.

8.1 $SU(2)$ structure

Recall the Killing spinor equation and the dilatino equation presented in chapter 3,

$$\begin{aligned} D_\mu\epsilon_I &= \frac{\mathrm{i}}{4\sqrt{2}}g(X+\tfrac{1}{3}X^{-3})\Gamma_\mu\Gamma_7\epsilon_I - \frac{\mathrm{i}}{16\sqrt{2}}X^{-1}F_{\nu\rho}(\Gamma_\mu{}^{\nu\rho}-6\delta_\mu{}^\nu\Gamma^\rho)\epsilon_I \\ &\quad -\frac{1}{48}X^2H_{\nu\rho\sigma}\Gamma^{\nu\rho\sigma}\Gamma_\mu\Gamma_7\epsilon_I + \frac{1}{16\sqrt{2}}X^{-1}F^i_{\nu\rho}(\Gamma_\mu{}^{\nu\rho}-6\delta_\mu{}^\nu\Gamma^\rho)\Gamma_7(\sigma^i)_I{}^J\epsilon_J \ , \end{aligned} \tag{8.1.1}$$

$$\begin{aligned} 0 &= -\mathrm{i}X^{-1}\partial_\mu X\Gamma^\mu\epsilon_I + \frac{1}{2\sqrt{2}}g\left(X-X^{-3}\right)\Gamma_7\epsilon_I + \frac{\mathrm{i}}{24}X^2H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_7\epsilon_I \\ &\quad -\frac{1}{8\sqrt{2}}X^{-1}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon_I - \frac{\mathrm{i}}{8\sqrt{2}}X^{-1}F^i_{\mu\nu}\Gamma^{\mu\nu}\Gamma_7(\sigma^i)_I{}^J\epsilon_J \ . \end{aligned} \tag{8.1.2}$$

Consider a Dirac spinor ϵ in six dimensions, such that $(\epsilon_1,\epsilon_2)=(\epsilon,\epsilon^c)$ solves the equations above. We may construct the following scalar bilinears

$$S \equiv \epsilon^\dagger\epsilon \ , \qquad \tilde{S} \equiv \epsilon^\dagger\Gamma_7\epsilon \ , \qquad f \equiv \epsilon^{\mathrm{T}}\epsilon \ . \tag{8.1.3}$$

Here we have chosen a basis for the gamma matrices in which they are purely imaginary and anti-symmetric, with charge conjugation matrix $\mathcal{C} = -\mathrm{i}\Gamma_7$. A short computation reveals that

$$\mathrm{d}(Xf) \;\; = \;\; -\mathrm{i}(Xf)\mathcal{A} \; . \tag{8.1.4}$$

The integrability condition for this equation immediately implies $\mathcal{F} = \mathrm{d}\mathcal{A} = 0$ unless $f \equiv 0$ (notice that X is nowhere zero). We will henceforth restrict our analysis to the case $f \equiv 0$, which is necessary for a non-trivial R-symmetry gauge field.[1]

We may then write

$$\epsilon \;\; = \;\; \epsilon_+ + \epsilon_- \; , \tag{8.1.5}$$

where $-\Gamma_7\epsilon_\pm = \pm\epsilon_\pm$, and furthermore the condition $f \equiv 0$ allows us to introduce [56]

$$\epsilon_+ \;\; = \;\; \sqrt{S}\cos\vartheta\,\eta_1 \; , \qquad \epsilon_- \;\; = \;\; \sqrt{S}\sin\vartheta\,\eta_2^* \; . \tag{8.1.6}$$

Here η_1, η_2 are two orthogonal unit norm chiral spinors, so that $\eta_1^\dagger\eta_1 = \eta_2^\dagger\eta_2 = 1$ and $\eta_2^\dagger\eta_1 = 0$. These each define a canonical $SU(3)$ structure, and together determine a canonical $SU(2)$ structure. Concretely, in six dimensions such a structure is specified by two one-forms K_1, K_2 and a triplet of two-forms J_i, $i = 1, 2, 3$, given by

$$\begin{aligned} K_1 - \mathrm{i}K_2 \;\; &\equiv \;\; -\frac{1}{2}\varepsilon^{\alpha\beta}\eta_\alpha^{\mathrm{T}}\Gamma_{(1)}\eta_\beta \; , \\ J_i \;\; &\equiv \;\; -\frac{\mathrm{i}}{2}\sigma_i^{\alpha\beta}\eta_\alpha^\dagger\Gamma_{(2)}\eta_\beta \; . \end{aligned} \tag{8.1.7}$$

Here we have introduced the notation $\Gamma_{(n)} \equiv \frac{1}{n!}\Gamma_{\mu_1\cdots\mu_n}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}$, where x^μ are local coordinates. We also define

$$\Omega \;\; \equiv \;\; J_2 + \mathrm{i}J_1 \; , \qquad J \;\; \equiv \;\; J_3 \; . \tag{8.1.8}$$

The canonical $SU(2)$ structure is thus determined by (K_1, K_2, J, Ω). We note that K_1 and K_2 are orthonormal one-forms, and both are orthogonal to J and Ω, with $J \wedge \Omega = 0$ and $2J \wedge J = \Omega \wedge \bar{\Omega}$.

[1] There are nevertheless interesting solutions for which $f \neq 0$. In particular the 1/2 BPS solution constructed in chapter 4 lies in this class.

The $SU(2)$ structure $(S, \vartheta, K_1, K_2, J, \Omega)$ that arises naturally from a supersymmetric solution is thus related to the canonical $SU(2)$ structure by the square norm S and angle ϑ, via (8.1.6). For completeness we note that $\tilde{S} = -S\cos 2\vartheta$.

Before proceeding, let us remark that the spinor ϵ is charged under the Abelian R-symmetry gauge field $\mathcal{A}$, and thus it is rotated by a phase under gauge transformations. The two-form Ω is then rotated by the square of this phase. As a consequence we more precisely have a $U(2)$ structure, as explained in [21]. Nevertheless, here we will continue to refer to this as an $SU(2)$ structure.

8.2 Differential constraints

We begin by introducing the one-form bilinear

$$K \;\equiv\; \epsilon^\dagger\Gamma_{(1)}\epsilon \;=\; S\sin 2\vartheta\, K_1 \ . \tag{8.2.1}$$

Using the Killing spinor equation and dilatino equation (8.1.2) one can show that K is a Killing one-form, so that the dual vector field $\xi \equiv K^\#$ is a Killing vector. We may hence introduce a local coordinate ψ, so that $\xi = \partial_\psi$ and the metric is independent of ψ. From (8.2.1) it follows that we may write

$$K_1 \;=\; S\sin 2\vartheta\,(\mathrm{d}\psi + \sigma) \ , \tag{8.2.2}$$

where $\mathcal{L}_\xi\sigma = 0 = i_\xi\sigma$. In fact, as shown in appendix E, all of the supergravity fields and $SU(2)$ structure are annihilated by $\mathcal{L}_\xi$, with the exception of the complex two-form Ω. The spinor ϵ is a spin^c spinor, charged under the Abelian R-symmetry gauge field $\mathcal{A}$, and provided one makes the gauge choice (8.2.5) below then also $\mathcal{L}_\xi\Omega = 0$. Thus the vector field $\xi = \partial_\psi$ generates a symmetry of the full solution.

The spinor equations (3.2.5), (3.2.6) impose further constraints on the supergravity fields and $SU(2)$ structure. A more detailed analysis may be found in appendix E, while here we simply summarize the results. The B-field and R-symmetry gauge field strength $\mathcal{F} = \mathrm{d}\mathcal{A}$ may be written as

$$B = \mathrm{i}\, K_1 \wedge \left[\frac{3}{\sqrt{2} S \sin 2\vartheta} \mathrm{d}(XS) + X^{-2} K_2 \right] + B_\perp \ , \tag{8.2.3}$$

$$\mathcal{F} = K_1 \wedge \frac{\sqrt{2}}{S \sin 2\vartheta} \mathrm{d}(XS \cos 2\vartheta) + \mathcal{F}_\perp \ , \tag{8.2.4}$$

where $B_\perp$ and $\mathcal{F}_\perp$ have zero interior contraction with ξ. In particular (8.2.4) allows us to write

$$\mathcal{A} = -\sqrt{2} X \cot 2\vartheta \, K_1 + \mathcal{A}_\perp \ , \tag{8.2.5}$$

where $i_\xi \mathcal{A}_\perp = 0$ and we have made a partial gauge choice for $\mathcal{A}$. We note that

$$\mathcal{F}_\perp = -\sqrt{2} X S \cos 2\vartheta \, \mathrm{d}\sigma + \mathrm{d}\mathcal{A}_\perp \ . \tag{8.2.6}$$

We may similarly write the component of $H = \mathrm{d}B$ perpendicular to ξ as

$$H_\perp \equiv \mathrm{i} \left[\tfrac{3}{\sqrt{2}} \mathrm{d}(XS) + X^{-2} S \sin 2\vartheta K_2 \right] \wedge \mathrm{d}\sigma + \mathrm{d}B_\perp \ . \tag{8.2.7}$$

Given these definitions, the spinor equations (8.1.2) imply the following set of differential constraints on the $SU(2)$ structure $(S, \vartheta, K_1, K_2, J, \Omega)$:

$$
\begin{aligned}
X^2 S^2 \sin^2 2\vartheta \, \mathrm{d}\sigma &= -\tfrac{2\sqrt{2}}{3} X^{-1} S \cos 2\vartheta \, J - \mathrm{i} X^4 S \sin 2\vartheta \, K_1 \lrcorner * H_\perp \\
&\quad + \sqrt{2} X S (\cos 2\vartheta \, \mathcal{F}_\perp + \tfrac{2}{3} \mathrm{i} B_\perp) \, , \\
\mathrm{d}(X^{-1} S \cos 2\vartheta \, J) &= -\tfrac{3}{2\sqrt{2}} \mathrm{d}[(XS)^2 \mathrm{d}\sigma] + \mathrm{i} X S \, \mathrm{d} B_\perp \\
&\quad + \tfrac{\sqrt{2}}{3} \mathrm{i} X^{-2} S \sin 2\vartheta \, [K_1 \lrcorner * B_\perp - K_2 \wedge B_\perp] \, , \\
\mathrm{d}(X^{-1} S J) &= -\sqrt{2} S \sin 2\vartheta \, J \wedge K_2 - \tfrac{3}{2\sqrt{2}} \cos 2\vartheta \, \mathrm{d}[(XS)^2 \mathrm{d}\sigma] \\
&\quad + \mathrm{i} X S \cos 2\vartheta \, \mathrm{d} B_\perp - \tfrac{1}{\sqrt{2}} X^{-2} S \sin 2\vartheta \, [K_1 \lrcorner * \mathcal{F}_\perp - K_2 \wedge \mathcal{F}_\perp] \, , \\
\mathrm{d}(S \sin 2\vartheta \, J \wedge K_2) &= 0 \, , \\
D_\perp (X^{-1} S \sin 2\vartheta \, \Omega) &= -\sqrt{2} S \Omega \wedge K_2 \, , \\
S^2 J \wedge \mathrm{d}\sigma &= -\sqrt{2} S \cos 2\vartheta (X + \tfrac{2}{3} X^{-3}) \tfrac{1}{2} J \wedge J + 2 S K_1 \lrcorner * \mathrm{d}\vartheta \\
&\quad + \tfrac{1}{\sqrt{2}} X^{-1} S J \wedge (\cos 2\vartheta \, \mathrm{d}\mathcal{A}_\perp + \tfrac{2}{3} \mathrm{i} B_\perp) \, , \\
S^2 \Omega \wedge \mathrm{d}\sigma &= -2 \mathrm{i} S \mathrm{d}\vartheta \wedge K_2 \wedge \Omega + \tfrac{1}{\sqrt{2}} X^{-1} S \Omega \wedge (\cos 2\vartheta \, \mathrm{d}\mathcal{A}_\perp + \tfrac{2}{3} \mathrm{i} B_\perp) \, , \\
0 &= X^4 K_2 \lrcorner \mathrm{d}(X^{-3} S \sin 2\vartheta) + \sqrt{2} S (X^2 - \tfrac{2}{3} X^{-2}) \\
&\quad + \tfrac{1}{\sqrt{2}} S J \lrcorner (\mathcal{F}_\perp + \tfrac{2}{3} \mathrm{i} \cos 2\vartheta \, B_\perp) \, .
\end{aligned}
\tag{8.2.8}
$$

Here the covariant derivative is $D_\perp = \mathrm{d} + \mathrm{i}\mathcal{A}_\perp \wedge$, and the interior contraction of a p-form ρ into a q-form λ (with $q \geq p$) is the $(q-p)$-form $(\rho \lrcorner \lambda)_{\mu_1 \cdots \mu_{q-p}} \equiv \frac{1}{p!} \rho^{\nu_1 \cdots \nu_p} \lambda_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{q-p}}$. Notice that the one-form σ effectively determines K_1 via (8.2.2), while the supergravity fields enter the equations via X, $\mathcal{A}_\perp$ and $B_\perp$.

8.3 Sufficiency

In this section we shall argue that (8.2.8) are in fact *equivalent* to the original spinor equations (8.1.2), and moreover as shown in appendix G these imply all but one component of the equations of motion (3.2.1), (3.2.3).

As in equation (8.1.5), we may decompose the Killing spinor as $\epsilon = \epsilon_+ + \epsilon_-$, where $\epsilon_\pm$ have definite chirality under Γ_7. Each of these defines an $SU(3)$ structure in six dimensions, which is equivalent to specifying the real two-forms $\mathcal{J}_\pm \equiv -\mathrm{i}\epsilon_\pm^\dagger\Gamma_{(2)}\epsilon_\pm$ and complex three-forms $\Omega_\pm \equiv \epsilon_\pm^{\mathrm{T}}\Gamma_{(3)}\epsilon_\pm$. For each choice of $\pm$, there exists a generalized connection with torsion $\nabla_\pm^{(T)}$ which preserves the corresponding structure, *i.e.* $\nabla_\pm^{(T)}\epsilon_\pm = 0$. One then defines the *intrinsic torsion* as $\tau_\pm \equiv \nabla_\pm^{(T)} - \nabla$, where ∇ is the Levi-Civita connection. The exterior derivatives of $\mathcal{J}_\pm$ and $\Omega_\pm$ determine completely the corresponding intrinsic torsions. One can thus regard the Killing spinor equation as an equation that relates the exterior derivatives of $\mathcal{J}_\pm$ and $\Omega_\pm$, on the left hand side of (3.2.5), to the supergravity fields on the right hand side. Since

$$
\begin{aligned}
\mathcal{J}_\pm &= \tfrac{1}{2}S(1\pm\cos 2\vartheta)(J\mp K_1\wedge K_2)\ , \\
\Omega_\pm &= \tfrac{1}{2}S(1\pm\cos 2\vartheta)\,\Omega\wedge(\mp K_1+\mathrm{i}K_2)\ ,
\end{aligned}
\tag{8.3.1}
$$

our equations (8.2.8) certainly contain this information, as they imply the exterior derivatives of all k-form bilinears, for $k\leq 3$ (this is clear from the analysis in appendix E). In fact they contain more than this information, as we have also used the dilatino constraint (3.2.6) to further simplify the equations.

It thus remains to show that (8.2.8) imply the dilatino equation (3.2.6). First we note that neither ϵ_+ nor ϵ_- can be identically zero. For if $\epsilon_\pm = 0$, respectively, then we in fact have an $SU(3)$ structure, rather than $SU(2)$ structure, and the bilinear $W\equiv \epsilon^{\mathrm{T}}\Gamma_{(3)}\epsilon = \Omega_\mp$ is the corresponding complex three-form. However, since the left hand side of equation (E.0.8) of appendix E is identically zero, we would deduce that $\Omega_\mp = 0$ and hence $\epsilon_\mp = 0$. Thus on an open dense subset where $\epsilon_\pm$ are both non-zero, we have that $\{\epsilon_\pm, \Gamma_\mu\epsilon_\pm^*\}$ span the positive and negative chirality spin bundles $\mathcal{S}^\pm$, respectively. In order for the dilatino equation to hold, it is therefore sufficient to check that the contraction of the right hand side of (3.2.6) with $\epsilon_\pm^\dagger$ and $\epsilon_\pm^{\mathrm{T}}\Gamma_\mu$ is zero. These are equivalent to two scalar and two one-form equations, respectively, that may be expressed in terms of bilinears. The corresponding equations may be found in appendix F. It is straightforward, but somewhat tedious, to show that these are indeed implied by (8.2.8).

We thus conclude that (8.2.8) are in fact necessary and sufficient for the original spinor equations

(3.2.5), (3.2.6) to hold.

8.4 Summary

We have shown that a real supersymmetric solution to Euclidean Romans supergravity, with non-trivial Abelian R-symmetry gauge field $\mathcal{A}$, is described by an $SU(2)$ structure $(S, \vartheta, K_1 = S\sin 2\vartheta(\mathrm{d}\psi+\sigma), K_2, J, \Omega)$ with corresponding metric

$$\mathrm{d}s^2 \;=\; S^2\sin^2 2\vartheta(\mathrm{d}\psi+\sigma)^2 + K_2^2 + g_{SU(2)} \ . \qquad (8.4.1)$$

Here we may complete K_1, K_2 to an orthonormal frame $\{e^a, e^5 \equiv K_1, e^6 \equiv K_2\}$, $a = 1,\ldots,4$, where

$$g_{SU(2)} \;=\; \sum_{a=1}^{4}(e^a)^2 \ , \quad J \;=\; e^1\wedge e^2 + e^3\wedge e^4 \ , \quad \Omega \;=\; (e^1+\mathrm{i}e^2)\wedge(e^3+\mathrm{i}e^4) \ . \qquad (8.4.2)$$

The vector field $\xi = \partial_\psi$ is a Killing vector, and all supergravity fields and the $SU(2)$ structure are annihilated by $\mathcal{L}_\xi$ in the gauge for which

$$\mathcal{A} \;=\; -\sqrt{2}X\cot 2\vartheta\, K_1 + \mathcal{A}_\perp \ . \qquad (8.4.3)$$

The Killing spinor equation and dilatino equation (8.1.2) are then equivalent to imposing the differential constraints (8.2.8) on this structure, where $B_\perp$ is the component of the B-field with zero interior contraction with ξ. Moreover, these imply all of the equations of motion (3.2.1), (3.2.3) provided we also impose

$$\begin{aligned} 0 \;=\; & X^4 S\sin 2\vartheta\,\mathrm{d}\sigma\wedge(K_1 \lrcorner *\mathrm{i}H_\perp) + \mathrm{d}\left[\frac{X^4}{S\sin 2\vartheta}K_1\lrcorner *\mathrm{d}(X^{-2}S\sin 2\vartheta\,K_2)\right] \\ & +\tfrac{2}{9}B_\perp\wedge B_\perp + \tfrac{1}{2}\mathcal{F}_\perp\wedge\mathcal{F}_\perp - \tfrac{4}{9}X^{-2}K_1\lrcorner *\left[\frac{3}{\sqrt{2}S\sin 2\vartheta}\mathrm{d}(XS) + X^{-2}K_2\right] \ . \end{aligned} \qquad (8.4.4)$$

This is the component of the B-field equation of motion in (3.2.1) that has zero interior contraction with ξ, where recall that $H_\perp$ is defined by (8.2.7).

Chapter 9

Applications of the $SU(2)$ structure

9.1 Expansion at a conformal boundary

In this section we determine the asymptotic form of the $SU(2)$ structure at a conformal boundary. The aim is to make contact with the results of [21]. A similar holographic approach to constructing rigid supersymmetric backgrounds in lower dimensions was followed in [47, 50, 48].

Given an asymptotically locally AdS solution we may introduce a radial coordinate r with the conformal boundary located at $r = \infty$. The bosonic fields then admit an expansion of the form

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{9}{2}\frac{\mathrm{d}r^2}{r^2} + r^2\left[g^{(0)}_{mn} + \frac{1}{r^2}g^{(2)}_{mn} + \cdots\right]\mathrm{d}x^m\mathrm{d}x^n \ , \\
X &= 1 + \frac{1}{r^2}X_2 + \cdots \ , \\
B &= rb - \frac{1}{r^2}\mathrm{d}r\wedge A^{(0)} + \cdots \ , \\
\mathcal{A} &= a + \cdots \ ,
\end{aligned} \tag{9.1.1}$$

where recall $H = \mathrm{d}B$ and $\mathcal{F} = \mathrm{d}\mathcal{A}$. The five-dimensional coordinates on the conformal boundary are denoted x^m, with $m = 1, 2, 3, 4, 5$. Some of the terms *a priori* present in these expansions are set to zero by the equations of motion.

In order to determine the corresponding expansion of the $SU(2)$ structure, for this section we introduce the following explicit basis for Cliff(6, 0):

$$\Gamma_m = \begin{pmatrix} 0 & \mathrm{i}\gamma_m \\ -\mathrm{i}\gamma_m & 0 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} 0 & -1_4 \\ -1_4 & 0 \end{pmatrix}, \quad \Gamma_7 = \begin{pmatrix} -1_4 & 0 \\ 0 & 1_4 \end{pmatrix}, \tag{9.1.2}$$

where γ_m are a Hermitian basis of Cliff(5, 0). Notice that (9.1.2) is different to the basis used in the rest of the thesis (where Γ_μ are purely imaginary), but instead coincides with the basis used in [21]. The asymptotic form of the metric implies the radial expansion of an orthonormal frame is

$$E^6 = -\frac{3}{\sqrt{2}}\frac{\mathrm{d}r}{r}, \qquad E^m = re^m + \cdots . \tag{9.1.3}$$

The Killing spinor then has the following asymptotic expansion

$$\epsilon = \sqrt{r}\begin{pmatrix} \chi \\ -\mathrm{i}\chi \end{pmatrix} + \frac{1}{\sqrt{r}}\begin{pmatrix} \varphi \\ \mathrm{i}\varphi \end{pmatrix} + \cdots . \tag{9.1.4}$$

From this, together with $S \equiv \epsilon^\dagger\epsilon$ and the definitions in (E.0.1), we deduce the following asymptotic expansion for the $SU(2)$ structure:

$$\begin{aligned}
S &= 2S^{(0)}(x)\, r + \cdots , \\
\vartheta &= \frac{\pi}{4} + \frac{\vartheta^{(0)}(x)}{r} + \cdots , \\
K_1 &= K_1^{(0)}(x)\, r + \cdots , \\
K_2 &= K_2^{(0)}(x) - \frac{3}{\sqrt{2}}\frac{\mathrm{d}r}{r} + \cdots , \\
J &= J^{(0)}(x)\, r^2 + \cdots , \\
\Omega &= \Omega^{(0)}(x)\, r^2 + \cdots ,
\end{aligned} \tag{9.1.5}$$

where the ellipses denote subleading terms. Inserting these expansions into (8.2.8) reduces to the following independent equations, at leading order in r:

$$
\begin{aligned}
\mathrm{d}S^{(0)} &= -\tfrac{\sqrt{2}}{3}\left(S^{(0)}K_2^{(0)} + \mathrm{i}S^{(0)}K_1^{(0)} \lrcorner\, b\right) ,\\
\mathrm{d}(S^{(0)}\vartheta^{(0)}) &= -\tfrac{1}{2\sqrt{2}}S^{(0)}K_1^{(0)} \lrcorner\, \mathrm{d}a\,,\\
\mathrm{d}(S^{(0)}K_1^{(0)}) &= \tfrac{2\sqrt{2}}{3}\left[2\vartheta^{(0)}S^{(0)}J^{(0)} + S^{(0)}K_1^{(0)}\wedge K_2^{(0)} + \mathrm{i}S^{(0)}b - \tfrac{\mathrm{i}}{2}S^{(0)}K_1^{(0)} \lrcorner\,(*b)\right] ,\\
\mathrm{d}(S^{(0)}K_2^{(0)}) &= \mathrm{i}S^{(0)}K_1^{(0)} \lrcorner\, \mathrm{d}b - \mathrm{i}S^{(0)}K_1^{(0)} \lrcorner\, \mathrm{d}(\log S^{(0)})b\,,\\
\mathrm{d}(S^{(0)}J^{(0)}) &= -\sqrt{2}K_2^{(0)}\wedge(S^{(0)}J^{(0)})\,,\\
\mathrm{d}(S^{(0)}\Omega^{(0)}) &= -\mathrm{i}\left(a - 2\sqrt{2}\vartheta^{(0)}K_1^{(0)} - \mathrm{i}\sqrt{2}K_2^{(0)}\right)\wedge(S^{(0)}\Omega^{(0)})\,. \qquad (9.1.6)
\end{aligned}
$$

Here $*$ denotes the Hodge duality operator for the boundary metric $g^{(0)}$. We also note that the flux equation of motion (8.4.4) does not impose an independent constraint at leading order. The set of equations (9.1.6) is precisely the starting point for the purely field theory analysis of rigid supersymmetric five-manifold backgrounds carried out in [21].

9.2 BPS Wilson loops

The expectation value of Wilson loops in $USp(2N)$ SCFTs have been computed when the gauge theory is placed on the round five-sphere [45] or $SU(3)\times U(1)$ squashed five-spheres in chapter 6. Romans supergravity solutions dual to these backgrounds have also been constructed and successfully compared with the large N gauge theory results. In this section we compute the regularised string action dual to the Wilson loops for *any* Romans solution with ball topology and $U(1)^3$ symmetry, confirming one of the conjectures made previously.

As shown in chapter 6, the relevant string action is

$$
S_{\text{string}} = \int_{\Sigma_2} X^{-2}\mathrm{vol}_2 + \mathrm{i}B - \frac{3}{\sqrt{2}}\mathrm{length}(\partial\Sigma_2)\ , \qquad (9.2.1)
$$

where the boundary counterterm regularizes the divergence arising from the infinite boundary length. We begin by writing

$$
B \equiv B_1\wedge K_1 + B_\perp\ . \qquad (9.2.2)
$$

Comparing to (8.2.3) we see that

$$X^{-2}K_2 \;=\; -\frac{3}{\sqrt{2}S\sin 2\vartheta}\mathrm{d}(XS)+\mathrm{i}B_1\;. \tag{9.2.3}$$

It is natural to define the radial coordinate

$$\rho \;\equiv\; XS\;. \tag{9.2.4}$$

Then

$$X^{-2}K_2 \;=\; -\frac{3X}{\sqrt{2}\sin 2\vartheta}\frac{\mathrm{d}\rho}{\rho}+\mathrm{i}B_1\;. \tag{9.2.5}$$

Notice that in general B_1 has a component in the $\mathrm{d}\rho$ direction, and also $\mathrm{d}\rho$ is not orthogonal to J and Ω. However, we may still consider substituting (9.2.5) into the bilinears, at the expense of introducing the unknown B_1. From the point of view of asymptotically locally AdS solutions this is natural, since to leading order at large ρ we see from (9.1.5) that K_2 is in the $\mathrm{d}\rho$ direction. Let us next wedge (9.2.5) with K_1. This reads

$$X^{-2}K_1\wedge K_2+\mathrm{i}(B-B_{\perp}) \;=\; \frac{3}{\sqrt{2}}\mathrm{d}\rho\wedge(\mathrm{d}\psi+\sigma)\;. \tag{9.2.6}$$

The left hand side is precisely the (unregularized) action of a string wrapping the K_1–K_2 direction, while the right hand side is exact on the string worldsheet. In appendix H we show that such a string is supersymmetric. Notice that

$$\|\partial_\psi\| \;=\; S\sin 2\vartheta \;=\; \rho X^{-1}\sin 2\vartheta \;=\; \rho+O(1/\rho)\;. \tag{9.2.7}$$

Here we have used the asymptotic expansions in section 9.1. Since the string wraps the ∂_ψ direction, the boundary length is

$$\mathrm{length}(\partial\Sigma_2) \;=\; \|\partial_\psi\|\int_{S^1}\mathrm{d}\psi\;. \tag{9.2.8}$$

Integrating by parts the bulk action in (9.2.1), we see that the boundary counterterm simply cancels against the bulk contribution at infinity, leaving

$$S_{\mathrm{string}} \;=\; -\frac{3}{\sqrt{2}}\rho_{\mathrm{origin}}\int_{S^1}\mathrm{d}\psi\;, \tag{9.2.9}$$

where

$$\rho_{\text{origin}} = (XS)\mid_{\text{origin}} . \tag{9.2.10}$$

Here $\rho \in [\rho_{\text{origin}}, \infty)$. We next claim that for a solution with ball topology and $U(1)^3$ isometry

$$(XS)\mid_{\text{origin}} = \frac{b_1 + b_2 + b_3}{\sqrt{2}} . \tag{9.2.11}$$

Here we write the supersymmetric Killing vector as

$$\partial_\psi = \sum_{i=1}^{3} b_i \partial_{\varphi_i} , \tag{9.2.12}$$

where φ_i, $i = 1, 2, 3$, have period 2π, and the orientations (and hence signs) will be fixed shortly. Combining (9.2.11) with (9.2.9) for a Wilson loop wrapping the φ_i circle we obtain

$$S_{\text{string}} = -9\pi \frac{b_1 + b_2 + b_3}{3b_i} , \tag{9.2.13}$$

where $\int_{S^1} \mathrm{d}\psi = 2\pi/b_i$. This is precisely the Wilson loop conjecture made in chapter 6

Thus it remains to prove (9.2.11). Geometrically, the b_i arise as the skew eigenvalues of the two-form $\mathrm{d}K$ at the origin (recall that $K = S \sin 2\vartheta\, K_1$ is a Killing one-form). That is, raising an index of $\mathrm{d}K$ to obtain a skew-symmetric 6×6 matrix in an orthonormal frame, at the origin we have

$$(\mathrm{d}K)\mid_{\text{origin}} = \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{pmatrix} , \qquad R_i = \begin{pmatrix} 0 & -b_i \\ b_i & 0 \end{pmatrix} . \tag{9.2.14}$$

This follows from a simple local calculation. Specifically, at the origin we may introduce three sets of polar coordinates (ρ_i, φ_i), $i = 1, 2, 3$, and write the leading order flat metric as

$$\mathrm{d}s^2_{\text{flat}} = \sum_{i=1}^{3} \mathrm{d}\rho_i^2 + \rho_i^2 \mathrm{d}\varphi_i^2 . \tag{9.2.15}$$

One can then compute $\mathrm{d}K$ at the origin using this local metric, where $K = \sum_{i=1}^{3} b_i \rho_i^2 \mathrm{d}\varphi_i$ is the dual one-form to ∂_ψ. In the orthonormal frame

$$e_{2i-1} = \mathrm{d}\rho_i , \qquad e_{2i} = \rho_i \mathrm{d}\varphi_i , \qquad i = 1, 2, 3 , \tag{9.2.16}$$

at the origin this gives precisely (9.2.14). Our solution is also equipped with a six-dimensional almost complex structure, which as a two-form reads

$$\mathcal{J} \quad = \quad K_1 \wedge K_2 + J \ . \tag{9.2.17}$$

In the same frame this reads

$$\mathcal{J} \quad = \quad \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} , \qquad \varepsilon \ = \ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{9.2.18}$$

Thus $\mathcal{J}(e_1) = e_2$, *etc.* Notice this fixes the orientations of the φ_i. Then

$$\mathcal{J} \lrcorner \mathrm{d}K \mid_{\text{origin}} \quad = \quad 2(b_1 + b_2 + b_3) \ . \tag{9.2.19}$$

Let us now look at computing the same quantity using the bilinear equations. We have

$$K_1 \lrcorner \mathrm{d}K \quad = \quad \frac{1}{X^2}\left[-\frac{1}{S\sin 2\vartheta}\mathrm{d}(X^2 S^2 \sin^2 2\vartheta) + 2XS\sin 2\vartheta \, \mathrm{d}X \right] \ . \tag{9.2.20}$$

K has norm $S\sin 2\vartheta$, which by definition is zero at the origin. Contracting K_2 into (9.2.20) and restricting to the origin we hence find

$$(K_1 \wedge K_2) \lrcorner \mathrm{d}K \mid_{\text{origin}} \quad = \quad -2K_2 \lrcorner \mathrm{d}(S\sin 2\vartheta) \mid_{\text{origin}} \ , \tag{9.2.21}$$

where we have assumed that X is regular at the origin (and we shall make similar regularity assumptions for other fields in what follows). We next compute

$$J \lrcorner \mathrm{d}K \quad = \quad (S\sin 2\vartheta)^2 J \lrcorner \mathrm{d}\sigma \ , \tag{9.2.22}$$

which thus tends to zero at the origin. Finally contracting K_2 into (E.0.24), and restricting to the origin, we find

$$K_2 \lrcorner \mathrm{d}(S\sin 2\vartheta) \mid_{\text{origin}} \quad = \quad -\sqrt{2}(XS) \mid_{\text{origin}} \ . \tag{9.2.23}$$

Combined with (9.2.21), this shows that

$$\mathcal{J} \lrcorner \mathrm{d}K \mid_{\text{origin}} \quad = \quad 2\sqrt{2}(XS) \mid_{\text{origin}} \ , \tag{9.2.24}$$

which together with (9.2.19) proves (9.2.11).

Chapter 10

New Solutions

The system of equations for the $SU(2)$ structure in chapter 8 is too complicated to solve in general; to find solutions one needs to make some additional assumptions. In this chapter we consider an ansatz that naturally generalizes the 1/4 BPS solutions (and their 1/2 BPS limit) found in chapter 4, that is, we generalize the $\mathbb{CP}^2$ to a Kähler-Einstein base. Following that, we analyze what happens when we consider as an ansatz a fibration over the product of two Riemann surfaces.

Later in this chapter, we give some details of a new analytic supersymmetric solution to Euclidean six-dimensional Romans supergravity. This corresponds to the 3/4 BPS squashed sphere, constructed earlier as a perturbation expansion, but that can be obtained in a closed form using our new set of equations.

10.1 Squashed Sasaki-Einstein solutions

We begin by making the following ansatz for the supergravity fields[1]

$$
\begin{aligned}
\mathrm{d}s^2 &= \alpha^2(r)\mathrm{d}r^2 + \gamma^2(r)(\mathrm{d}\psi+\sigma)^2 + \beta^2(r)\mathrm{d}s^2_{\mathrm{KE}} \ , \\
B &= p(r)\mathrm{d}r\wedge(\mathrm{d}\psi+\sigma) + \tfrac{1}{2}q(r)\mathrm{d}\sigma \ , \\
\mathcal{A} &= f(r)(\mathrm{d}\psi+\sigma) - 3\mathrm{d}\psi \ , \\
X &= X(r) \ .
\end{aligned} \tag{10.1.1}
$$

Here we take $\mathrm{d}s^2_{\mathrm{KE}}$ to be a four-dimensional positively curved Kähler-Einstein metric, so that a constant r hypersurface is a squashed Sasaki-Einstein five-manifold. Concretely, this means that $\mathrm{d}\psi+\sigma$ is a global contact one-form on such a hypersurface, with

$$
\mathrm{d}\sigma = 2\omega_{\mathrm{KE}} \ . \tag{10.1.2}
$$

The ansatz (10.1.1) reduces to our earlier one on taking the Kähler-Einstein metric to be the Fubini-Study metric on $\mathbb{CP}^2$. Notice also that in writing (10.1.1) we have taken the supersymmetric Killing vector ∂_ψ to coincide with the Reeb vector field of the squashed Sasaki-Einstein manifold.

Comparing to chapter 8, and identifying the four-dimensional $SU(2)$ structure metric in (8.4.1) with $\beta^2(r)\mathrm{d}s^2_{\mathrm{KE}}$, allows us to identify

$$
S\sin 2\vartheta = \gamma(r) \ , \quad K_2 = -\alpha(r)\mathrm{d}r \ , \quad \Omega = \beta^2(r)\Omega_{\mathrm{KE}} \ , \quad J = -\beta^2(r)\omega_{\mathrm{KE}} \ , \tag{10.1.3}
$$

where Ω_{KE} satisfies[2]

$$
\mathrm{d}\Omega_{\mathrm{KE}} = -3\mathrm{i}\sigma\wedge\Omega_{\mathrm{KE}} \ . \tag{10.1.4}
$$

We take $S=S(r)$, $\vartheta=\vartheta(r)$. From the remaning supergravity fields, we similarly read off

$$
f(r) = 3-\sqrt{2}XS\cos 2\vartheta \ , \qquad \mathcal{F}_\perp = 2f(r)\omega_{\mathrm{KE}} \ , \qquad B_\perp = q(r)\omega_{\mathrm{KE}} \ . \tag{10.1.5}
$$

[1] Recall that the formula (8.4.3) for the gauge field $\mathcal{A}$ requires a specific gauge choice. However, in chapter 4 this was presented in a different gauge. This accounts for the factor of $-3\mathrm{d}\psi$ in (10.1.1).

[2] We have chosen sign conventions so as to agree with those of [1].

Substituting these into the differential constraints (8.2.8) and flux equation of motion (8.4.4) then reduces to the following independent ODEs:

$$\begin{aligned}
0 &= \mathrm{i}X^3\left(2p-q'\right)\sin 2\vartheta + \tfrac{2\sqrt{2}}{3}\alpha\left[\mathrm{i}q + (9+\beta^2X^{-2})\cos 2\vartheta\right] - \alpha XS(3+\cos 4\vartheta)\ ,\\
0 &= \frac{\mathrm{d}}{\mathrm{d}r}(X^{-1}S\beta^2\cos 2\vartheta) - 3\sqrt{2}XS\frac{\mathrm{d}}{\mathrm{d}r}(XS) + \mathrm{i}XSq'\ ,\\
0 &= \frac{\mathrm{d}}{\mathrm{d}r}(X^{-1}S\beta^2\sin 2\vartheta) - \sqrt{2}S\alpha\beta^2\ ,\\
0 &= -2XS + 3\sqrt{2}\cos 2\vartheta + \mathrm{i}\tfrac{\sqrt{2}}{3}q + \tfrac{1}{\sqrt{2}}\left(\tfrac{2}{3}X^{-2}+X^2\right)\beta^2\cos 2\vartheta - \beta^2X\alpha^{-1}\vartheta'\ ,\\
0 &= -\sqrt{2}\alpha S\left[\left(3X^4+1\right)\beta^2 + 18X^2\right]\sin 2\vartheta + X^2\left(12XS\beta' + \sqrt{2}\mathrm{i}p\beta\right)\beta\ ,\\
0 &= -\frac{p\beta^4\csc 2\vartheta}{\alpha X^2 S} + 6\sqrt{2}qXS - \mathrm{i}q^2 - \frac{6\sqrt{2}\mathrm{i}S\beta^2\cos 2\vartheta}{X} + 18\mathrm{i}X^2S^2 - 81\mathrm{i}\ .
\end{aligned} \qquad (10.1.6)$$

Notice that as a consequence of parametrization invariance one is free to specify the function $\beta = \beta(r)$. Hence (10.1.6) are six coupled ODEs for the six functions $(X, S, \vartheta, \alpha, p, q)$. Furthermore, notice that they are independent of the choice of Kähler-Einstein metric, and are thus equivalent to the equations studied in chapter 4. There we constructed a two-parameter family of 1/4 BPS solutions, as a series expansion both around the conformal boundary at $r = \infty$, and as an expansion around Euclidean AdS. Specifically, the parameters are

$$f_0 \equiv f(r)|_{\text{boundary}}\ , \qquad s^{-1} \equiv \left.\frac{\gamma(r)}{\beta(r)}\right|_{\text{boundary}}\ . \qquad (10.1.7)$$

We hence automatically construct new solutions, with an arbitrary squashed Sasaki-Einstein five-manifold, with squashing parameter s, as conformal boundary. Setting $s = 1$ and $f_0 = 0$, the conformal boundary is a Sasaki-Einstein manifold with metric $\mathrm{d}s^2_{\mathrm{SE}} = (\mathrm{d}\psi + \sigma)^2 + \mathrm{d}s^2_{\mathrm{KE}}$, and in the bulk the only non-trivial field is the metric, which is a "hyperbolic cone"

$$\mathrm{d}s^2_6 = \frac{\mathrm{d}r^2}{1+\frac{2}{9}r^2} + r^2\mathrm{d}s^2_{\mathrm{SE}}\ . \qquad (10.1.8)$$

When $\mathrm{d}s^2_{\mathrm{SE}}$ is the round five-sphere this is simply Euclidean AdS_6, while more generally (10.1.8) has an isolated Calabi-Yau cone singularity at $r = 0$. The solutions with general s and f_0 have the same behaviour near the tip of the cone/origin, and thus in general these supergravity solutions have a Calabi-Yau singularity. Nevertheless, this singularity does not lead to any UV divergences in the holographic free energy or Wilson loop VEVs. Although we were unable to solve the system (10.1.6) analytically, see the end of section 10.3 for further discussion.

Any solution to Romans $F(4)$ supergravity uplifts to a solution of massive type IIA supergravity, as a warped product $M_6 \times S^4$ [14]. For an asymptotically locally AdS solution M_6, these are expected to be the gravity duals to a certain family of $USp(2N)$ gauge theories, defined on the conformal boundary of M_6. The gauge theories arise from a system of N D4-branes, N_f of D8-branes and an orientifold plane. This data is captured in the six-dimensional effective Newton constant [7]

$$G_N = \frac{15\pi\sqrt{8-N_f}}{4\sqrt{2}N^{5/2}} . \tag{10.1.9}$$

Recall that the two-parameter family of solutions constructed in this section reduce to the 1/4 BPS family in chapter 4 when the Kähler-Einstein metric is taken to be the Fubini-Study metric on $\mathbb{CP}^2$. The computation of the holographic free energy then very closely follows that previous one. The upshot is that

$$\mathcal{F}_{\text{gravity}} = I_{\text{renormalized}} = -\frac{27}{4\pi G_N} \cdot \text{vol(SE)} , \tag{10.1.10}$$

is independent of the two parameters s and f_0. Notice that the volume vol(SE) appearing in (10.1.10) is that of the Sasaki-Einstein metric, which is the conformal boundary metric when $s = 1$, even though (10.1.10) holds for all s.

Comparison to field theory

We would like to compare (10.1.10) with the corresponding large N field theory calculation. This involves computing the localized partition function of the $USp(2N)$ gauge theories on a squashed Sasaki-Einstein background, and taking the $N \to \infty$ limit. In [54] the perturbative partition function of an arbitrary $\mathcal{N} = 1$ supersymmetric gauge theory was computed on a general $U(1)^3$–invariant Sasaki-Einstein five-manifold. For a gauge theory with gauge group G and a matter hypermultiplet in an arbitrary representation $\mathcal{R}$, the localized perturbative partition function is

$$Z^{\text{SE}}_{\text{pert}} = \int_{\mathfrak{t}} \mathrm{d}a\, \mathrm{e}^{-S_{\text{cl}}} \frac{\prod_\alpha S_3^{\text{SE}}[\,\mathrm{i}\alpha(a)\,;\,\vec{\xi}\,]}{\prod_\rho S_3^{\text{SE}}[\,\mathrm{i}\rho(a) + \frac{3}{2}\,;\,\vec{\xi}\,]} . \tag{10.1.11}$$

The integration in a is over the Cartan $\mathfrak{t}$ of the gauge group. The products are over roots α of G and weights ρ of the representation $\mathcal{R}$, and we have denoted by $S_{\rm cl}$ the classical action evaluated on the localization locus. Furthermore $S_3^{\rm SE}[\,x\,;\,\vec{\xi}\,]$ is a generalized version of the triple-sine function

$$S_3^{\rm SE}[\,x\,;\,\vec{\xi}\,] \;\equiv\; \prod_{\vec{m}} (\vec{m}\cdot\vec{\xi}+x)(\vec{m}\cdot\vec{\xi}+\vec{\xi}\cdot\vec{\xi}-x)\,. \tag{10.1.12}$$

Here $\vec{m}=(m_1,m_2,m_3)$ runs over the charge lattice of holomorphic functions on the Calabi-Yau cone over the Sasaki-Einstein five-manifold, where m_i is the charge under the ith $U(1)$ symmetry. Furthermore, we have written the supersymmetric (Reeb) vector field as

$$\xi \;=\; \sum_{i=1}^{3}\xi_i\partial_{\varphi_i}\;, \tag{10.1.13}$$

where $\vec{\xi}=(\xi_1,\xi_2,\xi_3)$ and ∂_{φ_i} generate the $U(1)^3$ isometry. For example, for the round S^5 the Calabi-Yau cone is simply $\mathbb{C}^3$, with a basis of holomorphic functions $z_1^{m_1}z_2^{m_2}z_3^{m_3}$, where $m_i\in\mathbb{Z}_{\geq 0}$. In this case, (10.1.12) reduces to the standard triple-sine function.

We are interested in evaluating (10.1.11) for the $USp(2N)$ gauge theories, in the large N limit. This involves the asymptotics of the hypermultiplet and vectormultiplet contributions computed in [54]:

$$\begin{aligned}
\log S_3^{\rm SE}[\,x\,;\,\vec{\xi}\,] \;&\sim\; -\mathrm{i}\pi\,\mathrm{sgn}(\mathrm{Im}\,x)\left[\left(\frac{x^3}{6}+\frac{3x}{4}\right)\frac{\mathrm{vol(SE)}}{\pi^3}+\frac{x}{24\pi}\sum_I\beta_I\right]\,,\\
\log S_3^{\rm SE}[\,x+\tfrac{3}{2}\,;\,\vec{\xi}\,] \;&\sim\; \mathrm{i}\pi\,\mathrm{sgn}(\mathrm{Im}\,x)\left[\left(\frac{x^3}{6}-\frac{3x}{8}\right)\frac{\mathrm{vol(SE)}}{\pi^3}+\frac{x}{24\pi}\sum_I\beta_I\right]\,.
\end{aligned} \tag{10.1.14}$$

Here β_I are certain parameters defined in [54], which will not enter the final result.[3] We may then compute the leading contribution to the partition function at large N using a saddle point method. One specifies an element of the Cartan subalgebra of $USp(2N)$ by its eigenvalues $\{\lambda_1,\ldots,\lambda_N\}$. In the large N saddle point these behave as $\lambda_n\sim N^{1/2}x_n$. One then introduces an eigenvalue density

$$\rho(x) \;=\; \frac{1}{N}\sum_n\delta(x-x_n)\,, \tag{10.1.15}$$

which has support on a finite interval $[0,x_\star]$. Solving the saddle point approximation to the above

[3] β_I is the length of the Ith closed Reeb orbit.

matrix model, we find

$$\rho(x) \;=\; \frac{4(8-N_f)x}{9}\,, \quad \text{and} \quad x_\star \;=\; \frac{3}{\sqrt{2}\sqrt{8-N_f}}\,, \tag{10.1.16}$$

which leads to the final result for the large N free energy

$$\mathcal{F}_{\text{gauge theory}} \;=\; -\frac{9\sqrt{2}}{5\pi^2\sqrt{8-N_f}}\text{vol(SE)}N^{5/2}+o(N^{5/2})\,. \tag{10.1.17}$$

This precisely agrees with (10.1.10).

The field theory computation above is for the Sasaki-Einstein conformal boundary, with $s=1$ and $f_0=0$. On the other hand, in [21] we conjectured that the partition function should depend only on the holomorphic foliation generated by the Killing vector ξ. Since this is independent of s and f_0, this conjecture implies that (10.1.17) holds for the entire two-parameter family of 1/4 BPS backgrounds. Since (10.1.17) agrees with (10.1.10), this lends credence to the conjecture. We also regard this as evidence that the 1/4 BPS family of supergravity backgrounds is the correct holographic dual, in spite of the Calabi-Yau singularity at the origin.

BPS Wilson loops

Finally, let us discuss the computation of the VEV of BPS Wilson loops on both sides of the correspondence. Following a similar computation to that in [21], in the large N matrix model for the gauge theory this is given by

$$\langle\, W\,\rangle \;=\; \int_0^{x_\star} \mathrm{e}^{\lambda(x)\beta_I}\,\rho(x)\mathrm{d}x\;, \tag{10.1.18}$$

where β_I is the length of the closed Reeb orbit wrapped by the Wilson loop.[4] At large N one hence obtains

$$\log\langle\, W\,\rangle \;=\; x_\star\beta_I N^{1/2}+o(N^{1/2})\;. \tag{10.1.19}$$

On the other hand, in the dual supergravity solution this corresponds to a fundamental string wrapping the circle of length β_I, together with the radial direction r. We find that the regularized

[4]Recall that the computation of [54] is valid for a $U(1)^3$-invariant Sasaki-Einstein manifold, for which the index I runs over the rays of the corresponding polyhedral cone.

action is

$$S_{\rm string} \;=\; -\frac{3}{\sqrt{2}\sqrt{8-N_f}}\beta_I N^{1/2}\ . \qquad (10.1.20)$$

This should be identified with $-\log\langle W\rangle$ in field theory, and we find perfect agreement.

10.2 Fibration over the product of two Riemann surfaces

After generalizing $\mathbb{CP}^2$ to a Kähler-Einstein manifold, the next natural generalization is to allow the four-dimensional base to be a product of two Riemann surfaces with different curvatures. When the curvature of these two manifolds are the same, we return to the Kähler-Einstein case.

We start by making the following ansatz for the supergravity fields

$$\begin{aligned} {\rm d}s^2 &= \alpha^2(r){\rm d}r^2 + \gamma^2(r)({\rm d}\psi+\sigma)^2 + \beta_1^2(r){\rm d}\Sigma_1^2 \ + \beta_2^2(r){\rm d}\Sigma_2^2\ ,\\ B &= p(r){\rm d}r\wedge({\rm d}\psi+\tilde{\sigma}) + \omega\ q(r)\ ,\\ \mathcal{A} &= a\ {\rm d}\psi + f(r)({\rm d}\psi+\bar{\sigma})\ ,\\ X &= X(r)\ , \end{aligned} \qquad (10.2.1)$$

where ${\rm d}\Sigma_1^2$ and ${\rm d}\Sigma_2^2$ are the metrics on the Riemann surfaces. The one-form σ is given by

$$\sigma = c_1\ \sigma_1 + c_2\ \sigma_2\ , \qquad (10.2.2)$$

and the other one-forms $\tilde{\sigma}$ and $\bar{\sigma}$ are also combinations of σ_1 and σ_2, such that $\sigma_i = vol_{\Sigma_i}$. As in the previous case, we use ${\rm d}\sigma = 2\omega$. The elements of the $SU(2)$ structure can then be written as

$$S\sin 2\xi \;=\; \gamma(r)\ ,\quad K_2 \;=\; -\alpha(r){\rm d}r\ , \qquad (10.2.3)$$

as the two dimensional fibration over the the four dimensional base stays the same, with $S\equiv S(r)$ and $\xi\equiv\xi(r)$, and

$$\Omega \;=\; \beta_1(r)\beta_2(r)\Omega_{\rm RS}\ ,\quad J \;=\; -\beta_1^2(r)J_1 - \beta_2^2(r)J_2\ , \qquad (10.2.4)$$

where Ω_{RS} is a product of Ω_1 and Ω_2 such that each of them respect

$$d\Omega_i \;=\; {\rm i}\ k_i\ \sigma_i\wedge\Omega_i. \qquad (10.2.5)$$

From the supergravity fields, we deduce that

$$
\begin{aligned}
f(r) &= -a-\sqrt{2}SX\cos 2\xi \ , \\
\mathcal{F}_\perp &= d\bar{\sigma}f(r)-\partial_r f(r)dr\wedge(\sigma-\bar{\sigma}) \ , \\
B_\perp &= \omega\ q(r)+p(r)\mathrm{d}r\wedge(\tilde{\sigma}-\sigma) \ . \qquad (10.2.6)
\end{aligned}
$$

Substituting these into the differential constraints (8.2.8) and flux equation of motion (8.4.4), one quickly realizes that $\sigma\equiv\tilde{\sigma}\equiv\bar{\sigma}$. Then the set of equations reduces to the following system of independent ODEs:

$$
\begin{aligned}
0 &= -2\alpha{\beta_2}^2\left(\sqrt{2}\ k_1X^2(3\ a\ \cos 2\xi-\mathrm{i}q)-2\sqrt{2}\ a\ \cos 2\xi{\beta_1}^2+3k_1\left(\cos^2 2\xi+1\right)SX^3\right) \\
&\quad +3\ k_2\ {\beta_1}^2\sin 2\xi X^5\left(3\sqrt{2}\left(XS'+SX'\right)-\mathrm{i}q'\right)-6\ k_2\ \alpha{\beta_1}^2\sin^2 2\xi SX^3 \ , \\
0 &= -2\alpha{\beta_1}^2\left(\sqrt{2}\ k_2X^2(3\ a\ \cos 2\xi-\mathrm{i}q)-2\sqrt{2}\ a\ \cos 2\xi{\beta_2}^2+3k_2\left(\cos^2 2\xi+1\right)SX^3\right) \\
&\quad +3\ k_1\ {\beta_2}^2\sin 2\xi X^5\left(3\sqrt{2}\left(XS'+SX'\right)-\mathrm{i}q'\right)-6\ k_1\ \alpha{\beta_2}^2\sin^2 2\xi SX^3 \ , \\
0 &= 6\ a\ \beta_1{\beta_2}^2\left(\cos 2\xi\left(X\left(\beta_1S'+2S\beta_1'\right)-\beta_1SX'\right)-2\beta_1\sin 2\xi\ SX\xi'\right) \\
&\quad +\mathrm{i}\ k_1\ S\ {\beta_2}^2\left(\sqrt{2}\ q\ \alpha\ \sin 2\xi+3X^3\left(q'+3\sqrt{2}\mathrm{i}\left(XS'+SX'\right)\right)\right)-\sqrt{2}\mathrm{i}\ k_2\ q\ S\ \alpha{\beta_1}^2\sin 2\xi \ , \\
0 &= 6\ a\ \beta_2{\beta_1}^2\left(\cos 2\xi\left(X\left(\beta_2S'+2S\beta_2'\right)-\beta_2SX'\right)-2\beta_2\sin 2\xi\ SX\xi'\right) \\
&\quad +\mathrm{i}\ k_2\ S\ {\beta_1}^2\left(\sqrt{2}\ q\ \alpha\ \sin 2\xi+3X^3\left(q'+3\sqrt{2}\mathrm{i}\left(XS'+SX'\right)\right)\right)-\sqrt{2}\mathrm{i}\ k_1\ q\ S\ \alpha{\beta_2}^2\sin 2\xi \ , \\
0 &= \alpha\ \sin 2\xi\ S\left(\sqrt{2}\ a\ \left(k_1\ {\beta_2}^2-{\beta_1}^2\left(k_2+2\ {\beta_2}^2X^2\right)\right)+2\cos 2\xi\ SX\left(k_1\ {\beta_2}^2-k_2\ {\beta_1}^2\right)\right) \\
&\quad +{\beta_2}^2\left(2\ a\ \beta_1\left(\beta_1\ XS'-\beta_1\ SX'+2\ SX\ \beta_1'\right)+k_1\cos 2\xi\ SX^3\left(-3\sqrt{2}\left(XS'+SX'\right)+\mathrm{i}q'\right)\right) \\
0 &= \alpha\ \sin 2\xi\ S\left(\sqrt{2}\ a\ \left(k_2\ {\beta_1}^2-{\beta_2}^2\left(k_1+2\ {\beta_1}^2X^2\right)\right)+2\cos 2\xi\ SX\left(k_2\ {\beta_1}^2-k_1\ {\beta_2}^2\right)\right) \\
&\quad +{\beta_1}^2\left(2\ a\ \beta_2\left(\beta_2\ XS'-\beta_2\ SX'+2\ SX\ \beta_2'\right)+k_2\cos 2\xi\ SX^3\left(-3\sqrt{2}\left(XS'+SX'\right)+\mathrm{i}q'\right)\right) \\
0 &= S\left(-X\left(2\ \cos 2\xi\ \beta_1\beta_2\ \xi'+\sin 2\xi\ \beta_2\beta_1'+\sin 2\xi\ \beta_1\beta_2'\right)+\sin 2\xi\ \beta_1\beta_2\ X'+\sqrt{2}\ \alpha\beta_1\beta_2\ X^2\right) \\
&\quad -\beta_1\beta_2\ \sin 2\xi\ XS' \ ,
\end{aligned}
$$

$$
\begin{aligned}
0 &= + X^2\left(-6X\left(2\ a\ {\beta_1}^2{\beta_2}^2\ \xi' + \alpha\ S\left(k_1{\beta_2}^2+k_2{\beta_1}^2\right)\right)+\sqrt{2}\mathrm{i}\ q\ \alpha\left(k_1{\beta_2}^2+k_2{\beta_1}^2\right)\right) \\
&\quad \sqrt{2}\ a\ \cos 2\xi\ \alpha\left({\beta_2}^2\left(2\ {\beta_1}^2\left(3\ X^4+2\right)-3k_1X^2\right)-3k_2{\beta_1}^2X^2\right)\ , \\
0 &= -\sin 2\xi\ SX\left(-4\ a\ {\beta_1}^2{\beta_2}^2\left(3X^4+2\right)\xi'+\sqrt{2}\mathrm{i}\ q\ \alpha X^3\left(k_1{\beta_2}^2+k_2{\beta_1}^2\right)-6\ \alpha\ SX^4\left(k_1{\beta_2}^2+k_2{\beta_1}^2\right)\right) \\
&\quad -\mathrm{i}SX^3\left(k_1{\beta_2}^2+k_2{\beta_1}^2\right)\left(q'+3\sqrt{2}\mathrm{i}\left(XS'+SX'\right)\right)-4\ a\ \cos 2\xi\ S\left(3X^5+X\right){\beta_1}^2\beta_2\beta_2' \\
&\quad +2\ a\ \cos 2\xi\ \beta_1\beta_2\left(\beta_2\left(\beta_1\left(\left(2-3X^4\right)SX'-\left(3X^4+2\right)XS'\right)-2S\left(3X^5+X\right)\beta_1'\right)\right)\ , \\
0 &= 2\ \alpha{\beta_1}^2{\beta_2}^2\ \sin 2\xi\ S\left(k_1k_2X^4\left(9\left(a^2+2\cos 2\xi\ SX\left(\sqrt{2}a+\cos 2\xi\ SX\right)\right)+q^2\right)+4\ a^2{\beta_1}^2{\beta_2}^2\right) \\
&\quad -12\sqrt{2}\ a^2\ {\beta_1}^4{\beta_2}^4\ X^2\left(XS'+SX'\right)-18\mathrm{i}\ p\ \sin^2 2\xi\ S^2X^8\left({k_1}^2{\beta_2}^4+{k_2}^2{\beta_1}^4\right) \\
&\quad +9\mathrm{i}\sin^2 2\xi S^2X^8q'\left({k_1}^2{\beta_2}^4+{k_2}^2{\beta_1}^4\right)\ . \qquad (10.2.7)
\end{aligned}
$$

This is a set of ten coupled ODEs for seven functions $X,\ S,\ \xi,\ \alpha,\ \beta_2,\ p,\ q$, and three constants $c_1,\ c_2$, and a. We can therefore proceed to try constructing solutions to this system of equations.

An exact solution to this system is given by

$$
\begin{aligned}
\alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}}\ , \qquad \beta_1(r) = \beta_2 = \frac{3\sqrt{6r^2-1}}{\sqrt{2}}\ , \qquad \xi(r) = \arccos\left(\frac{1}{\sqrt{6}\ r}\right)\ , \\
p(r) &= q(r) = 0\ , \qquad X(r) = 1\ , \qquad S(r) = 3\sqrt{3}\ r\ , \qquad (10.2.8)
\end{aligned}
$$

where β_1 is chosen due to reparametrization invariance and b_2 is a free parameter associated with the size of the second Riemann surface. In particular, we notice that

$$
\gamma(r) = \frac{3\sqrt{6r^2-1}}{\sqrt{2}} \qquad \text{and} \qquad f(r) = 0\ ,
$$

so that only the metric is non-trivial, and this can be realized as a hyperbolic cone with radial coordinate $r\in[\frac{1}{\sqrt{6}},\infty)$, with the conformal boundary at infinity $r=\infty$. One can now look at series solutions around this hyperbolic cone, which can be done in two ways, as series expansion at infinity, and as a perturbation of this exact solution, by lineazing the system of ODEs and solving them order by order in the perturbation parameter δ.

10.2.1 Expansion around the conformal boundary

Starting from a general series expansion around the conformal boundary at $r = \infty$ and imposing the equations above (10.2.7) we find

$$
\begin{aligned}
X(r) &= 1+\left(\frac{k_1}{72\ b_2^2}\left(\frac{2\ b_2^2\ k_0^{\ 4}\ k_1}{9\ a_0^{\ 2}}-b_2^{\ 2}\right)-\frac{13\ k_0^{\ 4}\ k_1^{\ 2}}{8748 a_0^{\ 2}}+k_1\left(-\frac{13\ k_0^{\ 4}\ k_1}{4374\ a_0^{\ 2}}-\frac{1}{72}\right)\right)\frac{1}{r^2}+\dots\ , \\
S(r) &= +2\ k_0^2\ r-\left(\frac{2\ k_0^{\ 6}\ k_1^{\ 2}}{243\ a_0^{\ 2}}+\frac{k_0^{\ 2}\ k_1}{18}+\frac{k_0^{\ 2}}{6}\right)\frac{1}{r}+\dots\ , \\
\xi(r) &= \frac{1}{2}\arccos\left(\frac{54\sqrt{2}\ a\ k_1\ k_0^{\ 2}\ r}{1458\ a^2\ r^2+k_1^{\ 2}\ k_0^{\ 4}}\right)+\dots , \\
\alpha(r) &= \frac{3}{\sqrt{2}}\frac{1}{r}+\left(\frac{\sqrt{2}\ k_0^{\ 4}\ k_1^{\ 2}}{243\ a_0^{\ 2}}+\frac{k_1}{12\sqrt{2}}+\frac{1}{2\sqrt{2}}\right)\frac{1}{r^3}+\dots\ , \\
\beta_2(r) &= b_2\ r-\frac{b_2}{12}\frac{1}{r}-\frac{b_2}{288}\frac{1}{r^3}+\dots\ , \\
p(r) &= -\frac{4\sqrt{2}\mathrm{i}\ k_0^{\ 6}\ k_1^{\ 2}}{243\ a_0^{\ 2}}\frac{1}{r^2}+\dots\ , \\
q(r) &= -2\sqrt{2}\mathrm{i}\ k_0^{\ 2}\ r+\left(\frac{8\sqrt{2}\mathrm{i}\ k_0^{\ 6}\ k_1^{\ 2}}{729\ a_0^{\ 2}}+\frac{\sqrt{2}\mathrm{i}\ k_0^{\ 2}\ k_1}{9}+\frac{\mathrm{i}k_0^{\ 2}}{3\sqrt{2}}\right)\frac{1}{r}+\dots\ ,
\end{aligned}
\qquad (10.2.9)
$$

and the constants are given by

$$
c_1 = -\frac{k_1}{a_0} \qquad c_2 = -\frac{b_2^{\ 2}\ k_1}{27\ a_0} \qquad a = a_0\ . \qquad (10.2.10)
$$

Notice that the solution has four free parameters a_0, k_0, k_1 and b_2.

As it was done earlier, one could now proceed into finding the expansion around the 0^{th} order solution, and plugging these two expansions into the expressions for the holographic free energy and for the Wilson loop in order to compute these BPS quantities. An analysis of the behaviour on the field theory side would also be required in order to check the duality between these two theories. This is still a work in progress.

10.3 Analytic 3/4 BPS solution

In this section we give some details of a new analytic supersymmetric solution to Euclidean six-dimensional Romans supergravity. This corresponds to the 3/4 BPS squashed sphere, constructed earlier as a perturbation expansion. As shown in chapter 4 an interesting family of solutions arises by considering the following $SU(3)\times U(1)$ symmetric ansatz for the supergravity fields

$$\begin{aligned}
\mathrm{d}s_6^2 &= \alpha^2(r)\mathrm{d}r^2+\gamma^2(r)(\mathrm{d}\tau+C)^2+\beta^2(r)\Big[\mathrm{d}\sigma^2+\frac{1}{4}\sin^2\sigma(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2)\\
&\quad+\frac{1}{4}\cos^2\sigma\sin^2\sigma(\mathrm{d}\beta+\cos\theta\mathrm{d}\varphi)^2\Big]\ ,\\
B &= p(r)\mathrm{d}r\wedge(\mathrm{d}\tau+C)+\frac{1}{2}q(r)\mathrm{d}C\ ,\\
A^i &= f^i(r)(\mathrm{d}\tau+C)\ ,
\end{aligned} \tag{10.3.1}$$

where

$$C \equiv -\frac{1}{2}\sin^2\sigma(\mathrm{d}\beta+\cos\theta\mathrm{d}\varphi)\ , \tag{10.3.2}$$

together with $X=X(r)$. The equations of motion for the background $SU(2)_R$ gauge field imply

$$f^i(r) = \kappa_i f(r)\ . \tag{10.3.3}$$

The equations for the other fields then depend only on the $SU(2)\sim SO(3)$ invariant $\kappa_1^2+\kappa_2^2+\kappa_3^2$, which we can set to one by rescaling $f(r)$. The set of equations for the fields involved in the ansatz have been listed in the appendix B. In addition, if the solution is supersymmetric there exists a Killing spinor. For the case of the 3/4 BPS solution the Killing spinor depends on four extra functions, denoted $k_i(r)$, $i=1,2,3,4$, which, together with the fields above, satisfy first order constraints as a result of supersymmetry. Although, as shown here, these constraints are equivalent to the original equations of motions (upon supplementing them with one extra second order equation), we found them more convenient in order to find an analytic form for the solution.

The solution depends on a single parameter s, the squashing parameter, but it is convenient to parametrize it in terms of $b_1=1+\sqrt{1-s^2}$ and $b_2=1-\sqrt{1-s^2}$, introduced in chapter 2. The high amount of supersymmetry implies a large number of constraints (many of them algebraic) which can be used to eliminate all the fields in favour of $k_2(r), k_3(r), X(r)$ and $\beta(r)$. For instance

$$\begin{aligned}
k_1(r) &= b_2(b_1+b_2)\frac{k_2(r)\beta(r)}{b_2k_2^2(r)+b_1k_3^2(r)}\ ,\\
k_4(r) &= b_1(b_1+b_2)\frac{k_3(r)\beta(r)}{b_2k_2^2(r)+b_1k_3^2(r)}\ ,\\
\gamma(r) &= (b_1+b_2)\frac{k_2(r)k_3(r)\beta(r)}{b_2k_2^2(r)+b_1k_3^2(r)}\ ,
\end{aligned} \tag{10.3.4}$$

while the expressions for the remaining fields are more complicated. As a consequence of reparametrization invariance we can demand that k_2, k_3 and X depend on r only through $\beta(r)$. It is then convenient to introduce a new variable ζ:

$$(b_1 + b_2)\sqrt{b_1 b_2}\,\beta(r) \;\equiv\; \zeta \; . \tag{10.3.5}$$

The remaining equations can be used to eliminate further fields and we end up with a single equation for $v(\zeta) \equiv \zeta^2 X^2(\zeta)$:

$$v'(\zeta) \;=\; 4\zeta^3 \frac{(b_1 + b_2)^2(b_1 + 2b_2)^2 + 2v(\zeta)}{\zeta^4 + 3(b_1 + b_2)^3(b_1 + 2b_2)v(\zeta) + 3v^2(\zeta)} \; , \tag{10.3.6}$$

which can be simply solved, for instance, with Mathematica. Equation (10.3.6) has two inequivalent solutions, each of them depending on a constant of integration. Of those only one has the correct boundary condition at infinity $v(\zeta) = \zeta^2 + \cdots$. The constant of integration can then be fixed by requiring regularity at the origin for $X(\zeta)$, which implies $v(0) = 0$. This fixes the solution uniquely. Although the explicit solution is too cumbersome to be written here, we give the expansion of $X(\zeta)$ for small and large values of ζ:

$$X(\zeta) \;=\; \left(\frac{2(b_1 + 2b_2)}{3(b_1 + b_2)}\right)^{1/4} + \cdots \; , \qquad \zeta \ll 1 \; , \tag{10.3.7}$$

$$\begin{aligned} X(\zeta) \;=\; & 1 - \frac{(b_1 - b_2)(b_1 + b_2)^2(b_1 + 2b_2)}{4}\frac{1}{\zeta^2} \\ & + \frac{(b_1 - b_2)(b_1 + b_2)^3(b_1 + 2b_2)^2}{2\sqrt{2}}\frac{1}{\zeta^3} + \cdots \; , \qquad \zeta \gg 1 \; . \end{aligned} \tag{10.3.8}$$

For instance, these expansions allow us to fix the parameter κ. We obtain

$$\kappa \;=\; \frac{2\sqrt{2}(3 - \sqrt{1 - s^2})^2(1 - s^2 + \sqrt{1 - s^2})}{27\sqrt{3}s^5} \; . \tag{10.3.9}$$

Finally, let us remark that although cumbersome, the solution contains only roots and rational functions.

Comments on the 1/4 and 1/2 BPS solutions

The 1/4 BPS squashed sphere solution considered previously is much harder to obtain, the reason being the smaller degree of supersymmetry. More precisely, the Killing spinor now depends on only two new functions $k_1(r)$ and $k_2(r)$, but the number of constraints is much smaller. A related issue is that now there are no natural "constants of motion" such as b_1 and b_2 to parametrize the solution with. Proceeding as before one can write two (third order and very cumbersome!) equations for two of the fields, for instance $X(\zeta)$ and $f(\zeta)$. After requiring regularity at the origin this should lead to a two-parameter family of solutions (s and f_0 introduced in (10.1.7)). These equations, however, are very complicated and we haven't managed to solve them exactly. Before proceeding, two comments are in order: first, these two equations can be solved in different limits, and reproduce the 1/4 BPS solution in the limits studied in chapter 4. Furthermore, in order to obtain these two equations it is necessary to supplement the bilinear equations with (8.4.4). Otherwise, we would obtain only one equation for two fields. This example shows that the differential constraints (8.2.8) do indeed need to be supplemented by equation (8.4.4).

We can also consider the special case $f(\zeta) = 0$. In this case the 1/4 BPS solution reduces to the 1/2 BPS solution studied in chapter 4. Although not covered by our analysis in this paper because the bilinear $\epsilon^{\mathrm{T}}\epsilon \neq 0$, the 1/2 BPS solution is a limit of the 1/4 BPS solution, where one of the two parameters, namely f_0, vanishes. The final equation for $X(\zeta)$, with $\beta(r) \equiv \zeta$ is still rather involved, but it can be solved analytically in an interesting limit. Denoting $X(0) = x_0$ one can explicitly check the solution takes the following form

$$v(\zeta) = v_0(x_0\zeta) + \frac{1}{x_0^4} v_1(x_0\zeta) + \cdots \,, \tag{10.3.10}$$

where recall $v(\zeta) \equiv \zeta^2 X^2(\zeta)$ and $v_0(y)$ satisfies a simple equation

$$v_0''(y) = 3\frac{v_0'(y)}{y} - \frac{(6 + v_0(y))v_0'(y)^2}{6v_0(y)} \,, \tag{10.3.11}$$

whose solution with correct boundary conditions is

$$v_0(y) = 1 + \mathcal{W}\left(\frac{y^4 - 72\,\mathrm{e}}{72\,\mathrm{e}}\right) \,. \tag{10.3.12}$$

Here $\mathcal{W}(z)$ is the Lambert W function or product logarithm, namely $\mathcal{W}(z)\mathrm{e}^{\mathcal{W}(z)} = z$. Hence, as opposed to the 3/4 BPS solution, this solution contains special functions.

Chapter 11

Classifying solutions with a zero B-field

Briefly, the idea is to take Euclidean Romans supergravity theory and set the two-form potential B to be zero. This will lead us to a simpler set of equations of motion, dilatino and Killing spinor equations.

We analysed earlier possible solutions to Romans supergravity, and a particular case where $B = 0$ was later discussed in [55] by other authors. One can expect, however, to find more possible solutions to this case. By following the procedure adopted in [56], here we classify these families of solutions.

11.1 Taking $B = 0$

The idea here is to take the two-form potential $B = 0$ in (8.2.8) and (8.4.4) and analyse these new equations.

Notice that for $B=0$ the set of differential constraints become

$$
\begin{aligned}
X^2 S^2 \sin^2 2\vartheta \,\mathrm{d}\sigma &= -\tfrac{2\sqrt{2}}{3} X^{-1} S \cos 2\vartheta \, J + \sqrt{2} X S \cos 2\vartheta \, \mathcal{F}_\perp \ , && (11.1.1)\\
\mathrm{d}(X^{-1} S \cos 2\vartheta \, J) &= -\tfrac{3}{2\sqrt{2}} \mathrm{d}[(XS)^2 \mathrm{d}\sigma] \ , && (11.1.2)\\
\mathrm{d}(X^{-1} S J) &= -\sqrt{2} S \sin 2\vartheta \, J \wedge K_2 - \tfrac{3}{2\sqrt{2}} \cos 2\vartheta \, \mathrm{d}[(XS)^2 \mathrm{d}\sigma] \\
&\quad - \tfrac{1}{\sqrt{2}} X^{-2} S \sin 2\vartheta \, [K_1 \lrcorner * \mathcal{F}_\perp - K_2 \wedge \mathcal{F}_\perp] \ , && (11.1.3)\\
\mathrm{d}(S \sin 2\vartheta \, J \wedge K_2) &= 0 \ , && (11.1.4)\\
D_\perp(X^{-1} S \sin 2\vartheta \, \Omega) &= -\sqrt{2} S \Omega \wedge K_2 \ , && (11.1.5)\\
S^2 J \wedge \mathrm{d}\sigma &= -\sqrt{2} S \cos 2\vartheta (X + \tfrac{2}{3} X^{-3}) \tfrac{1}{2} J \wedge J + 2 S K_1 \lrcorner * \mathrm{d}\vartheta \\
&\quad + \tfrac{1}{\sqrt{2}} X^{-1} S J \wedge \cos 2\vartheta \, \mathrm{d}\mathcal{A}_\perp \ , && (11.1.6)\\
S^2 \Omega \wedge \mathrm{d}\sigma &= -2\mathrm{i} S \mathrm{d}\vartheta \wedge K_2 \wedge \Omega + \tfrac{1}{\sqrt{2}} X^{-1} S \Omega \wedge \cos 2\vartheta \, \mathrm{d}\mathcal{A}_\perp \ , && (11.1.7)\\
0 &= X^4 K_2 \lrcorner \mathrm{d}(X^{-3} S \sin 2\vartheta) + \sqrt{2} S (X^2 - \tfrac{2}{3} X^{-2}) \\
&\quad + \tfrac{1}{\sqrt{2}} S J \lrcorner \mathcal{F}_\perp \ , && (11.1.8)
\end{aligned}
$$

and

$$
0 = \mathrm{d}\left[\frac{X^4}{S \sin 2\vartheta} K_1 \lrcorner * \mathrm{d}(X^{-2} S \sin 2\vartheta \, K_2) \right] + \tfrac{1}{2} \mathcal{F}_\perp \wedge \mathcal{F}_\perp \ . \qquad (11.1.9)
$$

These equations are necessary and sufficient to have a supersymmetric solution to the Euclidean equations of motion of Romans theory. It is also worth recalling that

$$
K_1 = S \sin 2\vartheta (\mathrm{d}\psi + \sigma) \ ,
$$

where ∂_ψ is the supersymmetric Killing vector that preserves all the structure, and

$$
\mathcal{F}_\perp = -\sqrt{2} X S \cos 2\vartheta \mathrm{d}\sigma + \mathrm{d}\mathcal{A}_\perp \ . \qquad (11.1.10)
$$

In order to analyse these equations, we will need to break the derivatives (and the remaining of each equation) into components. It is important to make sure we understand what the terms are "made of". We start by defining a radial coordinate in the K_2 direction, this will given by

$$
\rho = XS \ , \quad \text{such that} \quad \mathrm{d}\rho = \mathrm{d}(XS) \ . \qquad (11.1.11)
$$

The exterior derivative can be written as

$$\mathrm{d} = \mathrm{d}\psi\partial_\psi \ + \ \mathrm{d}\rho\partial_\rho \ + \ \mathrm{d}_4 \ , \tag{11.1.12}$$

i.e., in the K_1, K_2 and M_4 directions.

Another term that requires attention is the one-form σ. For instance, one may consider

$$\sigma = \sigma_4 \ + \ \sigma_\rho \mathrm{d}\rho \ , \tag{11.1.13}$$

where σ_4 is the σ component in the direction that is both perpendicular to K_1 and K_2. Its derivative is then given by

$$\mathrm{d}\sigma = \mathrm{d}_4\sigma_4 \ + \ \mathrm{d}\rho \wedge (\partial_\rho \sigma_4) \ + \ (\mathrm{d}_4\sigma_\rho) \wedge \mathrm{d}\rho. \tag{11.1.14}$$

Notice however that, if we reparametrise ψ (that enters in the definition of the one-form K_1), one can make a gauge choice of shifting it in such a way that

$$\psi \longrightarrow \psi - \int_\rho \sigma_\rho(y, x^i)\mathrm{d}y \ , \tag{11.1.15}$$

which leads to

$$\mathrm{d}\psi \longrightarrow \mathrm{d}\psi - \sigma_\rho \mathrm{d}\rho - \mathrm{d}_4 \int_\rho \sigma_\rho(y, x^i)\mathrm{d}y \ . \tag{11.1.16}$$

This way, in writing down $\mathrm{d}\psi + \sigma$, we have $\sigma_\rho \mathrm{d}\rho$ being cancelled, and it can simply write

$$\mathrm{d}\psi + \sigma \longrightarrow \mathrm{d}\psi + \sigma_4 - \mathrm{d}_4 \int_\rho \sigma_\rho(y, x^i)\mathrm{d}y \ , \tag{11.1.17}$$

and make

$$\sigma_4 - \mathrm{d}_4 \int_\rho \sigma_\rho(y, x^i)\mathrm{d}y \longrightarrow \sigma_4 \ , \tag{11.1.18}$$

as both terms are in the d_4 direction. This way, we are free to make a choice where $\sigma \equiv \sigma_4$. Notice however that we still have to consider $\mathrm{d}\sigma = \mathrm{d}_4\sigma_4 + \mathrm{d}\rho \wedge (\partial_\rho \sigma_4)$.

11.2 Conditions for a supersymmetric M_6

Notice that once we take the B field to be zero, we immediately get $B_1 = 0$, this gives us

$$\frac{3}{\sqrt{2}S \sin 2\vartheta}\mathrm{d}(XS) + X^{-2}K_2 = 0 \ . \tag{11.2.1}$$

As we defined, $\mathrm{d}(XS) = \mathrm{d}\rho$ (and now confirmed that K_2 is in fact in the $\mathrm{d}\rho$ direction). In fact

$$K_2 = -\frac{3X^2}{\sqrt{2}S\sin 2\vartheta}\mathrm{d}\rho \ . \tag{11.2.2}$$

Notice from (11.1.9), that the first term is zero, and it simply reduces to

$$\mathcal{F}_\perp \wedge \mathcal{F}_\perp = 0 \ . \tag{11.2.3}$$

Let us reparametrize J and Ω by introducing

$$\hat{J} = X^2 \ J \ , \quad \text{and} \quad \hat{\Omega} = X^2 \ \Omega \ . \tag{11.2.4}$$

Equation (11.1.1) may be used to eliminate the flux $\mathcal{F}_\perp$ in terms of the $SU(2)$ structure (one should consider $\cos 2\vartheta \equiv 0$ as a separate case)

$$\mathcal{F}_\perp = \frac{2}{3X^2}J + \frac{\rho\sin^2 2\vartheta}{\sqrt{2}\cos 2\vartheta}\mathrm{d}\sigma \ . \tag{11.2.5}$$

From equation (11.1.4), one reads

$$\mathrm{d}(\hat{J}\wedge \mathrm{d}\rho) = 0 \ . \tag{11.2.6}$$

which reads

$$\mathrm{d}_4\hat{J} = 0 \ , \tag{11.2.7}$$

$$\partial_\psi \hat{J} = 0 \ , \tag{11.2.8}$$

Next, equation (11.1.2) reads

$$\mathrm{d}\left(\frac{\rho\cos 2\vartheta}{X^4}\hat{J}\right) = -\frac{3}{\sqrt{2}}\rho \ \mathrm{d}\rho\wedge \mathrm{d}\sigma \ . \tag{11.2.9}$$

This is equivalent to

$$\mathrm{d}_4\left(\frac{\cos 2\vartheta}{X^4}\right) = 0 \ , \tag{11.2.10}$$

so that we can say $\cos 2\vartheta = X^4\lambda(\rho)$. We then get an equivalent to equation (2.36) in [56], namely

$$\partial_\rho(\rho\lambda(\rho)\hat{J}) = -\frac{3}{\sqrt{2}}\rho \mathrm{d}_4\sigma \ . \tag{11.2.11}$$

Notice that from this equation also follows that $\mathrm{d}_4\sigma$ has no components proportional to Ω (but it still could have an anti-self-dual part).

Similarly, equation (11.1.5) reads

$$\mathrm{d}_4\left(\frac{\sin 2\vartheta}{X^4}\hat{\Omega}\right) = -\mathrm{i}\mathcal{A}_\perp \wedge \frac{\sin 2\vartheta}{X^4}\hat{\Omega}\ , \tag{11.2.12}$$

and

$$\partial_\rho\left(\frac{\rho\sin 2\vartheta}{X^4}\hat{\Omega}\right) = \frac{3}{\sin 2\vartheta}\hat{\Omega}\ . \tag{11.2.13}$$

Here again we have used gauge freedom to remove the part of $\mathcal{A}_\perp$ proportional to $\mathrm{d}\rho$, so that $\mathcal{A}_\perp = \mathcal{A}_4$. These equations imply that the geometry at constant ρ (and ψ) is (conformally) Kähler, with Kähler metric $\hat{g}_4$ associated to $\hat{J}$ and $\hat{\Omega}$. Moreover, since the derivative of $\hat{\Omega}$ in the ρ direction is proportional to $\hat{\Omega}$, this shows that the associated complex structure $\hat{I}$ is independent of ρ, $\partial_\rho\hat{I} = 0$. Since

$$\mathrm{d}_4\hat{\Omega} = \mathrm{i}\hat{P}\wedge\hat{\Omega}, \tag{11.2.14}$$

where $\hat{P}$ is the canonical Ricci one-form potential, we identify

$$\mathcal{A}_\perp = -\hat{P} - \hat{I}\cdot\mathrm{d}_4\log\left(\frac{\sin 2\vartheta}{X^4}\right)\ . \tag{11.2.15}$$

Note that this can be rewritten as

$$\mathcal{A}_\perp = -\hat{P} - \hat{I}\cdot\mathrm{d}_4\log\tan 2\vartheta\ . \tag{11.2.16}$$

Next, we turn to equation (11.1.6). Multiplying by X^2 and substituting from (11.1.10), this reads

$$X^2S^2\sin^2 2\vartheta\Omega\wedge\mathrm{d}\sigma = -2\mathrm{i}S\mathrm{d}\vartheta\wedge K_2\wedge\Omega + \frac{1}{\sqrt{2}}XS\Omega\wedge\cos 2\vartheta\mathcal{F}_\perp\ . \tag{11.2.17}$$

Subtracting $\frac{1}{2}\Omega$(11.1.1), then gives

$$\frac{1}{2}\rho^2\sin^2 2\vartheta\omega\wedge\mathrm{d}\sigma = -2\mathrm{i}S\mathrm{d}\vartheta\wedge K_2\wedge\Omega\ . \tag{11.2.18}$$

And therefore

$$\Omega\wedge\mathrm{d}_4\sigma = 0\ , \tag{11.2.19}$$

and

$$\Omega \wedge \left(\partial_\rho \sigma + \frac{6\sqrt{2}X^4}{\rho^2 \sin^3 2\vartheta} \mathrm{i}\ \mathrm{d}_4\vartheta \right) = 0 \ . \tag{11.2.20}$$

This implies that the one-form in brackets is a $(1,0)-$form, and hence

$$\partial_\rho \sigma = -\frac{6\sqrt{2}X^4}{\rho^2 \sin^3 2\vartheta} \hat{I} \dot{\mathrm{d}}_4 \vartheta \ . \tag{11.2.21}$$

Next we turn to equation (11.1.6). One finds that the component of this equation in the $\mathrm{d}\rho$ direction is precisely equivalent to equation (11.2.21). The remainder of equation (11.1.6) is equivalent to

$$\frac{1}{2} S^2 \sin^2 2\vartheta J \lrcorner \, \mathrm{d}_4 \sigma = -\sqrt{2}\rho \cos 2\vartheta + \frac{2\sqrt{2}\rho^2 \sin 2\vartheta}{3X^4} \partial_\rho \vartheta \ . \tag{11.2.22}$$

Finally we turn to the scalar equation (11.1.8). After a computation, one remarkably finds precisely equation (11.2.22) plus a (generically) non-zero function times $\partial_\rho(\rho\lambda(\rho))$. One concludes that

$$\partial_\rho(\rho\lambda(\rho)) = 0 \longrightarrow \lambda(\rho) = \frac{c}{\rho} \ , \tag{11.2.23}$$

where c is an integration constant. We have thus solved for one of the functions in the problem! One might be suspicious of this, given the cancellations that occur in the above calculation, but $\lambda(\rho)$ has nothing to cancel against to give an identity, and one can check that equation (11.2.23) is indeed true for the four-parameter family of BPS black hole solutions discussed in [55]. This is a highly non-trivial check. Notice that now one can write

$$\rho = cX^4 \sec 2\vartheta \ . \tag{11.2.24}$$

We thus really have only one free function in the problem, and we can take it to be X.

We have now analysed all the content of all the equations, apart from equations (11.1.3) and (11.1.9). After quite a lengthy calculation, and using many of the equations above, one can show that the $\mathrm{d}\rho$ component of equation (11.1.3) is precisely equivalent to (11.2.22). Notice that the anti-self-dual part of $\mathcal{F}_4$ enters, which is related to $(\mathrm{d}_4\sigma)^-$ via (11.2.5), but this combines with $\mathrm{d}_4\sigma$, and in the end only the self-dual part of the it enters, and it is proportional to J, as $\Omega \wedge \mathrm{d}_4\sigma = 0$.

The remainder of equation (11.1.3) is easier to compute, and one finds an equivalent to (11.2.21). Thus (11.1.3) is implied by all the other equations, and hence imposes nothing new.

It thus remains only to impose the equation of motion (11.1.9). The two components read

$$\partial_\rho \sigma \wedge \mathcal{F}_4 = 0 \ . \tag{11.2.25}$$

where from (11.2.5), we have

$$\mathcal{F}_4 = \frac{2}{3X^2} J + \frac{\rho \sin^2 2\vartheta}{\sqrt{2}\cos 2\vartheta} \mathrm{d}_4 \sigma \ , \tag{11.2.26}$$

together with the scalar equation

$$\|(\mathrm{d}_4\sigma)^-\| = \frac{2\cos 2\vartheta}{\rho \sin^2 2\vartheta} \left[\frac{2\rho \sin 2\vartheta}{3X^2 \cos 2\vartheta} \partial_\rho \vartheta + \frac{1}{X^2} \left(\frac{2}{3} - X^4 \right) \right] \ . \tag{11.2.27}$$

We conclude by noting that a few equations are redundant. First (11.2.21) is precisely the $\mathrm{d}\rho$ component of d(11.2.16). Here we have the second equation

$$\mathrm{d}\mathcal{A}_\perp = \frac{2}{3X^4} \hat{J} + \frac{\rho}{\sqrt{2}} (\cos 2\vartheta + \sec 2\vartheta) \mathrm{d}\sigma \ , \tag{11.2.28}$$

which may be combined with (11.2.25) to obtain Einstein-like equation (involving the Ricci form of the Kähler metric). To see this, recall that $\hat{\mathcal{P}} = \frac{1}{2} \hat{I} \mathrm{d}_2 \log \sqrt{\det \hat{g}}$. But since also $\hat{\Omega} \wedge \hat{\bar{\Omega}} = 4 \hat{\mathrm{vol}} = 2 \hat{J} \wedge \hat{J}$ is automatically true, it follows by taking ∂_ρ that

$$\hat{J} \lrcorner \partial_\rho \hat{J} = \hat{\bar{\Omega}} \lrcorner \partial_\rho \hat{\Omega} \tag{11.2.29}$$

is an identity. Using this, one can check that equations (11.2.11) and (11.2.13) in fact imply equation (11.2.22). The latter is hence implied by the other equations, and it is redundant.

Summary

We can now collect together the necessary and sufficient conditions to have a supersymmetric solution. The metric is given

$$\mathrm{d}s_6^2 = K_1^2 + K_2^2 + g_{SU(2)} \ , \tag{11.2.30}$$

where now $g_{SU(2)}$ is a Kähler manifold. We can rewrite it as

$$\mathrm{d}s_6^2 = X^{-2} \left(\rho^2 \sin^2 2\vartheta (\mathrm{d}\psi + \sigma)^2 + \frac{9X^8}{2\rho^2 \sin^2 2\vartheta} \mathrm{d}\rho^2 + \hat{g_{ij}} \mathrm{d}x^i \mathrm{d}x^j \right) \ , \tag{11.2.31}$$

where $\hat{g}$ is a one-parameter family of Kähler metrics depending only on ρ, for which the complex strucuture $\hat{I}$ is independent of ρ. The vector ∂_ψ is Killing, and preserves all of the structure. The functions X and ϑ are related by

$$X^4 = \frac{\rho}{c}\cos 2\vartheta \ , \tag{11.2.32}$$

where c is a non-zero constant, so that we may substitute

$$\sin^2 2\vartheta = 1 - \frac{c^2X^8}{\rho^2} \ . \tag{11.2.33}$$

The evolution equations for the Kähler structure are

$$\partial_\rho \hat{J} = -\frac{3}{\sqrt{2}c}\rho \mathrm{d}_4\sigma \tag{11.2.34}$$

and

$$\partial_\rho\left(\tan 2\vartheta\hat{\Omega}\right) = \frac{3}{c\sin 2\vartheta}\hat{\Omega} \ . \tag{11.2.35}$$

Notice that $\Omega\wedge \mathrm{d}_4\sigma = 0$ is consistent with equation (11.2.34) and the fact the $\hat{J}$ must remain type $(1,1)$ as the complex structure is independent of ρ.

From (11.2.16), we have the Einstein-like equation

$$\hat{\mathcal{R}} \equiv \mathrm{d}_4\hat{\mathcal{P}} = -\frac{2}{3X^4}\hat{J} - \frac{\rho}{\sqrt{2}}(\cos 2\vartheta + \sec 2\vartheta)\mathrm{d}_4\sigma - \mathrm{d}_4\cdot\hat{I}\cdot\mathrm{d}_4\log\tan 2\vartheta \ , \tag{11.2.36}$$

together with

$$\partial_\rho\sigma = -\frac{6\sqrt{2}X^4}{\rho^2\sin^3 2\vartheta}\hat{I}\dot{\mathrm{d}}_4\vartheta \ . \tag{11.2.37}$$

Finally, we must impose the B-field equation of motion components

$$\partial_\rho\sigma\wedge\left[\frac{2}{3X^4}\hat{J} + \frac{\rho\sin^2 2\vartheta}{\sqrt{2}\cos 2\vartheta}\mathrm{d}_4\sigma\right] = 0 \ , \tag{11.2.38}$$

and

$$\|(\mathrm{d}_4\sigma)^-\| = \frac{2\cos 2\vartheta}{\rho\sin^2 2\vartheta}\left[\frac{2\rho\sin 2\vartheta}{3X^2\cos 2\vartheta}\partial_\rho\vartheta + \frac{1}{X^2}\left(\frac{2}{3} - X^4\right)\right] \ . \tag{11.2.39}$$

The norm here is with respect to g_4 (rather than $\hat{g}_4$). The supersymmetry equations above are almost exactly the same (essentially up to numerical factors) to the equations in [56].

11.3 Complex M_6 (Setting $\mathrm{d}_4\vartheta = 0$)

In [56], remarkably, the equations of this form were solved in closed form, with the additional assumption of $\mathrm{d}_4\vartheta = 0$, leading to new solutions. It is then natural, due to the similarity of the system, to make the same assumption here.

In order to have a six-dimensional complex manifold with Hermitian metric, we require the three-form given by $\Omega_{(3)} = \Omega \wedge (K_1 + \mathrm{i}K_2)$ to have a derivative given by

$$\mathrm{d}\Omega_{(3)} = A \wedge \Omega_{(3)} + v \wedge \Omega \wedge (K_1 - \mathrm{i}K_2), \tag{11.3.1}$$

with $v = 0$. This restriction will imply that $\mathrm{d}\sigma \equiv \mathrm{d}_4\sigma$, and implies that $\mathrm{d}_4X = \mathrm{d}_4\vartheta = \mathrm{d}_4S = 0$. From this one can deduce that

$$\hat{P} = \mathcal{A}_4. \tag{11.3.2}$$

Next, we may look at (11.2.36), which reads

$$\hat{\mathcal{R}} = -\frac{2}{3X^4}\hat{J} - \frac{\rho}{\sqrt{2}}(\cos 2\vartheta + \sec 2\vartheta)\mathrm{d}_4\sigma \ . \tag{11.3.3}$$

The Ricci scalar of the Kähler metric $\hat{g}_4$ is $\hat{R} = \hat{J}^{ij}\hat{\mathcal{R}}_{ij}$, so that using (11.2.22), we compute

$$\hat{R} = -\frac{4}{3X^4} - \rho(\cos 2\vartheta + \sec 2\vartheta)\left(\frac{2\cos 2\vartheta}{\rho \sin^2 2\vartheta} - \frac{4}{3X^4 \sin 2\vartheta}\partial_\rho\vartheta\right) \ . \tag{11.3.4}$$

Since the right hand side is a function only of ρ, we deduce that $\mathrm{d}_4\hat{R} = 0$, and $\hat{g}_4$ is a constant scalar curvature Kähler metric (for fixed ρ).

We may similarly compute $\hat{R}_{ij}\hat{R}^{ij} = \hat{\mathcal{R}}_{ij}\hat{\mathcal{R}}^{ij}$ from equation (11.3.3). Using again (11.2.22) to compute $\hat{J} \lrcorner \mathrm{d}_4\sigma$, and (11.2.39) to compute $\|(\mathrm{d}_4\sigma)^-\|^2$, the right hand side is again a function only of ρ, and we deduce that

$$\mathrm{d}_4(\hat{R}_{ij}\hat{R}^{ij}) = 0. \tag{11.3.5}$$

It follows that at fixed ρ, the Ricci tensor $\hat{R}_{ij}$ has two pairs of constant eigenvalues. It these eigenvalues are the same, this is a Kähelr-Einstein metric, while if they are distinct and M_4 is compact, then the Goldberg conjecture implies that M_4 is locally a product of two Riemann surfaces of (distinct) constant curvature.

We shall consider both cases separately.

11.3.1 Kähler-Einstein base solutions

For a Kähler -Einstein metric $\hat{\mathcal{R}} \propto \hat{J}$, with the constant of proportionality depending only on ρ. Thus $\mathrm{d}_4\sigma$ is also proportional to $\hat{J}$. One checks that there are no solutions with $\mathrm{d}_4\sigma = 0$, so, without loss of generality, we set

$$\mathrm{d}_4\sigma = \tilde{J} \ , \quad \hat{J} = F(\rho)\tilde{J} \ , \tag{11.3.6}$$

where $\partial_\rho \tilde{J} = 0$. Thus the rescaled Kähler metric $\tilde{J}$ is independent of ρ, and the Kähler-Einstein conditions reads

$$\hat{\mathcal{R}} = \tilde{\mathcal{R}} = \kappa \tilde{J} \ , \tag{11.3.7}$$

where $\kappa \in \mathbb{R}$ is a constant. Solving first equation (11.2.34), we find

$$F(\rho) = a - \frac{3\rho^2}{2\sqrt{2}c} \ , \tag{11.3.8}$$

where a in an integration constant. Substitutuing this into (11.3.3), and X in (11.2.32), one can find ϑ, given by

$$\cos(2\vartheta) = \left(-1 + \sqrt{1 - \frac{4\sqrt{2}ac}{3\kappa^2}}\right)\frac{\kappa}{\sqrt{2}\rho} \ . \tag{11.3.9}$$

Notice that at this point, all the functions have been completely determined. Next, solving (11.2.35), we can write a and c in terms of κ (where $\hat{\Omega} = F(\rho)\tilde{\Omega}$)

$$a = -\frac{3\kappa}{4} \ , \quad c = -\frac{\kappa}{\sqrt{2}} \ . \tag{11.3.10}$$

It follows that $X \equiv 1$. Finally, notice that $(\mathrm{d}_4\sigma)^- = 0$, and one can check that the right hand side of the equation (11.2.39) is in fact zero. At this point we have solved all the equations.

The final solution is therefore given by

$$F\rho = \frac{3\rho^2}{2\kappa} - \frac{3\kappa}{4} \ , \quad \cos(2\vartheta) = -\frac{\kappa}{\sqrt{2}\rho} \ , \quad X \equiv 1 \ . \tag{11.3.11}$$

The six-dimensional metric is

$$\mathrm{d}s_6^2 = \frac{9}{2(\rho^2 - \frac{\kappa^2}{2})}\mathrm{d}\rho^2 + \left(\rho^2 - \frac{\kappa^2}{2}\right)(\mathrm{d}\psi + \sigma)^2 + \frac{3}{2\kappa}\left(\rho^2 - \frac{\kappa^2}{2}\right)\tilde{g}_4 \ , \tag{11.3.12}$$

where $\mathrm{d}\sigma = \tilde{J}$, and $\tilde{g}_4$ is a constant (in ρ) Kähler-Einstein metric with $\tilde{\mathcal{R}} = \kappa\tilde{J}$. The gauge field $\mathcal{A}$ has $\mathrm{d}\mathcal{A} = -\mathrm{d}\tilde{\mathcal{R}}$, so that $\mathcal{A}$ is a connection on the canonical bundle of M_4.

The ρ coordinate in the metric (11.3.12) is somewhat peculiar. A better system of coordinates is set by making the change

$$r^2 \equiv \rho^2 - \frac{\kappa^2}{2} \ . \tag{11.3.13}$$

The metric then becomes

$$\mathrm{d}s_6^2 \simeq \frac{9}{\kappa^2 + 2r^2}\mathrm{d}r^2 + r^2\left[(\mathrm{d}\psi + \sigma)^2 + \frac{3}{2\kappa}\hat{g}_4\right] \ . \tag{11.3.14}$$

This is simply the hyperbolic cone over a regular Sasaki-Einstein manifold. Notice that the full gauge field $\mathcal{F} = 0$. Thus, this solution was known.

11.3.2 Product of two Riemann surfaces base solutions

Analogous to the Kähler-Einstein solutions, we can consider the metric $\hat{\mathcal{R}} \propto \hat{J}$, where

$$\hat{J} = F_1(\rho)\tilde{J}_1 + F_2(\rho)\tilde{J}_2 \ , \tag{11.3.15}$$

with $F_1(\rho)$ and $F_2(\rho)$ depending only on ρ. Then we also have $\mathrm{d}_4\sigma$ given by

$$\mathrm{d}_4\sigma = c_1\tilde{J}_1 + c_2\tilde{J}_2 \ , \tag{11.3.16}$$

with the factors c_1 and c_2 being constants. Here the two-forms $\tilde{J}_1$ and $\tilde{J}_2$ are such that $\partial_\rho\tilde{J}_1 = \partial_\rho\tilde{J}_2 = 0$.

Again the rescaled Kähler metric is independent of ρ and we can write

$$\hat{\mathcal{R}} = k_1\tilde{J}_1 + k_2\tilde{J}_2 \ , \tag{11.3.17}$$

where $k_i \in \mathbb{R}$ are constants.

Solving equation (11.2.34), we find

$$F_1(\rho) = -\frac{3\rho^2\ c_1}{2\sqrt{2}c} - a \quad \text{and} \quad F_2(\rho) = -\frac{3\rho^2\ c_2}{2\sqrt{2}c} - b \ , \tag{11.3.18}$$

where a and b are integration constants. We can now substitute this into (11.3.3), with X given by (11.2.32). Notice that we find two equations for ϑ. We find

$$\cos 2\vartheta = \frac{4\ a\ c}{-3k_1\ \rho + \sqrt{3\rho^2(3k_1^2 - 4\sqrt{2}a\ c\ c_1)}}, \tag{11.3.19}$$

and a constraint for b, given by

$$b = 6\left(\frac{2\sqrt{2}a^2\ c\ c_2\ \rho^2}{\left(3k_1\ \rho - \sqrt{3\rho^2(3k_1^2 - 4\sqrt{2}a\ c\ c_1)}\right)^2} + \frac{a\ k_2\ \rho}{3k_1\ \rho - \sqrt{3\rho^2(3k_1^2 - 4\sqrt{2}a\ c\ c_1)}}\right). \tag{11.3.20}$$

Now solving equation (11.2.35) order by order, one also finds constraints to c, a and c_2, namely

$$c = -\frac{k_1}{\sqrt{2}c_1}\ , \qquad a = -\frac{3k_1}{4}\ , \qquad c_2 = c_1\frac{k_2}{k_1}\ . \tag{11.3.21}$$

We then check equation (11.2.39) and confirm that it holds.

The final solution is a function of the parameters left, i.e., k_1, k_2 and c_1, and it is given by:

$$F_1(\rho) = \frac{3\ c_1^2\ \rho^2}{2\ k_1} - \frac{3\ k_1}{4} \quad \text{and} \quad F_2(\rho) = \frac{3\ c_1\ c_2\ k_2\ \rho^2}{2\ k_1} - \frac{3\ k_1\ c_2}{4\ c_1}\ , \tag{11.3.22}$$

and our starting functions

$$X = 1\ , \quad \text{and} \quad \cos 2\vartheta = -\frac{k_1}{\sqrt{2}\ c_1\ \rho}\ . \tag{11.3.23}$$

The six-dimensional metric is

$$\begin{aligned} \mathrm{d}s_6^2 =\ & \rho^2\left(1 - \frac{k_1^2}{2\ c_1\ \rho}\right)(\mathrm{d}\psi + \sigma)^2 + \frac{9}{2\ \rho^2}\left(1 - \frac{k_1^2}{2\ c_1\ \rho}\right)^{-1}\mathrm{d}\rho^2 \\ & + \left(\frac{3\ c_1^2\ \rho^2}{2\ k_1} - \frac{3\ k_1}{4}\right)\mathrm{d}\tilde{s}^2(C_{k_1}) + \left(\frac{3\ c_1\ c_2\ k_2\ \rho^2}{2\ k_1} - \frac{3\ k_1\ c_2}{4\ c_1}\right)\mathrm{d}\tilde{s}^2(C_{k_2})\ , \end{aligned} \tag{11.3.24}$$

where $\mathrm{d}\tilde{s}^2(C_{k_i})$ are the metrics on a torus ($C_0 \equiv T^2$), a sphere ($C_1 \equiv S^2$) and a hyperbolic space ($C_{-1} \equiv H^2$).

One can reparametrise the metric making $\mathrm{d}s_6^2 \longrightarrow k_1\mathrm{d}s_6^2$, so that we get

$$\begin{aligned} \mathrm{d}s_6^2 =\ & \rho^2 k_1\left(1 - \frac{k_1^2}{2\ c_1\ \rho}\right)(\mathrm{d}\psi + \sigma)^2 + \frac{9k_1}{2\ \rho^2}\left(1 - \frac{k_1^2}{2\ c_1\ \rho}\right)^{-1}\mathrm{d}\rho^2 \\ & + \left(\frac{3\ c_1^2\ \rho^2}{2} - \frac{3\ k_1^2}{4}\right)\mathrm{d}\tilde{s}^2(C_{k_1}) + \left(\frac{3\ c_1\ c_2\ \rho^2}{2} - \frac{3\ k_1^2\ c_2}{4\ c_1}\right)\mathrm{d}\tilde{s}^2(C_{k_2})\ . \end{aligned} \tag{11.3.25}$$

Notice that we can then rewrite the metric as

$$\begin{aligned}\mathrm{d}s_6^2 = \; & \rho^2 k_1 \left(1 - \frac{k_1^2}{2\ c_1\ \rho}\right)(\mathrm{d}\psi + \sigma)^2 + \frac{9k_1}{2\ \rho^2}\left(1 - \frac{k_1^2}{2\ c_1\ \rho}\right)^{-1}\mathrm{d}\rho^2 \\ & + \left(\frac{3\ c_1^2\ \rho^2}{2} - \frac{3\ k_1^2}{4}\right)\mathrm{d}\tilde{s}^2(C_{k_1}) + \left(\frac{3\ c_1^2\rho^2}{2} - \frac{3\ k_1^2}{4}\right)\frac{k_2}{k_1}\mathrm{d}\tilde{s}^2(C_{k_2})\ . \end{aligned} \tag{11.3.26}$$

Notice that the factor $\frac{k_1}{k_2}$ changes the curvature of C_{k_2} to C_{k_1}, what we see therefore is a reduction back to the case where the base is simply a Kähler-Einstein manifold. We conclude that a Kähler-Einstein manifold is the most general solution to Romans supergravity with zero B field (and $\mathrm{d}_4\vartheta = 0$), completely classifying solutions of this type.

Chapter 12

Discussion and Conclusions

In this thesis we have constructed supergravity duals to the $USp(2N)$ superconformal gauge theories on various five-dimensional squashed backgrounds. In the case of the $SU(3) \times U(1)$ squashed five-spheres, these constitute a one-parameter family of 3/4 BPS solutions, and a two-parameter family of generically 1/4 BPS. The latter include new supersymmetric squashed five-sphere geometries with the background $SU(2)_R$ gauge field turned off, and moreover these have enhanced 1/2 BPS supersymmetry. By holographically renormalizing the Euclidean Romans supergravity theory, we have computed the holographic free energy for our solutions. We then compared this to the large N limit of the partition function of the gauge theories, and found perfect agreement. Given a supersymmetric supergravity solution one can construct the Killing vector $K^\mu = \varepsilon^{IJ}\epsilon_I^T \mathcal{C}\gamma^\mu \epsilon_J$, where ϵ_I, $I = 1, 2$, is the $SU(2)_R$ doublet of Killing spinors. For our solutions the free energy takes the form

$$\mathcal{F} = \frac{(|b_1| + |b_2| + |b_3|)^3}{27|b_1 b_2 b_3|}\mathcal{F}_{\mathrm{AdS}_6} , \tag{12.0.1}$$

where we write the supersymmetric Killing vector as $K = \sum_{i=1}^3 b_i \partial_{\varphi_i}$, and ∂_{φ_i} are standard generators of $U(1)^3 \subset SU(3) \times U(1)$ acting on $S^5 \subset \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$. Given the corresponding $4d/3d$ results of [15, 25], it is then natural to conjecture that (12.0.1) holds for *any* supersymmetric supergravity solution with the topology of a six-ball and for which the supersymmetric Killing vector K may be written as $K = \sum_{i=1}^3 b_i \partial_{\varphi_i}$. We chose orientation conventions so that $b_i > 0$ for $i = 1, 2, 3$. More generally we expect the orientations of ∂_{φ_i} to be fixed as in [15], leading to the modulus signs in (12.0.1). We shall comment further on this below. We also conjecture that any supersymmetric gauge theory, with finite N, defined on the conformal boundary of such a supergravity solution depends only on b_1, b_2, b_3.

We have also computed certain BPS Wilson loops, both in supergravity and in the large N gauge theories, again finding agreement. In this case we find that one can write the Wilson loop VEV as

$$\log \langle W \rangle = \frac{|b_1| + |b_2| + |b_3|}{3|b_i|} \log \langle W \rangle_{\mathrm{AdS}_6} , \qquad (12.0.2)$$

where the Wilson loop wraps the φ_i circle. Again, it is natural to conjecture that (12.0.2) holds for general supergravity backgrounds with $U(1)^3$ symmetry and the topology of a six-ball. And, in fact, using the technology developed later, we have computed the VEV of the holographic dual of a supersymmetric Wilson loop for a general class of solutions, thus proving this conjecture. The former conjecture, making a prediction for the holographic free energy for the same class of backgrounds, has a more involved computation than that for the Wilson loop; in particular the structure of the counterterms is much more complicated. It would, nevertheless, be interesting to prove this conjecture. A general proof of the analogous formula to (12.0.2) for the Wilson loop VEV in four dimensions appears in [57].

We have also presented a systematic study of supersymmetric solutions to six-dimensional Euclidean Romans supergravity. These are characterized by an $SU(2)$ structure. We then used these results to study a number of different applications.

Our results raise a number of interesting questions and directions for future work. Firstly, the gravity duals to (squashed) Sasaki-Einstein backgrounds we constructed have isolated Calabi-Yau singularities. However, as we have seen, the singularity does not contribute additional (UV) divergences to the free energy and Wilson loop, and moreover the supergravity computations agree with the gauge theory results. It is thus natural to conjecture that these are the correct gravity duals. More precisely, although one expects some stringy degrees of freedom to be supported at the singularity, we expect that these should not contribute to leading order at large N. Notice in any case that the uplift to massive IIA is also singular (along the internal S^4), even for Euclidean AdS_6 [10, 11].

Finally, it would be interesting to construct further analytic solutions, including solutions with different topology.

Appendix A

Integrability conditions

Here we compute the integrability conditions for the Killing spinor equation (3.2.5) and dilatino equation (3.2.6) of the Euclidean Romans theory.

Recall that a supersymmetric solution must satisfy

$$\begin{aligned} D_\mu\epsilon_I &= \frac{\mathrm{i}}{4\sqrt{2}}g(X+\tfrac{1}{3}X^{-3})\Gamma_\mu\Gamma_7\epsilon_I - \frac{1}{48}X^2H_{\nu\rho\sigma}\Gamma^{\nu\rho\sigma}\Gamma_\mu\Gamma_7\epsilon_I \\ &\quad -\frac{\mathrm{i}}{16\sqrt{2}}X^{-1}F_{\nu\rho}(\Gamma_\mu{}^{\nu\rho}-6\delta_\mu{}^\nu\Gamma^\rho)\epsilon_I + \frac{1}{16\sqrt{2}}X^{-1}F^i_{\nu\rho}(\Gamma_\mu{}^{\nu\rho}-6\delta_\mu{}^\nu\Gamma^\rho)\Gamma_7(\sigma^i)_I{}^J\epsilon_J\ , \end{aligned} \tag{A.0.1}$$

$$\begin{aligned} \delta\lambda_I \ = \ 0 \ &= \ -\mathrm{i}X^{-1}\partial_\mu X\Gamma^\mu\epsilon_I + \frac{1}{2\sqrt{2}}g\left(X-X^{-3}\right)\Gamma_7\epsilon_I + \frac{\mathrm{i}}{24}X^2H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_7\epsilon_I \\ &\quad -\frac{1}{8\sqrt{2}}X^{-1}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon_I - \frac{\mathrm{i}}{8\sqrt{2}}X^{-1}F^i_{\mu\nu}\Gamma^{\mu\nu}\Gamma_7(\sigma^i)_I{}^J\epsilon_J\ , \end{aligned} \tag{A.0.2}$$

where λ_I is the dilatino field. Let us also record the component form of the Romans field equations in (3.2.1) and (3.2.3)

$$
\begin{aligned}
(E_g)_{\mu\nu} &\equiv R_{\mu\nu} - 4X^{-2}\partial_\mu X\partial_\nu X - g^2\left(\tfrac{1}{18}X^{-6} - \tfrac{1}{2}X^2 - \tfrac{2}{3}X^{-2}\right)g_{\mu\nu} \\
&\quad -\tfrac{1}{4}X^4(H_\mu{}^{\rho\sigma}H_{\nu\rho\sigma} - \tfrac{1}{6}g_{\mu\nu}H^{\rho\sigma\tau}H_{\rho\sigma\tau}) - \tfrac{1}{2}X^{-2}(F_\mu{}^\rho F_{\nu\rho} - \tfrac{1}{8}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}) \\
&\quad -\tfrac{1}{2}X^{-2}(F^i_\mu{}^\rho F^i_{\nu\rho} - \tfrac{1}{8}g_{\mu\nu}F^{i\rho\sigma}F^i_{\rho\sigma})\ , \\
(E_X) &\equiv \nabla^\mu(X^{-1}\partial_\mu X) + g^2\left(\tfrac{1}{2}X^2 - \tfrac{2}{3}X^{-2} + \tfrac{1}{6}X^{-6}\right) - \tfrac{1}{24}X^4H^{\mu\nu\rho}H_{\mu\nu\rho} \\
&\quad +\tfrac{1}{16}X^{-2}(F^{\mu\nu}F_{\mu\nu} + F^{i\mu\nu}F^i_{\mu\nu})\ , \\
(E_A)^\mu &\equiv \nabla_\nu(X^{-2}F^{\nu\mu}) - \tfrac{\mathrm{i}}{12}\varepsilon^{\mu\nu\rho\sigma\tau\kappa}F_{\nu\rho}H_{\sigma\tau\kappa}\ , \\
(E_{A^i})^\mu &\equiv D_\nu(X^{-2}F^{i\nu\mu}) - \tfrac{\mathrm{i}}{12}\varepsilon^{\mu\nu\rho\sigma\tau\kappa}F^i_{\nu\rho}H_{\sigma\tau\kappa}\ , \\
(E_B)^{\mu\nu} &\equiv \nabla_\rho(X^4H^{\rho\mu\nu}) - \tfrac{2}{3}gX^{-2}F^{\mu\nu} - \tfrac{\mathrm{i}}{8}\varepsilon^{\mu\nu\rho\sigma\tau\kappa}(F_{\rho\sigma}F_{\tau\kappa} + F^i_{\rho\sigma}F^i_{\tau\kappa})\ .
\end{aligned}
\tag{A.0.3}
$$

The equations of motion are then $E_{\text{field}} = 0$. In addition, the gauge fields satisfy Bianchi identities $B_{\text{field}} = 0$, where we define

$$
\begin{aligned}
(B_F)_{\mu\nu\rho} &\equiv \nabla_{[\mu}F_{\nu\rho]} - \frac{2}{9}gH_{\mu\nu\rho}\ , \\
(B_{F^i})_{\mu\nu\rho} &\equiv D_{[\mu}F^i_{\nu\rho]}\ , \\
(B_H)_{\mu\nu\rho\sigma} &\equiv \nabla_{[\mu}H_{\nu\rho\sigma]}\ .
\end{aligned}
\tag{A.0.4}
$$

Taking the commutator of the Killing spinor equation (A.0.1) we find the integrability condition to be

$$
\mathcal{I}_{\mu\nu I}{}^J\epsilon_J = 0\ , \tag{A.0.5}
$$

where

$$
\begin{aligned}
\mathcal{I}_{\mu\nu I}{}^J\epsilon_J &= \tfrac{1}{4}R_{\mu\nu\rho\sigma}\Gamma^{\rho\sigma}\epsilon_I + \tfrac{\mathrm{i}}{2}gF^i_{\mu\nu}(\sigma^i)_I{}^J\epsilon_J + \Big[-\tfrac{\mathrm{i}}{4\sqrt{2}}g(1-X^{-4})\partial_\mu X\Gamma_\nu\Gamma_7\epsilon_I \\
&+\tfrac{1}{24}X\partial_\mu XH^{\rho\sigma\tau}\Gamma_{\rho\sigma\tau}\Gamma_\nu\Gamma_7\epsilon_I + \tfrac{1}{48}X^2\nabla_\mu H^{\rho\sigma\tau}\Gamma_{\rho\sigma\tau}\Gamma_\nu\Gamma_7\epsilon_I \\
&-\tfrac{\mathrm{i}}{16\sqrt{2}}X^{-2}\partial_\mu XF_{\rho\sigma}J_\nu{}^{\rho\sigma}\epsilon_I + \tfrac{\mathrm{i}}{16\sqrt{2}}X^{-1}\nabla_\mu F_{\rho\sigma}J_\nu{}^{\rho\sigma}\epsilon_I \\
&+\tfrac{1}{16\sqrt{2}}X^{-2}\partial_\mu XF^i_{\rho\sigma}J_\nu{}^{\rho\sigma}\Gamma_7(\sigma^i)_I{}^J\epsilon_J - \tfrac{1}{16\sqrt{2}}X^{-1}\nabla_\mu F^i_{\rho\sigma}J_\nu{}^{\rho\sigma}\Gamma_7(\sigma^i)_I{}^J\epsilon_J \\
&-\tfrac{1}{32}g^2(\tfrac{1}{9}X^{-6}+\tfrac{2}{3}X^{-2}+X^2)\Gamma_\nu\Gamma_\mu\epsilon_I - \tfrac{1}{2304}X^4H^{\lambda\omega\theta}H^{\rho\sigma\tau}\Gamma_{\lambda\omega\theta}\Gamma_\nu\Gamma_{\rho\sigma\tau}\Gamma_\mu\epsilon_I \\
&+\tfrac{1}{512}X^{-2}F_{\omega\theta}F_{\rho\sigma}J_\nu{}^{\omega\theta}J_\mu{}^{\rho\sigma}\epsilon_I + \tfrac{1}{512}X^{-2}F^i_{\omega\theta}F^i_{\rho\sigma}J_\nu{}^{\omega\theta}J_\mu{}^{\rho\sigma}\epsilon_I \\
&+\tfrac{\mathrm{i}}{512}X^{-2}\varepsilon_{ijk}F^i_{\omega\theta}F^j_{\rho\sigma}J_\nu{}^{\omega\theta}J_\mu{}^{\rho\sigma}(\sigma^k)_I{}^J\epsilon_J \\
&+\tfrac{\mathrm{i}}{192\sqrt{2}}g(X^3+\tfrac{1}{3}X^{-1})H^{\rho\sigma\tau}\Big(\Gamma_\nu\Gamma_{\rho\sigma\tau}\Gamma_\mu - \Gamma_{\rho\sigma\tau}\Gamma_\nu\Gamma_\mu\Big)\epsilon_I \\
&+\tfrac{1}{128}gX^{-1}(X+\tfrac{1}{3}X^{-3})F_{\rho\sigma}\Big(\Gamma_\nu J_\mu{}^{\rho\sigma} - J_\nu{}^{\rho\sigma}\Gamma_\mu\Big)\Gamma_7\epsilon_I \\
&+\tfrac{\mathrm{i}}{128}gX^{-1}(X+\tfrac{1}{3}X^{-3})F^i_{\rho\sigma}\Big(\Gamma_\nu J_\mu{}^{\rho\sigma} + J_\nu{}^{\rho\sigma}\Gamma_\mu\Big)(\sigma^i)_I{}^J\epsilon_J \\
&+\tfrac{\mathrm{i}}{768\sqrt{2}}XF_{\rho\sigma}H^{\lambda\omega\theta}\Big(\Gamma_{\lambda\omega\theta}\Gamma_\nu J_\mu{}^{\rho\sigma} - J_\nu{}^{\rho\sigma}\Gamma_{\lambda\omega\theta}\Gamma_\mu\Big)\Gamma_7\epsilon_I \\
&-\tfrac{1}{768\sqrt{2}}XF^i_{\rho\sigma}H^{\lambda\omega\theta}\Big(\Gamma_{\lambda\omega\theta}\Gamma_\nu J_\mu{}^{\rho\sigma} - J_\nu{}^{\rho\sigma}\Gamma_{\lambda\omega\theta}\Gamma_\mu\Big)(\sigma^i)_I{}^J\epsilon_J \\
&+\tfrac{\mathrm{i}}{512}X^{-2}(F_{\rho\sigma}F^i_{\omega\theta} - F_{\omega\theta}F^i_{\rho\sigma})J_\nu{}^{\rho\sigma}J_\mu{}^{\omega\theta}\Gamma_7(\sigma^i)_I{}^J\epsilon_J - (\mu\leftrightarrow\nu)\Big]\ , \qquad \text{(A.0.6)}
\end{aligned}
$$

and we have defined the Clifford algebra element

$$
J_\mu{}^{\rho\sigma} \equiv \Gamma_\mu{}^{\rho\sigma} - 6\delta_\mu{}^\rho\Gamma^\sigma\ . \qquad \text{(A.0.7)}
$$

Taking the covariant derivative of the dilatino equation (A.0.2) and contracting with Γ^μ leads to

$$
\begin{aligned}
&\Gamma^\mu D_\mu(\delta\lambda_I) - \tfrac{\mathrm{i}}{2\sqrt{2}}g(X-\tfrac{7}{3}X^{-3})\Gamma_7\delta\lambda_I + \tfrac{1}{24}X^2H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_7\delta\lambda_I \qquad \text{(A.0.8)} \\
&+\tfrac{\mathrm{i}}{8\sqrt{2}}X^{-1}F_{\mu\nu}\Gamma^{\mu\nu}\delta\lambda_I + \tfrac{1}{8\sqrt{2}}X^{-1}F^i_{\mu\nu}\Gamma^{\mu\nu}\Gamma_7(\sigma^i)_I{}^J\delta\lambda_J \\
=\ &\mathrm{i}\,(E_X)\,\epsilon_I - \tfrac{1}{4\sqrt{2}}X\,(E_A)_\mu\,\Gamma^\mu\epsilon_I - \tfrac{\mathrm{i}}{4\sqrt{2}}X\,(E_{A^i})_\mu\,\Gamma^\mu\Gamma_7(\sigma^i)_I{}^J\epsilon_J + \tfrac{\mathrm{i}}{8}X^{-2}\,(E_B)_{\mu\nu}\,\Gamma^{\mu\nu}\Gamma_7\epsilon_I \\
&-\tfrac{1}{8\sqrt{2}}X^{-1}\,(B_F)_{\mu\nu\rho}\,\Gamma^{\mu\nu\rho}\epsilon_I - \tfrac{\mathrm{i}}{8\sqrt{2}}X^{-1}\,(B_{F^i})_{\mu\nu\rho}\,\Gamma^{\mu\nu\rho}\Gamma_7(\sigma^i)_I{}^J\epsilon_J \\
&+\tfrac{\mathrm{i}}{24}X^2\,(B_H)_{\mu\nu\rho\sigma}\,\Gamma^{\mu\nu\rho\sigma}\Gamma_7\epsilon_I\ .
\end{aligned}
$$

We may similarly contract $\mathcal{I}_{\mu\nu I}{}^{J}\epsilon_J$ with Γ^ν. After a very lengthy calculation we find

$$
\begin{aligned}
& \Gamma^\nu \mathcal{I}_{\mu\nu I}{}^{J}\epsilon_J + \tfrac{\mathrm{i}}{2}\Gamma_\mu\Gamma_\nu D^\nu(\delta\lambda_I) + 2\mathrm{i}X^{-1}\partial_\mu X\delta\lambda_I + \tfrac{1}{2\sqrt{2}}g(X - \tfrac{5}{3}X^{-3})\Gamma_\mu\Gamma_7\delta\lambda_I \\
& -\tfrac{\mathrm{i}}{16}X^2 H_{\mu\nu\rho}\Gamma^{\nu\rho}\Gamma_7\delta\lambda_I + \tfrac{\mathrm{i}}{16}X^2 H^{\nu\rho\sigma}\Gamma_{\mu\nu\rho\sigma}\Gamma_7\delta\lambda_I - \tfrac{1}{8\sqrt{2}}X^{-1}F^{\nu\rho}\Gamma_{\mu\nu\rho}\delta\lambda_I \\
& +\tfrac{1}{4\sqrt{2}}X^{-1}F_{\mu\nu}\Gamma^\nu\delta\lambda_I - \tfrac{\mathrm{i}}{4\sqrt{2}}X^{-1}F^i_{\mu\nu}\Gamma^\nu\Gamma_7(\sigma^i)_I{}^J\delta\lambda_J + \tfrac{\mathrm{i}}{8\sqrt{2}}X^{-1}F^{i\nu\rho}\Gamma_{\mu\nu\rho}\Gamma_7(\sigma^i)_I{}^J\delta\lambda_J \\
= \; & \tfrac{1}{2}\left(E_X\right)\Gamma_\mu\epsilon_I - \tfrac{1}{2}\left(E_g\right)_{\mu\nu}\Gamma^\nu\epsilon_I - \tfrac{1}{8}X^{-2}\left(E_B\right)^{\nu\rho}\Gamma_{\mu\nu\rho}\Gamma_7\epsilon_I \\
& -\tfrac{\mathrm{i}}{2\sqrt{2}}X\left(E_A\right)_\mu\epsilon_I + \tfrac{1}{2\sqrt{2}}X\left(E_{A^i}\right)_\mu\Gamma_7(\sigma^i)_I{}^J\epsilon_J - \tfrac{1}{24}X^2\left(B_H\right)^{\nu\rho\sigma\tau}\Gamma_{\mu\nu\rho\sigma\tau}\Gamma_7\epsilon_I \\
& -\tfrac{3\mathrm{i}}{4\sqrt{2}}X^{-1}\left(B_F\right)_{\mu\nu\rho}\Gamma^{\nu\rho}\epsilon_I + \tfrac{3}{4\sqrt{2}}X^{-1}\left(B_{F^i}\right)_{\mu\nu\rho}\Gamma^{\nu\rho}\Gamma_7(\sigma^i)_I{}^J\epsilon_J \; . \qquad \text{(A.0.9)}
\end{aligned}
$$

Appendix B

Supersymmetric supergravity solutions

B.1 The equations

The solutions summarized in chapter 4 arise from the following $SU(3)\times U(1)$ symmetric ansatz for the supergravity fields

$$\begin{aligned}
\mathrm{d}s_6^2 &= \alpha^2(r)\mathrm{d}r^2+\gamma^2(r)(\mathrm{d}\tau+C)^2+\beta^2(r)\Big[\mathrm{d}\sigma^2+\frac{1}{4}\sin^2\sigma(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2)\\
&\quad +\frac{1}{4}\cos^2\sigma\sin^2\sigma(\mathrm{d}\psi+\cos\theta\mathrm{d}\varphi)^2\Big]\ ,\\
B &= p(r)\mathrm{d}r\wedge(\mathrm{d}\tau+C)+\frac{1}{2}q(r)\mathrm{d}C\ ,\\
A^i &= f^i(r)(\mathrm{d}\tau+C)\ ,
\end{aligned} \tag{B.1.1}$$

together with $X=X(r)$. The equations of motion for the background $SU(2)_R$ gauge field imply

$$f^i(r)\ =\ \kappa_i f(r)\ . \tag{B.1.2}$$

The equations for the other fields then depend only on the $SU(2)\sim SO(3)$ invariant $\kappa_1^2+\kappa_2^2+\kappa_3^2$, which we can set to one by rescaling $f(r)$. Explicitly, one finds that substituting the ansatz (B.1.1) into the equations of motion (3.2.1) and Einstein equation (3.2.3) leads to following coupled system of ODEs:

$$\frac{\lambda\gamma X^4}{\alpha} = \mathrm{i}f^2 + \mathrm{i}\frac{q^2}{9} + \frac{p\beta^4}{9\alpha\gamma X^2} \,, \tag{B.1.3}$$

$$\left(\frac{\lambda\gamma X^4}{\alpha}\right)' = 2\mathrm{i}ff' + \mathrm{i}\left(\frac{2}{3}\right)^2 pq + \left(\frac{2}{3}\right)^2 \frac{q\alpha\gamma}{X^2} \,, \tag{B.1.4}$$

$$\left(\frac{\beta^4 f'}{2\alpha\gamma X^2}\right)' - \frac{4\alpha\gamma f}{X^2} = -2\mathrm{i}f\lambda \,, \tag{B.1.5}$$

$$\begin{aligned}\frac{\alpha}{\gamma\beta^4}\left(\frac{\gamma\beta^4 X'}{\alpha X}\right)' &= -\frac{1}{8X^2}\left(\frac{f'^2}{\gamma^2} + \frac{8\alpha^2 f^2}{\beta^4}\right) - \left(\frac{2}{3}\right)^2 \frac{1}{8X^2}\left(\frac{p^2}{\gamma^2} + 2\frac{\alpha^2 q^2}{\beta^4}\right) \\ &\quad + \frac{X^4\lambda^2}{2\beta^4} - \frac{\alpha^2}{6X^6} + \frac{2\alpha^2}{3X^2} - \frac{\alpha^2 X^2}{2} \,,\end{aligned} \tag{B.1.6}$$

$$-\frac{\beta''}{\beta} + \frac{\beta'}{\beta}\frac{(\alpha\gamma)'}{\alpha\gamma} - \frac{(\alpha\gamma)^2}{\beta^4} = \left(\frac{X'}{X}\right)^2 + \frac{X^4\lambda^2}{4\beta^4} \,, \tag{B.1.7}$$

$$\begin{aligned}&-\frac{\gamma''}{\gamma} + \frac{\beta''}{\beta} + \frac{\alpha'}{\alpha}\left(\frac{\gamma'}{\gamma} - \frac{\beta'}{\beta}\right) - 3\frac{\beta'}{\beta}\left(\frac{\gamma'}{\gamma} - \frac{\beta'}{\beta}\right) + \frac{6\alpha^2}{\beta^4}(\gamma^2 - \beta^2) \\ &= -\frac{X^4\lambda^2}{2\beta^4} + \frac{1}{2X^2}\left(\frac{f'^2}{\gamma^2} - \frac{4\alpha^2 f^2}{\beta^4}\right) + \left(\frac{2}{3}\right)^2 \frac{1}{2X^2}\left(\frac{p^2}{\gamma^2} - \frac{\alpha^2 q^2}{\beta^4}\right) \,,\end{aligned} \tag{B.1.8}$$

$$\begin{aligned}-\frac{\gamma''}{\gamma} + \frac{\alpha'}{\alpha}\frac{\gamma'}{\gamma} - 4\frac{\beta'}{\beta}\frac{\gamma'}{\gamma} + 4\frac{(\alpha\gamma)^2}{\beta^4} &= \frac{\alpha^2}{18X^6} - \frac{2\alpha^2}{3X^2} - \frac{\alpha^2 X^2}{2} - \frac{X^4\lambda^2}{2\beta^4} \\ &\quad + \frac{1}{2X^2}\left[\frac{f'^2}{\gamma^2} - \frac{1}{4}\left(\frac{f'^2}{\gamma^2} + \frac{8\alpha^2 f^2}{\beta^4}\right)\right] \\ &\quad + \left(\frac{2}{3}\right)^2 \frac{1}{2X^2}\left[\frac{p^2}{\gamma^2} - \frac{1}{4}\left(\frac{p^2}{\gamma^2} + \frac{2\alpha^2 q^2}{\beta^4}\right)\right] \,.\end{aligned} \tag{B.1.9}$$

where we have introduced $\lambda = q' - 2p$. These are seven equations for seven functions. In addition one can explicitly check that the equations are invariant under changes in the parametrization $r \to \rho(r)$.

B.2 General solutions

Before writing the general series solutions to the above coupled system of ODEs, let us present the solution for Euclidean AdS_6 in these coordinates:

$$\begin{aligned}\alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}} \,, \qquad \beta(r) = \gamma(r) = \frac{3\sqrt{6r^2-1}}{\sqrt{2}} \,, \\ p(r) &= q(r) = f(r) = 0 \,, \qquad X(r) = 1 \,.\end{aligned} \tag{B.2.1}$$

Here only the metric is non-trivial, and the above realizes Euclidean AdS_6 as a hyperbolic ball with radial coordinate $r \in [\frac{1}{\sqrt{6}}, \infty)$, with the conformal boundary at infinity $r = \infty$. The point $r = \frac{1}{\sqrt{6}}$ is the origin of the ball, where the transverse copies of S^5 collapse smoothly to zero. Notice in particular that the conformal boundary at $r = \infty$ is equipped with a *round* metric on S^5, which is conformally flat. We would like to find families of solutions that generalize (B.2.1) by allowing for a squashed five-sphere boundary, keeping the metric asymptotically locally Euclidean AdS near $r = \infty$. We define the squashing parameter by:

$$\lim_{r\to\infty} \frac{\gamma(r)}{r} \;=\; 3\sqrt{3}\,\frac{1}{s}\;, \tag{B.2.2}$$

so that $s = 1$ for the round sphere. Even though we did not manage to find solutions in closed form, the solutions can nevertheless be given as expansions around different limits. In general notice that we can use reparametrization invariance to set

$$\beta(r) \;=\; \frac{3\sqrt{6r^2-1}}{\sqrt{2}}\;, \tag{B.2.3}$$

which we assume henceforth. In particular we shall only seek solutions with the topology of a ball, so that from (B.2.3) necessarily $r = \frac{1}{\sqrt{6}}$ is the origin of the ball. Correspondingly, the fields must satisfy certain boundary conditions at this point in order that the full solution is smooth at the origin.

B.2.1 Expansion around the conformal boundary

When finding gravity duals to a given boundary theory, it is natural to perform an expansion around the conformal boundary at $r = \infty$. This also has the advantage that the squashing parameter can be explicitly seen in the solution. Starting from a general expansion and imposing the equations of motion in chapter B.1 we find

$$
\begin{aligned}
\alpha(r) &= \frac{3}{\sqrt{2}}\frac{1}{r} + \frac{486+q_0^2 s^2}{1944\sqrt{2}s^2}\frac{1}{r^3} + \dots\,, \\
\gamma(r) &= \frac{3\sqrt{3}}{s}r + \frac{-486+\left(243-q_0^2\right)s^2}{324\sqrt{3}s^3}\frac{1}{r} + \dots\,, \\
X(r) &= 1 + \frac{-486q_0 + 72\mathrm{i}\sqrt{6}q_0^2 s + 486 q_0 s^2 + 7q_0^3 s^2 + 5832 s^2 q_2}{11664 q_0 s^2}\frac{1}{r^2} + \frac{x_3}{r^3} + \dots\,, \\
p(r) &= \frac{q_0\left(54-\sqrt{6}\mathrm{i}q_0 s\right)}{162 s^2}\frac{1}{r^2} + \dots\,, \\
q(r) &= q_0 r + \frac{q_2}{r} + \frac{q_3}{r^2} + \dots\,, \\
f(r) &= f_0 - \frac{f_0\left(54-\sqrt{6}\mathrm{i}q_0 s\right)}{81 s^2}\frac{1}{r^2} + \frac{f_3}{r^3} + \dots\,.
\end{aligned}
\tag{B.2.4}
$$

In addition to the squashing parameter s, the solution depends on $q_0, f_0, f_3, q_2, q_3, x_3$ and an extra parameter α_5, which appears at higher order in the expansion for $\alpha(r)$. All other coefficients in the expansion are fixed in terms of these constants. Of course, some of these parameters will be fixed in the full solution by requiring the correct boundary conditions at the origin $r = \frac{1}{\sqrt{6}}$, but at this point they are arbitrary.

B.2.2 Expansion around Euclidean AdS

The family of solutions we seek should approach Euclidean AdS$_6$ (B.2.1) as we take the squashing parameter $s \to 1$. Hence it should be possible to expand the solutions around this limit in terms of a perturbation parameter δ. Thus we make the ansatz

$$
\begin{aligned}
\alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}} + \delta\ \alpha^{(1)}(r) + \delta^2\ \alpha^{(2)}(r) + \dots\,, \\
\gamma(r) &= \frac{3\sqrt{6r^2-1}}{\sqrt{2}} + \delta\ \gamma^{(1)}(r) + \delta^2\ \gamma^{(2)}(r) + \dots\,, \\
X(r) &= 1 + \delta\ X^{(1)}(r) + \delta^2\ X^{(2)}(r) + \dots\,, \\
p(r) &= \delta\ p^{(1)}(r) + \delta^2\ p^{(2)}(r) + \dots\,, \\
q(r) &= \delta\ q^{(1)}(r) + \delta^2\ q^{(2)}(r) + \dots\,, \\
f(r) &= \delta\ f^{(1)}(r) + \delta^2\ f^{(2)}(r) + \dots\,.
\end{aligned}
\tag{B.2.5}
$$

Substituting this expansion into the equations of motion and expanding in powers of δ, at each order we obtain a system of linear differential equations which can be solved in closed form with some effort. For instance, at first order we find

$$
\begin{aligned}
\alpha^{(1)}(r) &= -c_\gamma \frac{\left(1-54r^2+96\sqrt{6}r^3-324r^4+216r^6\right)}{\sqrt{6}r^2\left(6r^2-1\right)^{7/2}} \,, \\
\gamma^{(1)}(r) &= c_\gamma \frac{\left(-5+16\sqrt{6}r-90r^2+180r^4-216r^6\right)}{\left(6r^2-1\right)^{5/2}} \,, \\
X^{(1)}(r) &= c_x \frac{\left(1-2\sqrt{6}r+6r^2\right)}{\left(6r^2-1\right)^2} \,, \\
p^{(1)}(r) &= c_q \frac{\left(\sqrt{6}-16r+12\sqrt{6}r^2-12\sqrt{6}r^4\right)}{3\left(6r^2-1\right)^3} \,, \\
q^{(1)}(r) &= -c_q \frac{\left(-4+9\sqrt{6}r-24r^2-12\sqrt{6}r^3+36\sqrt{6}r^5\right)}{18\left(6r^2-1\right)^2} \,, \\
f^{(1)}(r) &= c_f \frac{\left(-3+8\sqrt{6}r-36r^2+36r^4\right)}{\left(6r^2-1\right)^2} \,.
\end{aligned} \tag{B.2.6}
$$

The constants of integration have been partially fixed by requiring regularity at the origin $r=\frac{1}{\sqrt{6}}$. In particular we have

$$
\begin{aligned}
\alpha^{(1)}(r) &\sim \left(r-\frac{1}{\sqrt{6}}\right)^{1/2} \,, \qquad \gamma^{(1)}(r) \sim \left(r-\frac{1}{\sqrt{6}}\right)^{3/2} \,, \\
p^{(1)}(r) &\sim 1 \sim X^{(1)}(r) \,, \qquad q^{(1)}(r) \sim \left(r-\frac{1}{\sqrt{6}}\right) \sim f^{(1)}(r) \,.
\end{aligned} \tag{B.2.7}
$$

Here $\rho \sim (r-\frac{1}{\sqrt{6}})^{1/2}$ is geodesic distance from the origin at $\rho=0$. We can furthermore fix an extra constant of integration by fixing a relation between δ and the squashing parameter s (such that $\delta \to 0$ as $s \to 1$). As seen in the next section it will be convenient not to do this uniformly.

B.3 Imposing supersymmetry

We are interested in solutions that preserve some supersymmetry. In order for this to happen, there should exist non-trivial eight-component Killing spinors ϵ_1, ϵ_2 solving the Killing spinor equation (3.2.5) and dilatino equation (3.2.6). We choose the frame

$$
\begin{aligned}
e^0 &= \alpha(r)\mathrm{d}r \,, \qquad e^1 = \gamma(r)(\mathrm{d}\tau + C) \,, \quad e^2 = \beta(r)\mathrm{d}\sigma \,, \\
e^3 &= \frac{1}{2}\beta(r)\sin\sigma\cos\sigma\tau_3 \,, \quad e^4 = \frac{1}{2}\beta(r)\sin\sigma\tau_2 \,, \quad e^5 = \frac{1}{2}\beta(r)\sin\sigma\tau_1 \,,
\end{aligned} \tag{B.3.1}
$$

and the following basis for six-dimensional gamma matrices

$$\Gamma_0 = \begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix} , \quad \Gamma_m = \begin{pmatrix} 0 & \mathrm{i}\gamma_m \\ -\mathrm{i}\gamma_m & 0 \end{pmatrix} , \quad m = 1, \ldots, 5 ,$$

$$\Gamma_7 = \begin{pmatrix} -1_4 & 0 \\ 0 & 1_4 \end{pmatrix} , \tag{B.3.2}$$

where 1_4 is the 4×4 unit matrix and γ_m are the five-dimensional gamma matrices given explicitly preciously.

The vanishing of the dilatino variation as well as each component of the integrability condition (A.0.6) for the Killing spinor equation have the following general structure

$$\begin{aligned} P\epsilon_1 + Q\epsilon_2 &= 0 , \\ R\epsilon_1 + S\epsilon_2 &= 0 , \end{aligned} \tag{B.3.3}$$

where P, Q, R, S are 8×8 matrices, whose components are in general complicated functions of the fields. After setting $f_i(r) = \kappa_i f(r)$ we observe the following $SU(2)_R$ structure

$$\begin{pmatrix} A + \kappa_3 B & (\kappa_1 - \mathrm{i}\kappa_2)B \\ (\kappa_1 + \mathrm{i}\kappa_2)B & A - \kappa_3 B \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = 0 , \tag{B.3.4}$$

in terms of 8×8 matrices A, B. We can then diagonalize the block matrix and consider the equivalent problem

$$\begin{pmatrix} A + B & 0 \\ 0 & A - B \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = 0 , \tag{B.3.5}$$

where we have without loss of generality set $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 1$. There are four independent conditions. One of these arises from the dilatino variation, whose matrices we denote by A_0, B_0, and the other three conditions arise from integrability of the Killing spinor equation, whose matrices we denote by A_M, B_M with $M \in \{12, 13, 34\}$ (all other components of the integrability condition (A.0.6) are equivalent to one of these). The dilatino condition as well as $M = 12$ and $M = 34$ have the following structure:

$$A \pm B \;=\; \begin{pmatrix} * & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & * \end{pmatrix} . \tag{B.3.6}$$

The existence of a non-trivial solution requires, for instance, $\det(A+B) = 0$. The above structure implies the determinant factorizes into four factors

$$\det(A+B) \;=\; F_1 F_2 F_3 F_4 \;=\; 0 \,, \tag{B.3.7}$$

where the factors F_i are complicated functions of the supergravity fields $\alpha(r)$, $\beta(r)$, $\gamma(r)$, $p(r)$, $q(r)$, $f(r)$, $X(r)$. F_1 and F_3 differ only by a change of sign in $f(r)$, and the same happens for F_2 and F_4. We find two distinct classes of solutions which we describe in the following.

B.3.1 3/4 BPS solutions

There is a class of solutions that satisfies

$$F_1 \;=\; F_2 \;=\; F_3 \;=\; 0 \,, \qquad F_4 \;\neq\; 0 \,. \tag{B.3.8}$$

These are a one-parameter family of solutions parametrized by the squashing parameter s. The solution expanded around the conformal boundary is given by

$$\begin{aligned}
\alpha(r) &= \frac{3}{\sqrt{2}}\frac{1}{r} + \frac{8+s^2}{36\sqrt{2}s^2}\frac{1}{r^3} + \dots \ , \\
\gamma(r) &= \frac{3\sqrt{3}}{s}r + \frac{-16+7s^2}{12\sqrt{3}s^3}\frac{1}{r} - \frac{-1280+1120s^2+241s^4}{2592\sqrt{3}s^5}\frac{1}{r^3} + \dots \ , \\
X(r) &= 1 + \frac{1-s^2-3\sqrt{1-s^2}}{54s^2}\frac{1}{r^2} + \frac{s^2\sqrt{1-s^2}\kappa}{12\left(1-s^2+\sqrt{1-s^2}\right)}\frac{1}{r^3} + \dots \ , \\
p(r) &= -\frac{\mathrm{i}\sqrt{\frac{2}{3}}\left(s^2+3\sqrt{1-s^2}-1\right)}{s^3}\frac{1}{r^2} + \dots \ , \\
q(r) &= -\frac{3\mathrm{i}\left(\sqrt{6}\sqrt{1-s^2}\right)}{s}r + \frac{\sqrt{\frac{2}{3}}\mathrm{i}\sqrt{1-s^2}\left(5s^2+9\sqrt{1-s^2}-5\right)}{3s^3}\frac{1}{r} + \dots \ , \\
f(r) &= \frac{1-s^2+\sqrt{1-s^2}}{s^2} + \frac{2\left(-2+2s^2-(2+s^2)\sqrt{1-s^2}\right)}{9s^4}\frac{1}{r^2} + \frac{\kappa}{r^3} + \dots \ .
\end{aligned} \tag{B.3.9}$$

The extra parameter κ is fixed by requiring regularity at the origin. The solution expanded around Euclidean AdS$_6$ has $c_\gamma = 0$, hence it is convenient to set the relation between the expansion parameter and the squashing parameter to be

$$\frac{1}{s} = 1 + \delta^2 \ . \tag{B.3.10}$$

With this choice the solution is given by

$$\begin{aligned}
\alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}} + \frac{\left(-5\sqrt{6}+330\sqrt{6}r^2-3744r^3+1620\sqrt{6}r^4+8640r^5-7560\sqrt{6}r^6+5184\sqrt{6}r^8\right)}{9\sqrt{2}r^2(6r^2-1)^{9/2}}\delta^2 + \dots \ , \\
\gamma(r) &= \frac{3\sqrt{6r^2-1}}{\sqrt{2}} \\
&\quad - \frac{\left(55\sqrt{2}-384\sqrt{3}r+1080\sqrt{2}r^2+768\sqrt{3}r^3-5400\sqrt{2}r^4+11232\sqrt{2}r^6-11664\sqrt{2}r^8\right)}{6(6r^2-1)^{7/2}}\delta^2 + \dots \ , \\
X(r) &= 1 - \frac{\left(\sqrt{2}\left(1-2\sqrt{6}r+6r^2\right)\right)}{3\left(6r^2-1\right)^2}\delta + \dots \ , \\
p(r) &= \frac{18\mathrm{i}\sqrt{2}\left(\sqrt{6}-16r+12\sqrt{6}r^2-12\sqrt{6}r^4\right)}{\left(6r^2-1\right)^3}\delta + \dots \ , \\
q(r) &= -\frac{3\mathrm{i}\sqrt{2}\left(-4+9\sqrt{6}r-24r^2-12\sqrt{6}r^3+36\sqrt{6}r^5\right)}{\left(6r^2-1\right)^2}\delta + \dots \ , \\
f(r) &= \frac{\sqrt{2}\left(-3+8\sqrt{6}r-36r^2+36r^4\right)}{\left(6r^2-1\right)^2}\delta + \dots \ .
\end{aligned} \tag{B.3.11}$$

We have computed the solution up to sixth order in δ. Comparing this expansion with the expansion around the conformal boundary we can compute the coefficient κ as a series expansion in δ. We obtain

$$\frac{3\sqrt{3}}{4}\kappa \;=\; \delta+\frac{\sqrt{2}}{3}\delta^2+\frac{113}{36}\delta^3+\frac{25}{9\sqrt{2}}\delta^4+\frac{1127}{288}\delta^5+\frac{35}{9\sqrt{2}}\delta^6+\dots \;. \qquad \text{(B.3.12)}$$

B.3.2 1/4 BPS solutions

There is another class of supersymmetric solutions that satisfies

$$F_1\;,\;F_2\;,\;F_3\;\neq\;0\;,\qquad F_4\;=\;0\;. \qquad \text{(B.3.13)}$$

These are a two-parameter family of solutions and are parametrized by the squashing parameter s and the background $SU(2)_R$ field at the conformal boundary, which is parametrized by f_0. The solution expanded around the conformal boundary is given by

$$\begin{aligned}
\alpha(r) &= \frac{3}{\sqrt{2}}\frac{1}{r}-\frac{f_0^2s^2+9\left(-2+s^2\right)-6f_0\left(-1+s^2\right)}{36\sqrt{2}}\frac{1}{r^3}+\dots\;,\\
\gamma(r) &= \frac{3\sqrt{3}}{s}r+\frac{2f_0^2s^2-12f_0\left(-1+s^2\right)+9\left(-3+2s^2\right)}{12\sqrt{3}s}\frac{1}{r}+\dots\;,\\
X(r) &= 1+\frac{18-3f_0-18s^2+12f_0s^2-2f_0^2s^2}{54}\frac{1}{r^2}+\dots\;,\\
p(r) &= \frac{\mathrm{i}\sqrt{\frac{2}{3}}(-3+f_0)\left(3+(-3+f_0)s^2\right)}{s}\frac{1}{r^2}+\dots\;,\\
q(r) &= -\frac{3\mathrm{i}\sqrt{6}\left(3+(-3+f_0)s^2\right)}{s}r\\
&\quad+\frac{\mathrm{i}\left(3+(-3+f_0)s^2\right)\left(f_0^2s^2+9\left(-1+s^2\right)-6f_0\left(1+s^2\right)\right)}{6\sqrt{6}s}\frac{1}{r}+\frac{\xi_1}{r^2}+\dots\;,\\
f(r) &= f_0+\frac{2(-3+f_0)f_0}{9}\frac{1}{r^2}+\frac{\xi_2}{r^3}+\dots\;.
\end{aligned} \qquad \text{(B.3.14)}$$

The constants ξ_1 and ξ_2 are fixed by requiring regularity at the origin. Note that a particular case corresponds to $f_0=0$. In this case the $SU(2)_R$ background field is turned off, but the solution is still supersymmetric with a squashed five-sphere at the conformal boundary. In this case $F_4=F_2=0$, so we have enhanced supersymmetry; that is, this one-parameter family of solutions with $f_0=0$ is 1/2 BPS.

As an expansion around Euclidean AdS we parametrize the solution in terms of the expansion parameter δ and an extra parameter ω, related to s and f_0 above by

$$\frac{1}{s}\;=\;1+\delta\;,\qquad f_0\;=\;\delta\,\omega\;. \qquad \text{(B.3.15)}$$

With this choice the solution is given by

$$
\begin{aligned}
\alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}} + \frac{\sqrt{3}\left(1-54r^2+96\sqrt{6}r^3-324r^4+216r^6\right)}{2r^2\left(6r^2-1\right)^{7/2}}\delta + \dots\ , \\
\gamma(r) &= \frac{3\sqrt{6r^2-1}}{\sqrt{2}} + \frac{\left(15-48\sqrt{6}r+270r^2-540r^4+648r^6\right)}{\sqrt{2}\left(6r^2-1\right)^{5/2}}\delta + \dots\ , \\
X(r) &= 1 + \frac{\left(1-2\sqrt{6}r+6r^2\right)(4+\omega)}{\left(6r^2-1\right)^2}\delta + \dots\ , \\
p(r) &= -\frac{18\mathrm{i}\sqrt{2}\left(-\sqrt{3}+8\sqrt{2}r-12\sqrt{3}r^2+12\sqrt{3}r^4\right)(6+\omega)}{\left(6r^2-1\right)^3}\delta + \dots\ , \\
q(r) &= -\frac{3\mathrm{i}\left(-4+9\sqrt{6}r-24r^2-12\sqrt{6}r^3+36\sqrt{6}r^5\right)(6+\omega)}{\left(6r^2-1\right)^2}\delta + \dots\ , \\
f(r) &= \frac{\left(-3+8\sqrt{6}r-36r^2+36r^4\right)\omega}{\left(6r^2-1\right)^2}\delta + \dots\ . \qquad \text{(B.3.16)}
\end{aligned}
$$

As before it can be checked explicitly that the solution is regular at $r = \frac{1}{\sqrt{6}}$. We have computed this solution explicitly up to fourth order in δ. Comparing this expansion with the expansion around the conformal boundary we deduce

$$
\begin{aligned}
\xi_1 &= 2\mathrm{i}(6+\omega)\delta - \frac{1}{5}\mathrm{i}\left(144+98\omega+13\omega^2\right)\delta^2 \qquad \text{(B.3.17)} \\
&+ \frac{\mathrm{i}\left(307719+209547\omega+41094\omega^2+1282\omega^3\right)}{9450}\delta^3 \\
&- \frac{\mathrm{i}\left(26693550+21683700\omega+6126111\omega^2+771474\omega^3+51568\omega^4\right)}{623700}\delta^4 + \dots\ , \\
\xi_2 &= \frac{2}{3}\sqrt{\frac{2}{3}}\omega\delta - \frac{2}{45}\left(-\sqrt{6}\omega+2\sqrt{6}\omega^2\right)\delta^2 + \frac{\left(-999\sqrt{6}\omega-594\sqrt{6}\omega^2+244\sqrt{6}\omega^3\right)}{42525}\delta^3 \\
&+ \frac{\left(32724\sqrt{6}\omega+26082\sqrt{6}\omega^2+6105\sqrt{6}\omega^3+935\sqrt{6}\omega^4\right)}{1403325}\delta^4 + \dots\ . \qquad \text{(B.3.18)}
\end{aligned}
$$

B.4 Killing spinors

Having found the above supersymmetric solutions we now proceed to solve the dilatino equation (3.2.6) and Killing spinor equation (3.2.5) for the Killing spinors ϵ_I, $I = 1, 2$.

3/4 **BPS solution**

For the 3/4 to

$$\epsilon_1 = a_+^{(1)}\,e^{i\frac{\tau}{2}} \begin{pmatrix} k_2(r)\left[\cos\sigma + i\lambda_+(s)e^{i\frac{\psi}{2}}S_+^{(1)}\sin\sigma\right] \\ 0 \\ ik_3(r)\left[\sin\sigma - i\lambda_+(s)e^{i\frac{\psi}{2}}S_+^{(1)}\cos\sigma\right] \\ ik_3(r)\lambda_+(s)\,e^{-i\frac{\psi}{2}}S_+^{(2)} \\ -ik_4(r)\left[\cos\sigma + i\lambda_+(s)e^{i\frac{\psi}{2}}S_+^{(1)}\sin\sigma\right] \\ 0 \\ k_1(r)\left[\sin\sigma - i\lambda_+(s)e^{i\frac{\psi}{2}}S_+^{(1)}\cos\sigma\right] \\ k_1(r)\,\lambda_+(s)e^{-i\frac{\psi}{2}}S_+^{(2)} \end{pmatrix}, \tag{B.4.1}$$

$$\epsilon_2 = a_-^{(1)}\,e^{-i\frac{\tau}{2}} \begin{pmatrix} 0 \\ ik_4(r)\left[\cos\sigma - i\lambda_-(s)e^{-i\frac{\psi}{2}}S_-^{(1)}\sin\sigma\right] \\ -k_1(r)\lambda_-(s)\,e^{i\frac{\psi}{2}}S_-^{(2)} \\ k_1(r)\left[\sin\sigma + i\lambda_-(s)e^{-i\frac{\psi}{2}}S_-^{(1)}\cos\sigma\right] \\ 0 \\ k_2(r)\left[\cos\sigma - i\lambda_(s)e^{-i\frac{\psi}{2}}S_-^{(1)}\sin\sigma\right] \\ ik_3(r)\,\lambda_-(s)e^{i\frac{\psi}{2}}S_-^{(2)} \\ -ik_3(r)\left[\sin\sigma + i\lambda_-(s)e^{-i\frac{\psi}{2}}S_-^{(1)}\cos\sigma\right] \end{pmatrix}, \tag{B.4.2}$$

where we have introduced

$$\begin{aligned} S_\pm^{(1)} &= S_\pm^{(1)}(\theta,\varphi) &= a_\pm^{(3)}e^{\pm i\frac{\varphi}{2}}\cos\frac{\theta}{2} - a_\pm^{(2)}e^{\mp i\frac{\varphi}{2}}\sin\frac{\theta}{2}\,, \\ S_\pm^{(2)} &= S_\pm^{(2)}(\theta,\varphi) &= a_\pm^{(2)}e^{\mp i\frac{\varphi}{2}}\cos\frac{\theta}{2} + a_\pm^{(3)}e^{\pm i\frac{\varphi}{2}}\sin\frac{\theta}{2}\,, \\ & \lambda_\pm(s) &= \frac{\pm 1 + \sqrt{1-s^2}}{s}\,. \end{aligned} \tag{B.4.3}$$

The Killing spinors contain in total six constants of integration $a_{\pm}^{(i)}$, $i = 1, 2, 3$. These constants of integration are generically complex, but imposing the symplectic Majorana condition $\mathcal{C}\epsilon_I^* = \varepsilon_I{}^J \epsilon_J$ enforces certain reality conditions. The functions $k_i(r)$ are functions of the radial coordinate only and can be expanded either around Euclidean AdS or around the boundary. For instance, expanding around the conformal boundary we obtain

$$
\begin{aligned}
k_1(r) &= \frac{-1+\sqrt{1-s^2}}{s}\sqrt{r} + \frac{1}{2\sqrt{6}}\frac{1}{\sqrt{r}} + \ldots\,, \\
k_2(r) &= \sqrt{r} - \frac{5\sqrt{1-s^2}-3}{6\sqrt{6}s}\frac{1}{\sqrt{r}} + \ldots\,, \\
k_3(r) &= \frac{-1+\sqrt{1-s^2}}{s}\sqrt{r} - \frac{1}{2\sqrt{6}}\frac{1}{\sqrt{r}} + \ldots\,, \\
k_4(r) &= \sqrt{r} + \frac{5\sqrt{1-s^2}-3}{6\sqrt{6}s}\frac{1}{\sqrt{r}} + \ldots\,,
\end{aligned}
\tag{B.4.4}
$$

Notice that the expansion of the Killing spinor around the boundary is precisely of the form

$$
\epsilon_I = \begin{pmatrix} \epsilon_I^+ \\ \epsilon_I^- \end{pmatrix} = \sqrt{r}\begin{pmatrix} \chi_I \\ -\mathrm{i}\chi_I \end{pmatrix} + \frac{1}{\sqrt{r}}\begin{pmatrix} \varphi_I \\ \mathrm{i}\varphi_I \end{pmatrix} + \cdots\,, \tag{B.4.5}
$$

which arises from the general analysis of section 7 and should of course hold for our particular solution. This allows us to immediately identify the boundary five-dimensional Killing spinor χ_I corresponding to our bulk solution. Note that this precisely agrees with (2.2.15).

1/4 **BPS solution**

For the 1/4 BPS solution we find

$$\epsilon_1 = c_+ \, \mathrm{e}^{-\frac{3 \mathrm{i} \tau}{2}} \begin{pmatrix} 0 \\ k_2(r) \\ 0 \\ 0 \\ 0 \\ -\mathrm{i}\, k_1(r) \\ 0 \\ 0 \end{pmatrix} , \qquad \epsilon_2 = -c_- \, \mathrm{e}^{\frac{3 \mathrm{i} \tau}{2}} \begin{pmatrix} k_1(r) \\ 0 \\ 0 \\ 0 \\ -\mathrm{i}\, k_2(r) \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{B.4.6}$$

The solution depends now on two constants of integration $c_{\pm}$. The functions of the radial coordinate admit the following expansion around the conformal boundary

$$\begin{aligned} k_1(r) &= \sqrt{r} + \frac{(f_0 - 3)s}{6\sqrt{6}} \frac{1}{\sqrt{r}} + \frac{5(f_0-3)^2 s^2 + 6(4f_0 - 9)}{432} \left(\frac{1}{r}\right)^{3/2} + \dots \, , \\ k_2(r) &= \sqrt{r} - \frac{(f_0 - 3)s}{6\sqrt{6}} \frac{1}{\sqrt{r}} + \frac{5(f_0-3)^2 s^2 + 6(4f_0 - 9)}{432} \left(\frac{1}{r}\right)^{3/2} + \dots \, . \end{aligned} \tag{B.4.7}$$

As before, the corresponding Killing spinors at the boundary can be identified. In this case they are indeed of the form (2.2.14), as expected. Finally, let us mention that the supersymmetry gets enhanced for the case $f_0 = 0$ (or equivalently $\omega = 0$). In this limit the gauge field vanishes and so the two Killing spinors ϵ_I for $I = 1, 2$ decouple and have the same structure. They read

$$\epsilon_I \;=\; \begin{pmatrix} c_I^{(2)}\, k_1(r) \mathrm{e}^{\frac{3\mathrm{i}\tau}{2}} \\ c_I^{(1)}\, k_2(r) \mathrm{e}^{-\frac{3\mathrm{i}\tau}{2}} \\ 0 \\ 0 \\ -\mathrm{i}\, c_I^{(2)}\, k_2(r) \mathrm{e}^{\frac{3\mathrm{i}\tau}{2}} \\ -\mathrm{i}\, c_I^{(1)}\, k_1(r) \mathrm{e}^{-\frac{3\mathrm{i}\tau}{2}} \\ 0 \\ 0 \end{pmatrix}, \tag{B.4.8}$$

where $c_I^{(j)}$ for $j = 1, 2$ are the integration constants and where the r-dependent functions $k_i(r)$ are the same as in the 1/4 BPS case, with $f_0 = 0$. This solution may thus be referred to as a 1/2 BPS solution.

Appendix C

Asymptotics of multiple sine functions

Let us start by defining Barnes' multiple zeta function,

$$\zeta_{\mathcal{N}}\left(s,w\mid\mathbf{a}\right)\;\equiv\;\sum_{m_1,\ldots,m_{\mathcal{N}}=0}^{\infty}\left(w+m_1a_1+\cdots m_{\mathcal{N}}a_{\mathcal{N}}\right)^{-s}\;, \tag{C.0.1}$$

where $\mathbf{a}=(a_1,\ldots,a_{\mathcal{N}})$, $\mathrm{Re}\,w>0$, $\mathrm{Re}\,s>\mathcal{N}$ and $a_1,\ldots,a_{\mathcal{N}}>0$. This function is meromorphic in s, with simple poles at $s=1,\ldots,\mathcal{N}$. One can then define the Barnes multiple gamma function $\Gamma_{\mathcal{N}}(w\mid\mathbf{a})\equiv\exp\left[\Psi_{\mathcal{N}}\left(w\mid\mathbf{a}\right)\right]$, where

$$\Psi_{\mathcal{N}}\left(w\mid\mathbf{a}\right)\;\equiv\;\frac{\mathrm{d}}{\mathrm{d}s}\zeta_{\mathcal{N}}\left(s,w\mid\mathbf{a}\right)|_{s=0}\;\;. \tag{C.0.2}$$

In order to compute the asymptotics of the multiple gamma function, and the closely related multiple sine function, we have to express this function in a more convenient way. In [58], it was observed that there is an expansion of $\Psi_{\mathcal{N}}(w)$ of the form

$$\begin{aligned}\Psi_{\mathcal{N}}\left(w\mid\mathbf{a}\right)\;&=\;\frac{(-1)^{\mathcal{N}+1}}{\mathcal{N}!}B_{\mathcal{N},\mathcal{N}}(w)\log w+(-1)^{\mathcal{N}}\sum_{k=0}^{\mathcal{N}-1}\frac{B_{\mathcal{N},k}(0)w^{\mathcal{N}-k}}{k!(\mathcal{N}-k)!}\sum_{\ell=1}^{\mathcal{N}-k}\frac{1}{\ell}\\&+\sum_{k=\mathcal{N}+1}^{\mathcal{M}}\frac{(-1)^k}{k!}B_{\mathcal{N},k}\left(0\right)w^{\mathcal{N}-k}(k-\mathcal{N}-1)!+\mathcal{R}_{\mathcal{N},\mathcal{M}}(w)\,,\end{aligned} \tag{C.0.3}$$

where

$$\mathcal{R}_{\mathcal{N},\mathcal{M}}(w)\;\equiv\;\int_0^{\infty}\frac{\mathrm{d}t}{t}\mathrm{e}^{-wt}\left(\prod_{j=1}^{\mathcal{N}}\left(1-\mathrm{e}^{-a_jt}\right)^{-1}-\sum_{k=0}^{\mathcal{M}}\frac{(-1)^k}{k!}B_{\mathcal{N},k}(0)t^{k-\mathcal{N}}\right)\;, \tag{C.0.4}$$

and $\mathcal{M} \geq \mathcal{N}$ as well as $\mathrm{Re}\, w > 0$. The functions $B_{\mathcal{N},\mathcal{M}}(w)$ are the so-called multiple Bernoulli polynomials and can be determined by expanding and solving the following relation

$$\frac{t^{\mathcal{N}} \mathrm{e}^{xt}}{\prod_{j=1}^{\mathcal{N}} \left(\mathrm{e}^{a_j t} - 1\right)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_{\mathcal{N},n}(x) \ , \tag{C.0.5}$$

for $B_{\mathcal{N},\mathcal{M}}(w)$. It was further shown in [58] that in the asymptotic limit $|w| \to \infty$ and $|\arg w| < \pi$ the remainder $\mathcal{R}_{\mathcal{N},\mathcal{M}}(w)$ behaves as

$$\mathcal{R}_{\mathcal{N},\mathcal{M}}(w) = \mathcal{O}\left(w^{\mathcal{N}-\mathcal{M}-1}\right) , \tag{C.0.6}$$

and hence in the asymptotic limit is suppressed by the first three terms in (C.0.3). Similarly, the third term in (C.0.3) behaves as

$$\sum_{k=\mathcal{N}+1}^{\mathcal{M}} \frac{(-1)^k}{k!} B_{\mathcal{N},k}(0)\, w^{\mathcal{N}-k} (k - \mathcal{N} - 1)! = \mathcal{O}\left(w^{-1}\right) \ , \tag{C.0.7}$$

in the asymptotic limit $|w| \to \infty$. Hence for our purposes we shall only focus on the asymptotics of the first two contributions to $\Psi_{\mathcal{N}}$.

We are interested in the asymptotic expansion of the so-called multiple sine function, which is defined in terms of the Gamma function as

$$S_{\mathcal{N}}(w \mid \mathbf{a}) \equiv \Gamma_{\mathcal{N}}(w \mid \mathbf{a})^{-1}\, \Gamma_{\mathcal{N}}(a_{\text{tot}} - w \mid \mathbf{a})^{(-1)^{\mathcal{N}}} , \tag{C.0.8}$$

where $a_{\text{tot}} = \sum_{i=1}^{\mathcal{N}} a_i$. To compute the large N limit of the free energy, we are interested in the asymptotics of the logarithm of these functions

$$\log S_{\mathcal{N}}(w \mid \mathbf{a}) = -\Psi_{\mathcal{N}}(w \mid \mathbf{a}) - \Psi_{\mathcal{N}}(a_{\text{tot}} - w \mid \mathbf{a})^{(-1)^{\mathcal{N}}} . \tag{C.0.9}$$

Focusing on the case $\mathcal{N} = 3$, we find the following Bernoulli polynomials

$$\begin{aligned}
B_{3,0}(x) &= \frac{1}{a_1 a_2 a_3} , \\
B_{3,1}(x) &= \frac{x}{a_1 a_2 a_3} - \frac{a_{\text{tot}}}{2 a_1 a_2 a_3} , \\
B_{3,2}(x) &= \frac{x^2}{a_1 a_2 a_3} - \frac{a_{\text{tot}}}{a_1 a_2 a_3} x + \frac{a_{\text{tot}}^2 + (a_1 a_2 + a_1 a_3 + a_2 a_3)}{6 a_1 a_2 a_3} , \\
B_{3,3}(x) &= \frac{x^3}{a_1 a_2 a_3} - \frac{3 a_{\text{tot}}}{2 a_1 a_2 a_3} x^2 + \frac{a_{\text{tot}}^2 + (a_1 a_2 + a_1 a_3 + a_2 a_3)}{6 a_1 a_2 a_3} x \\
&\quad - \frac{a_{\text{tot}} (a_1 a_2 + a_1 a_3 + a_2 a_3)}{4 a_1 a_2 a_3} .
\end{aligned} \tag{C.0.10}$$

We can then compute (C.0.3) and take the asymptotic limit of the logarithm of the triple sine function to obtain

$$\begin{aligned}\log S_3(w \mid \mathbf{a}) \quad = \quad & \operatorname{sign}\operatorname{Re} w \left[\frac{\mathrm{i}\pi}{6a_1a_2a_3} w^3 - \frac{\mathrm{i}\pi a_{\mathrm{tot}}}{4a_1a_2a_3} w^2 + \frac{\mathrm{i}\pi \left(a_{\mathrm{tot}}^2 + a_1a_2 + a_1a_3 + a_2a_3\right)}{12a_1a_2a_3} \right. \\ & \left. - \frac{\mathrm{i}\pi a_{\mathrm{tot}} \left(a_1a_2 + a_1a_3 + a_2a_3\right)}{24a_1a_2a_3} + \mathcal{O}\left(w^{-1}\right) \right] . \end{aligned} \tag{C.0.11}$$

This procedure generalizes to any choice of $\mathcal{N}$, and gives a straightforward method to obtain the asymptotics of these functions.

Appendix D

Useful identities

From the dilatino equation (3.2.6) one can derive the following useful identities

$$
\begin{aligned}
(\partial^\mu X)\epsilon^\dagger[\mathbb{A},\Gamma_\mu]_\mp\epsilon \;&=\; -\tfrac{\mathrm{i}}{2\sqrt{2}}\left(X^2-X^{-2}\right)\epsilon^\dagger[\mathbb{A},\Gamma_7]_\pm\epsilon+\tfrac{1}{24}X^3H^{\mu\nu\rho}\epsilon^\dagger[\mathbb{A},\Gamma_{\mu\nu\rho}\Gamma_7]_\pm\epsilon \\
&\quad +\tfrac{\mathrm{i}}{12\sqrt{2}}B^{\mu\nu}\epsilon^\dagger[\mathbb{A},\Gamma_{\mu\nu}]_\pm\epsilon-\tfrac{1}{8\sqrt{2}}\mathcal{F}^{\mu\nu}\epsilon^\dagger[\mathbb{A},\Gamma_{\mu\nu}\Gamma_7]_\pm\epsilon\,, \qquad \text{(D.0.1)}
\end{aligned}
$$

$$
\begin{aligned}
(\partial^\mu X)\epsilon^{\mathrm{T}}[\mathbb{A},\Gamma_\mu]_\mp\epsilon \;&=\; -\tfrac{\mathrm{i}}{2\sqrt{2}}\left(X^2-X^{-2}\right)\epsilon^{\mathrm{T}}[\mathbb{A},\Gamma_7]_\mp\epsilon+\tfrac{1}{24}X^3H^{\mu\nu\rho}\epsilon^{\mathrm{T}}[\mathbb{A},\Gamma_{\mu\nu\rho}\Gamma_7]_\pm\epsilon \\
&\quad +\tfrac{\mathrm{i}}{12\sqrt{2}}B^{\mu\nu}\epsilon^{\mathrm{T}}[\mathbb{A},\Gamma_{\mu\nu}]_\mp\epsilon-\tfrac{1}{8\sqrt{2}}\mathcal{F}^{\mu\nu}\epsilon^{\mathrm{T}}[\mathbb{A},\Gamma_{\mu\nu}\Gamma_7]_\pm\epsilon\,. \qquad \text{(D.0.2)}
\end{aligned}
$$

Here $\mathbb{A}\in \mathrm{Cliff}(6,0)$ is an arbitrary element of the Clifford algebra, while $[\,\cdot\,,\,\cdot\,]_-$ denotes a commutator and $[\,\cdot\,,\,\cdot\,]_+$ denotes an anti-commutator.

Appendix E

Differential conditions for bilinears

We may introduce the following bilinears in the spinor ϵ:

$$\begin{aligned}
K &\equiv \epsilon^\dagger\Gamma_{(1)}\epsilon &=& \; S\sin 2\vartheta\, K_1 \ , \\
\tilde{K} &\equiv \mathrm{i}\epsilon^\dagger\Gamma_{(1)}\Gamma_7\epsilon &=& \; -S\sin 2\vartheta\, K_2 \ , \\
Y &\equiv \mathrm{i}\epsilon^\dagger\Gamma_{(2)}\epsilon &=& \; S(\cos 2\vartheta\, K_1\wedge K_2 - J) \ , \\
\tilde{Y} &\equiv \mathrm{i}\epsilon^\dagger\Gamma_{(2)}\Gamma_7\epsilon &=& \; S(-K_1\wedge K_2 + \cos 2\vartheta\, J) \ , \\
Z &\equiv \epsilon^{\mathrm{T}}\Gamma_{(2)}\Gamma_7\epsilon &=& \; -S\sin 2\vartheta\,\Omega \ , \\
V &\equiv \mathrm{i}\epsilon^\dagger\Gamma_{(3)}\epsilon &=& \; -S\sin 2\vartheta\, K_1\wedge J \ , \\
\tilde{V} &\equiv \epsilon^\dagger\Gamma_{(3)}\Gamma_7\epsilon &=& \; -S\sin 2\vartheta\, K_2\wedge J \ , \\
W &\equiv \epsilon^{\mathrm{T}}\Gamma_{(3)}\epsilon &=& \; S(-\cos 2\vartheta\, K_1 + \mathrm{i}\, K_2)\wedge\Omega \ , \\
\tilde{W} &\equiv \epsilon^{\mathrm{T}}\Gamma_{(3)}\Gamma_7\epsilon &=& \; S(K_1 - \mathrm{i}\,\cos 2\vartheta\, K_2)\wedge\Omega \ .
\end{aligned} \tag{E.0.1}$$

Here $(K_1, K_2, J, \Omega\,)$ is the canonical $SU(2)$ structure defined in section 8.1.

A straightforward but lengthy calculation shows that the Killing spinor equation (3.2.5) and dilatino equation (3.2.6) imply the following differential constraints on the bilinears in (E.0.1):

$$\mathrm{d}(XS) \;=\; \tfrac{\sqrt{2}}{3}(X^{-2}\tilde{K} - \mathrm{i}K\,\lrcorner\, B)\,, \tag{E.0.2}$$

$$\mathrm{d}(X\tilde{S}) \;=\; -\tfrac{1}{\sqrt{2}}K\,\lrcorner\,\mathcal{F}\,, \tag{E.0.3}$$

$$\mathrm{d}(X^2K) = -\tfrac{2\sqrt{2}}{3}X^{-1}\tilde{Y} - \mathrm{i}X^4 K \lrcorner *H - \sqrt{2}X(\tilde{S}\mathcal{F} - \mathrm{i}\tfrac{2}{3}SB)\,, \qquad (E.0.4)$$

$$\mathrm{d}(X^{-2}\tilde{K}) = -\mathrm{i}K \lrcorner H\,, \qquad (E.0.5)$$

$$\mathrm{d}(X^{-1}Y) = -\sqrt{2}\tilde{V} + \mathrm{i}(X\tilde{S})H + \tfrac{1}{\sqrt{2}}X^{-2}(K \lrcorner *\mathcal{F} + \mathcal{F}\wedge\tilde{K})\,, \qquad (E.0.6)$$

$$\mathrm{d}(X^{-1}\tilde{Y}) = \mathrm{i}(XS)H + \mathrm{i}\tfrac{\sqrt{2}}{3}X^{-2}(K \lrcorner *B + B\wedge\tilde{K})\,, \qquad (E.0.7)$$

$$D(X^{-1}Z) = -\mathrm{i}\sqrt{2}\,W\,, \qquad (E.0.8)$$

$$\mathrm{d}V = \sqrt{2}(X + \tfrac{1}{3}X^{-3}) * Y + \mathrm{i}\tfrac{\sqrt{2}}{3}X^{-1}(\tilde{S} * B + B\wedge Y)$$
$$-\tfrac{1}{\sqrt{2}}X^{-1}(S * \mathcal{F} + \mathcal{F}\wedge\tilde{Y})\,, \qquad (E.0.9)$$

$$\mathrm{d}\tilde{V} = 0\,, \qquad (E.0.10)$$

$$DW = -\tfrac{1}{\sqrt{2}}X^{-1}\mathcal{F}\wedge Z\,, \qquad (E.0.11)$$

$$D\tilde{W} = -\sqrt{2}(X + \tfrac{1}{3}X^{-3}) * Z - \mathrm{i}\tfrac{\sqrt{2}}{3}X^{-1}B\wedge Z\,, \qquad (E.0.12)$$

$$\mathrm{d}\Big[(X + \tfrac{1}{3}X^{-3}) * Y\Big] = \tfrac{\sqrt{2}}{3}\mathrm{i}B\wedge\tilde{V} - \tfrac{\mathrm{i}}{3}X^{-1}H\wedge Y + \tfrac{1}{3\sqrt{2}}X^{-4}(*\mathcal{F})\wedge\tilde{K}\,. \qquad (E.0.13)$$

Here the covariant derivatives are $D = \mathrm{d} + \mathrm{i}\mathcal{A}\wedge$, and the contraction of a p-form ρ into a q-form λ (with $q \geq p$) is the $(q-p)$-form $(\rho \lrcorner \lambda)_{\mu_1\cdots\mu_{q-p}} \equiv \frac{1}{p!}\rho^{\nu_1\cdots\nu_p}\lambda_{\nu_1\cdots\nu_p\mu_1\cdots\mu_{q-p}}$.

In addition to (E.0.2) – (E.0.13) it is also straightforward to show that K is a Killing one-form, so that the dual vector field $\xi \equiv K^{\#}$ is a Killing vector. We may hence introduce a local coordinate ψ, so that $\xi = \partial_\psi$ and the metric is independent of ψ. Since $K = S\sin 2\vartheta\, K_1$, where K_1 has unit length, we may thus write

$$K_1 = S\sin 2\vartheta\,(\mathrm{d}\psi + \sigma)\,, \qquad (E.0.14)$$

where $\mathcal{L}_\xi\sigma = 0 = i_\xi\sigma$ and $\mathcal{L}_\xi(S\sin 2\vartheta) = 0$.

In order to analyse the equations (E.0.2) – (E.0.13) further we write

$$B = B_1\wedge K_1 + B_\perp\,, \qquad \mathcal{F} = \mathcal{F}_1\wedge K_1 + \mathcal{F}_\perp\,, \qquad (E.0.15)$$

where $B_1, B_\perp, \mathcal{F}_1, \mathcal{F}_\perp$ are chosen to have zero contraction with K_1. The bilinear (E.0.2) then determines

$$B_1 = -\frac{3\mathrm{i}}{\sqrt{2}S\sin 2\vartheta}\mathrm{d}(XS) - \mathrm{i}X^{-2}K_2\,. \qquad (E.0.16)$$

Similarly the bilinear (E.0.3) is equivalent to

$$\mathcal{F}_1 \quad = \quad -\frac{\sqrt{2}}{S\sin 2\vartheta}\mathrm{d}(XS\cos 2\vartheta) \ . \qquad \text{(E.0.17)}$$

Contracting these last two equations with K_1, one concludes that $\mathcal{L}_\xi(XS) = 0 = \mathcal{L}_\xi\vartheta$. Notice also that setting $\mathbb{A} = 1_8$ in (D.0.1) and taking the anti-commutator leads immediately to $\mathcal{L}_\xi X = 0$. Having imposed (E.0.2), a short computation shows that equation (E.0.5) is equivalent to $\mathcal{L}_\xi B = 0$. One can also deduce from (E.0.5) that $\mathcal{L}_\xi K_2 = 0$, and similarly from (E.0.3) it follows that $\mathcal{L}_\xi \mathcal{F} = 0$. We may then write

$$\mathcal{A} \quad = \quad -\sqrt{2}X\cot 2\vartheta\, K_1 + \mathcal{A}_\perp \ . \qquad \text{(E.0.18)}$$

Notice here we have made a partial gauge choice for $\mathcal{A}$. Then

$$\mathcal{F}_\perp \quad = \quad -\sqrt{2}XS\cos 2\vartheta\,\mathrm{d}\sigma + \mathrm{d}\mathcal{A}_\perp \ . \qquad \text{(E.0.19)}$$

Next one can show that equation (E.0.4) is equivalent to

$$\begin{aligned} X^2S^2\sin^2 2\vartheta\,\mathrm{d}\sigma \quad &= \quad -\tfrac{2\sqrt{2}}{3}X^{-1}S\cos 2\vartheta\, J - \mathrm{i}X^4 S\sin 2\vartheta\, K_1 \lrcorner *H_\perp \\ &\quad +\sqrt{2}XS(\cos 2\vartheta\,\mathcal{F}_\perp + \tfrac{2}{3}\mathrm{i}B_\perp) \ . \qquad \text{(E.0.20)} \end{aligned}$$

Here we have defined

$$H_\perp \quad \equiv \quad \mathrm{i}\left[\tfrac{3}{\sqrt{2}}\mathrm{d}(XS) + X^{-2}S\sin 2\vartheta K_2\right]\wedge \mathrm{d}\sigma + \mathrm{d}B_\perp \ . \qquad \text{(E.0.21)}$$

The contractions of (E.0.6) and (E.0.7) with K_1 imply that $\mathcal{L}_\xi J = 0$. Equation (E.0.6) is then equivalent to

$$\begin{aligned} \mathrm{d}(X^{-1}SJ) \quad &= \quad -\sqrt{2}S\sin 2\vartheta\, J\wedge K_2 - \tfrac{3}{2\sqrt{2}}\cos 2\vartheta\,\mathrm{d}[(XS)^2\mathrm{d}\sigma] + \mathrm{i}XS\cos 2\vartheta\,\mathrm{d}B_\perp \\ &\quad -\tfrac{1}{\sqrt{2}}X^{-2}S\sin 2\vartheta\,[K_1 \lrcorner *\mathcal{F}_\perp - K_2\wedge\mathcal{F}_\perp] \ . \qquad \text{(E.0.22)} \end{aligned}$$

Similarly, one can show that (E.0.7) is equivalent to

$$\begin{aligned} \mathrm{d}(X^{-1}S\cos 2\vartheta\, J) \quad &= \quad -\tfrac{3}{2\sqrt{2}}\mathrm{d}[(XS)^2\mathrm{d}\sigma] + \mathrm{i}XS\,\mathrm{d}B_\perp \\ &\quad +\tfrac{\sqrt{2}}{3}\mathrm{i}X^{-2}S\sin 2\vartheta\,[K_1 \lrcorner *B_\perp - K_2\wedge B_\perp] \ . \qquad \text{(E.0.23)} \end{aligned}$$

The contraction of equation (E.0.8) with K_1, in the gauge in which $\mathcal{A}$ is given by (E.0.18), simply gives $\mathcal{L}_\xi \Omega = 0$. Equation (E.0.8) is then equivalent to

$$D_\perp (X^{-1} S \sin 2\vartheta\, \Omega) \;=\; -\sqrt{2} S \Omega \wedge K_2 \;, \tag{E.0.24}$$

where $D_\perp \equiv \mathrm{d} + \mathrm{i}\mathcal{A}_\perp \wedge$.

Finally we move onto the three-form bilinears. Equation (E.0.10) states

$$\mathrm{d}(S \sin 2\vartheta J \wedge K_2) \;=\; 0 \;. \tag{E.0.25}$$

The contraction of K_1 into (E.0.11) is equivalent to (E.0.24), while the remainder of this equation turns out to be the integrability condition for (E.0.24). Next one can show that K_1 contracted into (E.0.9) is implied by (E.0.22) and (E.0.23), while the remainder of this equation reads

$$\begin{aligned} -S^2 \sin^2 2\vartheta\, J \wedge \mathrm{d}\sigma \;&=\; \sqrt{2} S \cos 2\vartheta (X + \tfrac{2}{3} X^{-3}) \tfrac{1}{2} J \wedge J - 2 S K_1 \lrcorner * \mathrm{d}\vartheta \\ &\quad - \tfrac{1}{\sqrt{2}} X^{-1} S J \wedge (\cos 2\vartheta\, \mathcal{F}_\perp + \tfrac{2}{3} \mathrm{i} B_\perp) \;. \end{aligned} \tag{E.0.26}$$

Next we find that K_1 contracted into (E.0.12) is implied by (E.0.24). Using (E.0.24) the remainder of this equation reads

$$S^2 \sin^2 2\vartheta\, \Omega \wedge \mathrm{d}\sigma \;=\; -2\mathrm{i} S \mathrm{d}\vartheta \wedge K_2 \wedge \Omega + \tfrac{1}{\sqrt{2}} X^{-1} S \Omega \wedge (\cos 2\vartheta\, \mathcal{F}_\perp + \tfrac{2}{3} \mathrm{i} B_\perp) \;. \tag{E.0.27}$$

The contraction of K_1 into (E.0.13) can again be shown to follow from equations derived so far, while the remaining content of this equation is (on using various other equations) equivalent to

$$X^4 K_2 \lrcorner \mathrm{d}(X^{-3} S \sin 2\vartheta) + \sqrt{2} S (X^2 - \tfrac{2}{3} X^{-2}) + \tfrac{1}{\sqrt{2}} S J \lrcorner (\mathcal{F}_\perp + \tfrac{2}{3} \mathrm{i} \cos 2\vartheta B_\perp) \;=\; 0 \;. \tag{E.0.28}$$

Appendix F

More on the dilatino equation

In the Abelian case of interest, the dilatino equation (3.2.6) may be written as $\delta\chi = 0$, where we have introduced

$$\begin{aligned} \delta\chi \equiv & -\mathrm{i}X^{-1}\partial_\mu X\Gamma^\mu\epsilon + \tfrac{1}{2\sqrt{2}}\left(X - X^{-3}\right)\Gamma_7\epsilon + \tfrac{\mathrm{i}}{24}X^2 H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_7\epsilon \\ & -\tfrac{1}{12\sqrt{2}}X^{-1}B_{\mu\nu}\Gamma^{\mu\nu}\epsilon - \tfrac{\mathrm{i}}{8\sqrt{2}}X^{-1}\mathcal{F}_{\mu\nu}\Gamma^{\mu\nu}\Gamma_7\epsilon \ . \end{aligned} \qquad \text{(F.0.1)}$$

Recall here that $A^1_\mu = A^2_\mu = 0$, while $\mathcal{A}_\mu \equiv A^3_\mu$, with curvature $\mathcal{F} = \mathrm{d}\mathcal{A}$. The right hand side of (F.0.1) is an 8-component spinor, and thus $\delta\chi = 0$ comprise 8 algebraic equations for $\epsilon = \epsilon_+ + \epsilon_-$.

We begin by noting that neither of the definite chirality projections ϵ_+ nor ϵ_- can be identically zero. For if $\epsilon_\pm = 0$, respectively, then we in fact have an $SU(3)$ structure, rather than $SU(2)$ structure, and the bilinear $W \equiv \epsilon^{\mathrm{T}}\Gamma_{(3)}\epsilon = \Omega_\mp$ is the corresponding complex three-form. However, since the left hand side of equation (E.0.8) of appendix E is identically zero in this case, we would deduce that $\Omega_\mp = 0$ and hence $\epsilon_\mp = 0$.

On an open dense subset where $\epsilon_\pm$ are both non-zero, we then have that $\{\epsilon_\pm, \Gamma_\mu\epsilon^*_\pm\}$ span the positive and negative chirality spin bundles $\mathcal{S}^\pm$, respectively. Recall from (8.1.6) that $\epsilon_+ = \sqrt{S}\cos\vartheta\,\eta_1$, $\epsilon_- = \sqrt{S}\sin\vartheta\,\eta_2^*$, where η_1 and η_2 have unit norm. In an orthonormal frame $(e^1, \ldots, e^4, e^5 \equiv K_1, e^6 \equiv K_2)$ in which the canonical $SU(2)$ structure defined by η_1 and η_2 is given by (8.4.2), one can easily check that $\{\epsilon_+, \Gamma_1\epsilon^*_+, \Gamma_3\epsilon^*_+, \Gamma_5\epsilon^*_+\}$ form a basis for $\mathcal{S}^+$, while $\{\epsilon_-, \Gamma_1\epsilon^*_-, \Gamma_3\epsilon^*_-, \Gamma_5\epsilon^*_-\}$ form a basis for $\mathcal{S}^-$. Thus in order for the dilatino equation $\delta\chi = 0$ to hold, it is sufficient to check that the contraction of (F.0.1) with $\epsilon_\pm^\dagger$ and $\epsilon_\pm^{\mathrm{T}}\Gamma_\mu$ is zero. These are equivalent to two scalar and two one-form equations, respectively, that may be expressed in terms of the bilinears (E.0.1). Specifically, we may take the two scalar contractions to be

$$\begin{aligned}
\epsilon^\dagger\delta\chi &= -\mathrm{i}X^{-1}\partial_\mu X K^\mu + \tfrac{1}{2\sqrt{2}}\left(X - X^{-3}\right)\tilde{S} + \tfrac{\mathrm{i}}{24}X^2 H_{\mu\nu\rho}\tilde{V}^{\mu\nu\rho} \\
&\quad + \tfrac{\mathrm{i}}{12\sqrt{2}}X^{-1}B_{\mu\nu}Y^{\mu\nu} - \tfrac{1}{8\sqrt{2}}X^{-1}\mathcal{F}_{\mu\nu}\tilde{Y}^{\mu\nu}\,, \\
\epsilon^\dagger\Gamma_7\delta\chi &= X^{-1}\partial_\mu X\tilde{K}^\mu + \tfrac{1}{2\sqrt{2}}\left(X - X^{-3}\right)S - \tfrac{1}{24}X^2 H_{\mu\nu\rho}V^{\mu\nu\rho} \\
&\quad + \tfrac{\mathrm{i}}{12\sqrt{2}}X^{-1}B_{\mu\nu}\tilde{Y}^{\mu\nu} - \tfrac{1}{8\sqrt{2}}X^{-1}\mathcal{F}_{\mu\nu}Y^{\mu\nu}\,,
\end{aligned} \tag{F.0.2}$$

while the two one-form contractions are

$$\begin{aligned}
\epsilon^{\mathrm{T}}\Gamma_\sigma\delta\chi &= \tfrac{\mathrm{i}}{8}X^2 H_{\mu\nu\sigma}Z^{\mu\nu} - \tfrac{1}{12\sqrt{2}}X^{-1}B^{\mu\nu}W_{\mu\nu\sigma} - \tfrac{\mathrm{i}}{8\sqrt{2}}X^{-1}\mathcal{F}^{\mu\nu}\tilde{W}_{\mu\nu\sigma}\,, \\
\epsilon^{\mathrm{T}}\Gamma_\sigma\Gamma_7\delta\chi &= -\mathrm{i}X^{-1}\partial^\mu X Z_{\mu\sigma} - \tfrac{1}{8}X^2(*H)_{\mu\nu\sigma}Z^{\mu\nu} \\
&\quad - \tfrac{1}{12\sqrt{2}}X^{-1}B^{\mu\nu}\tilde{W}_{\mu\nu\sigma} - \tfrac{\mathrm{i}}{8\sqrt{2}}X^{-1}\mathcal{F}^{\mu\nu}W_{\mu\nu\sigma}\,.
\end{aligned} \tag{F.0.3}$$

The dilatino equation $\delta\chi = 0$ is thus equivalent to the the right hand sides of (F.0.2) and (F.0.3) being zero. A tedious, but straightforward, calculation shows that $\delta\chi = 0$ is implied by the differential constraints (8.2.8).

Appendix G

Integrability conditions for the $SU(2)$ structure

For what follows it will be convenient to record the component form of the Romans field equations in (3.2.1) and (3.2.3):

$$
\begin{aligned}
(\mathscr{E}_g)_{\mu\nu} &\equiv R_{\mu\nu} - 4X^{-2}\partial_\mu X\partial_\nu X - \left(\tfrac{1}{18}X^{-6} - \tfrac{1}{2}X^2 - \tfrac{2}{3}X^{-2}\right) g_{\mu\nu} \\
&\quad -\tfrac{1}{4}X^4(H_\mu{}^{\rho\sigma}H_{\nu\rho\sigma} - \tfrac{1}{6}g_{\mu\nu}H^{\rho\sigma\tau}H_{\rho\sigma\tau}) - \tfrac{2}{9}X^{-2}(B_\mu{}^\rho B_{\nu\rho} - \tfrac{1}{8}g_{\mu\nu}B^{\rho\sigma}B_{\rho\sigma}) \\
&\quad -\tfrac{1}{2}X^{-2}(F^i_\mu{}^\rho F^i_{\nu\rho} - \tfrac{1}{8}g_{\mu\nu}F^{i\rho\sigma}F^i_{\rho\sigma}) \ , \\
(\mathscr{E}_X) &\equiv \nabla^\mu(X^{-1}\partial_\mu X) + \left(\tfrac{1}{2}X^2 - \tfrac{2}{3}X^{-2} + \tfrac{1}{6}X^{-6}\right) - \tfrac{1}{24}X^4H^{\mu\nu\rho}H_{\mu\nu\rho} \\
&\quad +\tfrac{1}{16}X^{-2}(\tfrac{4}{9}B^{\mu\nu}B_{\mu\nu} + F^{i\mu\nu}F^i_{\mu\nu}) \ , \\
(\mathscr{E}_A)^\mu &\equiv \nabla_\nu(X^{-2}B^{\nu\mu}) - \tfrac{\mathrm{i}}{12}\varepsilon^{\mu\nu\rho\sigma\tau\kappa}B_{\nu\rho}H_{\sigma\tau\kappa} \ , \\
(\mathscr{E}_{A^i})^\mu &\equiv D_\nu(X^{-2}F^{i\nu\mu}) - \tfrac{\mathrm{i}}{12}\varepsilon^{\mu\nu\rho\sigma\tau\kappa}F^i_{\nu\rho}H_{\sigma\tau\kappa} \ , \\
(\mathscr{E}_B)^{\mu\nu} &\equiv \nabla_\rho(X^4H^{\rho\mu\nu}) - \tfrac{4}{9}X^{-2}B^{\mu\nu} - \tfrac{\mathrm{i}}{8}\varepsilon^{\mu\nu\rho\sigma\tau\kappa}(\tfrac{4}{9}B_{\rho\sigma}B_{\tau\kappa} + F^i_{\rho\sigma}F^i_{\tau\kappa}) \ . \qquad \text{(G.0.1)}
\end{aligned}
$$

The equations of motion are then $\mathscr{E}_{\text{field}} = 0$. The field A is the Stueckelberg one-form, that we set to zero using the gauge symmetry of the theory. Its equation of motion $\mathscr{E}_A = 0$ follows from taking the divergence of the B-field equation of motion $\mathscr{E}_B = 0$. We also introduce

$$\begin{aligned}(\mathscr{B}_F)_{\mu\nu\rho} &\equiv \nabla_{[\mu}B_{\nu\rho]} - \frac{1}{3}H_{\mu\nu\rho} \ , \\ (\mathscr{B}_{F^i})_{\mu\nu\rho} &\equiv D_{[\mu}F^i_{\nu\rho]} \ , \\ (\mathscr{B}_H)_{\mu\nu\rho\sigma} &\equiv \nabla_{[\mu}H_{\nu\rho\sigma]} \ . \end{aligned} \tag{G.0.2}$$

Note that $\mathscr{B}_{\text{field}}$ vanish automatically as a consequence of the Bianchi identities. For the Abelian case studied in the main text recall that $F^1_{\mu\nu} = F^2_{\mu\nu} = 0$ while $\mathcal{F}_{\mu\nu} \equiv F^3_{\mu\nu}$.

In what follows we will show that supersymmetry together with $(\mathscr{E}_B)_\perp = 0$ imply the equations of motion for all the fields. We begin by taking the exterior derivative of (E.0.4) to obtain

$$0 \ = \ -\tfrac{2\sqrt{2}}{3}\mathrm{d}(X^{-1}\tilde{Y}) - \mathrm{i}\,\mathrm{d}(X^4 K \lrcorner *H) + \tfrac{2\sqrt{2}}{3}\mathrm{i}\,\mathrm{d}[XSB] - \sqrt{2}\mathcal{F}\wedge \mathrm{d}(X\tilde{S}) \, . \tag{G.0.3}$$

Using (E.0.2), (E.0.3) and (E.0.7) then gives

$$0 \ = \ -\mathrm{i}\,\mathrm{d}(X^4 K \lrcorner *H)) - \tfrac{4}{9}\mathrm{i}K \lrcorner *B + \tfrac{4}{9}B\wedge(K\lrcorner B) + \mathcal{F}\wedge(K\lrcorner\mathcal{F}) \, . \tag{G.0.4}$$

Since $\mathcal{L}_\xi(X^4 * H) = 0$ it hence follows that $K_1 \lrcorner \mathscr{E}_B = 0$. Recall that

$$\mathscr{E}_B \ = \ K_1 \wedge (K_1 \lrcorner \mathscr{E}_B) + (\mathscr{E}_B)_\perp \ . \tag{G.0.5}$$

In general it is not true that supersymmetry implies $(\mathscr{E}_B)_\perp = 0$. We henceforth impose this equation, and continue our analysis by taking the exterior derivative of (E.0.13). After a computation we find this implies

$$\tfrac{2}{3}\mathrm{i}\left[\mathrm{d}(X^{-2} * B) + \mathrm{i}B\wedge H\right](X\tilde{S}) - \left[\mathrm{d}(X^{-2} * \mathcal{F}) + \mathrm{i}\mathcal{F}\wedge H\right](XS) \ = \ 0 \ . \tag{G.0.6}$$

Since $\mathscr{E}_B = 0$ implies $\mathscr{E}_A = 0$, (G.0.6) implies $\mathscr{E}_{\mathcal{A}} = 0$.

To obtain the remaining equations of motion, we may use the integrability conditions for the dilatino equation (3.2.6) and Killing spinor equation (3.2.5):

$$
\begin{aligned}
0 \;=\;& \mathrm{i}\left(\mathscr{E}_X\right)\epsilon_I - \tfrac{1}{6\sqrt{2}} X \left(\mathscr{E}_A\right)_\mu \Gamma^\mu \epsilon_I - \tfrac{\mathrm{i}}{4\sqrt{2}} X \left(\mathscr{E}_{A^i}\right)_\mu \Gamma^\mu \Gamma_7 (\sigma_i)_I{}^J \epsilon_J + \tfrac{\mathrm{i}}{8} X^{-2} \left(\mathscr{E}_B\right)_{\mu\nu} \Gamma^{\mu\nu}\Gamma_7 \epsilon_I \\
& - \tfrac{1}{12\sqrt{2}} X^{-1} \left(\mathscr{B}_F\right)_{\mu\nu\rho} \Gamma^{\mu\nu\rho}\epsilon_I - \tfrac{\mathrm{i}}{8\sqrt{2}} X^{-1} \left(\mathscr{B}_{F^i}\right)_{\mu\nu\rho} \Gamma^{\mu\nu\rho}\Gamma_7 (\sigma_i)_I{}^J \epsilon_J \\
& + \tfrac{\mathrm{i}}{24} X^{2} \left(\mathscr{B}_H\right)_{\mu\nu\rho\sigma} \Gamma^{\mu\nu\rho\sigma}\Gamma_7 \epsilon_I \ , \qquad \text{(G.0.7)}
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& \tfrac{1}{2}\left(\mathscr{E}_X\right)\Gamma_\mu \epsilon_I - \tfrac{1}{2}\left(\mathscr{E}_g\right)_{\mu\nu}\Gamma^\nu \epsilon_I - \tfrac{1}{8} X^{-2}\left(\mathscr{E}_B\right)^{\nu\rho}\Gamma_{\mu\nu\rho}\Gamma_7\epsilon_I \\
& - \tfrac{\mathrm{i}}{3\sqrt{2}} X \left(\mathscr{E}_A\right)_\mu \epsilon_I + \tfrac{1}{2\sqrt{2}} X \left(\mathscr{E}_{A^i}\right)_\mu \Gamma_7 (\sigma_i)_I{}^J \epsilon_J - \tfrac{1}{24} X^2 \left(\mathscr{B}_H\right)^{\nu\rho\sigma\tau}\Gamma_{\mu\nu\rho\sigma\tau}\Gamma_7\epsilon_I \\
& - \tfrac{\mathrm{i}}{2\sqrt{2}} X^{-1}\left(\mathscr{B}_F\right)_{\mu\nu\rho}\Gamma^{\nu\rho}\epsilon_I + \tfrac{3}{4\sqrt{2}} X^{-1}\left(\mathscr{B}_{F^i}\right)_{\mu\nu\rho}\Gamma^{\nu\rho}\Gamma_7 (\sigma_i)_I{}^J \epsilon_J \ . \qquad \text{(G.0.8)}
\end{aligned}
$$

Since $\mathscr{B}_{\text{field}} = 0$, and given the results above, (G.0.7) immediately implies $\mathscr{E}_X = 0$. Using this, and contracting (G.0.8) with $\epsilon^\dagger \Gamma^\upsilon$, we deduce the Einstein equation $\mathscr{E}_g = 0$.

Appendix H

Supersymmetry of the fundamental string

We have seen that it is possible to get the exact Wilson loop and that the computation can be simplified using the $SU(2)$ structure we set up. In particular, a key formula is the action of the fundamental string, which can be written in terms of the two form $\mathrm{d}[XS(\mathrm{d}\psi + \sigma)]$.

What we have not shown is that this fundamental string is in fact supersymmetric. This is quite important, since the Wilson loop is supposed to be a BPS object. Unfortunately, just taking a fundamental string and putting it in a 2-manifold Σ_2 such that its action can be written as

$$S_{\text{string}} = \int_{\Sigma_2} X^{-2}\mathrm{vol}_2 + \mathrm{i}B - \frac{3}{\sqrt{2}}\mathrm{length}(\partial\Sigma_2) \ , \qquad \text{(H.0.1)}$$

does not imply supersymmetry.

A very concrete calculation was made for a ten-dimensional space of the form $AdS_6 \times S^4$ in [45]. Here the fundamental string is sitting at the pole $\alpha = \pi/2$. Note that to generalize this to our case, one has to replace the AdS_6 for a more generic space $\mathcal{M}_6$.

In order to show the supersymmetry of the embedding, one needs to take the full ten-dimensional geometry and show that the projection condition is satisfied, i.e., that the ten-dimensional Killing spinor satisfies

$$\mathbf{P}\epsilon = 0, \qquad \text{(H.0.2)}$$

where (see [59])

$$\mathbf{P} \equiv \frac{1}{2}\Big(1 - \frac{\mathrm{i}}{2}\varepsilon^{ij}\partial_i X^M \partial_j X^N \Gamma_{MN}\Big). \tag{H.0.3}$$

Here Γ_{MN} is our Clifford algebra in the ten dimensions and $\partial_i X^M \partial_j X^N$ is the pull back of the gamma matrices to the world volume.

Both spinor and gamma matrices can be written as a product of the six-dimensional quantities with the four-dimensional quantities, i.e.,

$$\epsilon_{(10)} = \epsilon_{(6)} \otimes \epsilon_{(4)} \quad \text{and} \tag{H.0.4}$$

$$\Gamma_{(10)} = \gamma_{(6)} \otimes \rho_{(4)}. \tag{H.0.5}$$

This way, we can limit ourselves to check the six-dimensional case.

The Clifford algebra in the internal space S^4 can be written more specifically as

$$\begin{aligned} \rho_0 &= \sigma \otimes \mathbf{1}_2, \\ \rho_i &= \sigma_3 \otimes \rho_i, \end{aligned} \tag{H.0.6}$$

where ρ_i are the Pauli matrices, in the S^3. The chirality operator here will be given by

$$\rho_4 = \rho_0\rho_1\rho_2\rho_3 = -\mathrm{i}\sigma_1 \otimes \mathbf{1}_2. \tag{H.0.7}$$

One can then rewrite the gamma matrices in ten dimensions as

$$\Gamma_m = \gamma_m \otimes \rho_4 \tag{H.0.8}$$

$$\Gamma_a = \mathbf{1}_8 \otimes \rho_a. \tag{H.0.9}$$

Where the ρ_4 in (H.0.8) is necessary so that Γ_m and Γ_a anticommute among each other.

One can also write

$$\Gamma_{10} = \mathrm{i}\gamma_1 \ldots \gamma_6 \otimes \rho_0 \ldots \rho_3. \tag{H.0.10}$$

The imaginary unit assures that $\Gamma_{10}^2 = 1$. The spinor in ten dimensions is supposed to be chiral, i.e., $\epsilon_{(10)}$ is such that $\Gamma_{10}\epsilon_{(10)} = \epsilon_{(10)}$

Now notice that the Wilson loop wraps K_1 and K_2, that are in the γ_5 and γ_6 directions respectively (if we take in account [56]), that is where our fundamental string is sitting. So that the projection in ten dimensions can be written as

$$(1 - \mathrm{i}\Gamma_{56})\epsilon_{(10)} = 0, \tag{H.0.11}$$

which is equivalent to

$$(1 - \mathrm{i}\gamma_{56} \otimes \rho_4)\epsilon_{(10)} = 0, \tag{H.0.12}$$

In [45], one see that the dilatino equation in ten dimensions reduces to a condition involving only the S^4 space, namely

$$\epsilon = [\cos\alpha\Gamma^6 - \sin\alpha\Gamma^{6789}]\epsilon. \tag{H.0.13}$$

Recall that our fundamental string is sitting at the pole $\alpha = \pi/2$, so that this equation reduces to

$$\epsilon = -\rho_4\epsilon. \tag{H.0.14}$$

Note that, since $\Gamma_{10}\epsilon = \epsilon$, this also implies that $\gamma_7\epsilon_{(6)} = -\epsilon_{(6)}$. One can then split the equation (H.0.12) into the four and the six dimensional parts and sneak in a γ_7 in the six dimensional part. I.e.,

$$(1 + \mathrm{i}\gamma_7\gamma_{56} \otimes \rho_4)\epsilon_{(6)} = 0, . \tag{H.0.15}$$

This can be regarded as the projection condition over our 6d spinor.

Indeed, recall that our 6d spinor can be written as

$$\epsilon = \epsilon_+ + \epsilon_-, \tag{H.0.16}$$

where

$$\epsilon_+ = \sqrt{S}\cos\alpha\ \ \eta_1, \tag{H.0.17}$$

$$\epsilon_- = \sqrt{S}\sin\alpha\ \ \eta_2^*. \tag{H.0.18}$$

We also have that

$$\gamma_7\eta_1 = -\eta_1 \tag{H.0.19}$$

and

$$\gamma_7 \eta_2^* = \eta_2^*. \tag{H.0.20}$$

Considering still [56], one sees that

$$-\gamma_{56} \eta_1 = \mathrm{i} \eta_1 \tag{H.0.21}$$

and

$$-\gamma_{56} \eta_2 = \mathrm{i} \eta_2, \tag{H.0.22}$$

or yet

$$\gamma_{56} \eta_2^* = \mathrm{i} \eta_2^* \tag{H.0.23}$$

Putting this all together, one finds that

$$-\mathrm{i} \gamma_7 \gamma_{56} \epsilon = \epsilon. \tag{H.0.24}$$

So one can write

$$(1 + \mathrm{i} \gamma_7 \gamma_{56}) \epsilon = 0, \tag{H.0.25}$$

which is our projection condition being satisfied. It is clear then that our fundamental string is indeed supersymmetric.

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