

Contributions to the model theory of henselian fields



Konstantinos Kartas
Lady Margaret Hall
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2022

Acknowledgements

I wholeheartedly thank my supervisors, E. Hrushovski and J. Koenigsmann. I will never forget how generous they have been, both with their time and also with their ideas. I would like to thank T. Scanlon for his hospitality while I was a visitor at UC Berkeley and for several stimulating discussions. Likewise, my research stay in Münster has been a pleasure, thanks to F. Jahnke who has been an excellent host and collaborator. I am grateful to all the participants in a study group that took place at the Fields Institute, for their meticulous readings of my work in chapter 3. I also wish to thank F.-V. Kuhlmann, D. Rössler and M. Temkin for helpful discussions related to chapter 5. Finally, I would like to thank Margarita for invaluable support throughout this journey.

During this research, the author was funded by EPSRC grant EP/20998761 and was also supported by the Onassis Foundation - Scholarship ID: F ZP 020-1/2019-2020.

Abstract

The thesis addresses certain problems in the model theory of henselian fields, with a special focus on decidability. Some new methods are introduced along the way, which are of independent interest. The following results are obtained:

- A transfer theorem for perfectoid fields, saying that a perfectoid field K is decidable relative to its tilt K^\flat , modulo a subtle (albeit natural) condition on K . As an application, we prove that the fields $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p(\zeta_{p^\infty})$ admit decidable maximal immediate extensions, thereby obtaining some of the first few decidability results for tame fields of mixed characteristic.
- A model-theoretic proof of the Fontaine-Wintenberger theorem, which states that the absolute Galois groups of K and K^\flat are canonically isomorphic. (joint with F. Jahnke).
- A general existential Ax-Kochen/Ershov principle for tamely ramified fields in all characteristics, conditional to a certain form of resolution of singularities. This extends well-known existential Ax-Kochen/Ershov principles in residue characteristic 0 and also unramified mixed characteristic. It also encompasses the (conditional) existential decidability result of Denef-Schoutens for $\mathbb{F}_p((t))$, which we also strengthen by replacing the assumption of (global) resolution with local uniformization.
- An undecidability result for the asymptotic theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ in the language of valued fields with a cross-section. The proof goes via reduction to positive characteristic, à la Krasner-Kazhdan-Deligne, ultimately adapting Pheidas' proof of the undecidability of $\mathbb{F}_p((t))$ with a cross-section. This answers a variant of a question of Derakhshan-Macintyre.
- An undecidability result for power series fields $k((t))$, equipped with a *total residue* map $\text{res} : k((t)) \rightarrow k$, which picks out the constant term of the Laurent series. Becker-Denef-Lipschitz showed that $\text{res} : k((t)) \rightarrow k$ is definable in the language of rings with a parameter for t , when the base field k is *finite*. We show that $(k((t)), \text{res})$ is undecidable, whenever k is *infinite*.

Contents

1	Preliminaries	1
1.1	Logic	1
1.2	Algebraic geometry	6
1.3	Valued fields	12
2	Ax-Kochen/Ershov principles	19
2.1	A result by van den Dries	21
2.2	Existential AKE in mixed characteristic	24
2.3	Decidability	26
2.4	Ax-Kochen/Ershov with cross-section	27
3	Perfectoid fields	33
3.1	Perfectoid fields	37
3.2	Relative decidability for perfectoid fields	50
3.3	Applications: Tame fields of mixed characteristic	58
3.4	Applications: Congruences modulo p	61
3.5	Final remarks	63
4	Interlude: Fontaine-Wintenberger	67
4.1	Taming perfectoid fields	69
4.2	Elementary Fontaine-Wintenberger	72
4.3	Classical Fontaine-Wintenberger	73
5	Tamely ramified fields	79
5.1	Tamely ramified extensions	81
5.2	RV-Hensel's Lemma	84
5.3	Existential transfer for tamely ramified fields	88
5.4	Applications	91
5.5	Revisiting Denef-Schoutens	96
5.6	Local uniformization	100

6	The theory of all p-adic fields	105
6.1	Local field approximation	107
6.2	Truncations of $(\mathbb{N}; 0, 1, +, _p)$	109
6.3	Proof of the main Theorem	114
6.4	Final remarks	115
7	Valued fields with a total residue map	117
7.1	Valued Fields with a total residue map	119
7.2	Undecidability of $k((t))$ with a total residue map	126

Notation

- If (K, v) is a valued field, we denote by \mathcal{O}_v the valuation ring. If the valuation is clear from the context, we shall also write \mathcal{O}_K .
- We write Γ_v for the value group and k_v for the residue field. If the valuation v is clear from the context, we also denote them by Γ and k respectively.
- When (K, v) is of mixed characteristic, we always write p for the residue characteristic $\text{char}(k)$.
- The notation $p^n\mathcal{O}_v$ stands for the ideal of \mathcal{O}_v generated by the element p^n . If both the field in question and the valuation v are clear from the context, we shall write (p^n) for $p^n\mathcal{O}_v$.
- \mathbb{Z}_p^{ur} : The valuation ring of (\mathbb{Q}_p^{ur}, v_p) , the maximal unramified extension of \mathbb{Q}_p equipped with the unique extension of the p -adic valuation.
- \mathbb{Z}_p^{ab} : The valuation ring of (\mathbb{Q}_p^{ab}, v_p) , the maximal Galois extension of \mathbb{Q}_p whose Galois group over \mathbb{Q}_p is abelian.
- We write $\mathbb{F}_p((t))^{1/p^\infty}$ (resp. $\overline{\mathbb{F}}_p((t))^{1/p^\infty}$) for the perfect hull of $\mathbb{F}_p((t))$ (resp. $\overline{\mathbb{F}}_p((t))$).
- For a given language L , we denote by Sent_L the set of L -sentences and by Form_L the set of L -formulas. If M is an L -structure and $A \subseteq M$ is an arbitrary subset, we write $L(A)$ for the language L enriched with a constant symbol c_a for each element $a \in A$. The L -structure M can be updated into an $L(A)$ -structure in the obvious way.
- Given a language L , an L -structure M and $A \subseteq M$, we write $L(A)$ for the language L enriched with constant symbols for the elements of A . We write $\text{Diag}_M(A)$ for the atomic diagram of A in M .
- Given L -structures M, N with a common substructure A , we use the notation $M \equiv_{\exists, A} N$ in L to mean that the structures M and N are existentially elementary equivalent in the language $L(A)$.
- We write $L_{\text{rings}} = \{0, 1, +, \cdot\}$ for the language of rings, $L_{\text{oag}} = \{0, +, <\}$ for the language of ordered abelian groups, $L_{\text{val}} = L_{\text{rings}} \cup \{\mathcal{O}\}$ for the language of valued fields (where \mathcal{O} is a unary predicate \mathcal{O} for the valuation ring), $L_t = L_{\text{val}} \cup \{t\}$, where t is a constant symbol whose intended interpretation will always be clear from the context.

- A local ring (R, \mathfrak{m}_R) may be viewed as an L_{loc} -structure, where L_{loc} is the *language of local rings* consisting of the language of rings L_{rings} together with a unary predicate \mathfrak{m} , whose intended interpretation is the maximal ideal $\mathfrak{m}_R \subseteq R$.
- Let X be a scheme over a discrete valuation ring R with residue field κ and fraction field K . Let s be the closed point of $\text{Spec}R$ and η be its generic point. We denote by X_s the special fiber $X \times_{\text{Spec}R} \text{Spec}(\kappa)$ and by X_K the generic fiber $X \times_{\text{Spec}R} \text{Spec}K$.
- Given an R -algebra A , we denote by $X(A)$ the set of A -integral points, i.e., the set of morphisms $\text{Spec}(A) \rightarrow X$ over $\text{Spec}R$. If $X = \text{Spec}B$, where B is a finitely generated R -algebra of the form $B = R[x_1, \dots, x_n]/(f_1, \dots, f_m)$, this can be identified with the set of tuples $(a_1, \dots, a_n) \in A^n$ such that $f_1(a_1, \dots, a_n) = \dots = f_m(a_1, \dots, a_n) = 0$. If $\text{Frac}(A) = L$, we also speak about the underlying L -rational point of an A -integral point $P : \text{Spec}(A) \rightarrow X$, which is simply the induced morphism $\text{Spec}(L) \rightarrow X_K$.

1

Preliminaries

1.1 Logic

We collect some basic definitions and facts from logic.

1.1.1 Decidability

Introduction

Fix a *countable* language L and let Sent_L be the set of well-formed L -sentences, identified with \mathbb{N} via some Gödel numbering. Let M be an L -structure. Recall that M is decidable if we have an algorithm to decide whether $M \models \phi$, for any given $\phi \in \text{Sent}_L$. More formally, let $\chi_M : \text{Sent}_L \rightarrow \{0, 1\}$ be the characteristic function of $\text{Th}(M) \subseteq \text{Sent}_L$. We say that M is *decidable* if χ_M is recursive.

Uniform decidability

Definition 1.1.2. For each $n \in \mathbb{N}$, let $f_n : \mathbb{N} \rightarrow \mathbb{N}$ be a function. The sequence $(f_n)_{n \in \omega}$ is uniformly recursive if the function $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} : (n, m) \mapsto f_n(m)$ is recursive.

This concept is best illustrated with a *non-example*:

Example 1.1.3. Let $A \subseteq \mathbb{N}$ be non-recursive. For each $n \in \mathbb{N}$, let $\delta_n : \mathbb{N} \rightarrow \mathbb{N} : m \mapsto \delta_{nm}$, where δ_{nm} is the Kronecker symbol. For each $n \in \mathbb{N}$, define $f_n : \mathbb{N} \rightarrow \mathbb{N}$ to be (1) δ_n if $n \in A$ and (2) identically 0 if $n \notin A$. One readily verifies that each individual f_n is recursive. On the other hand, the sequence $(f_n)_{n \in \omega}$ is not uniformly

recursive. Indeed, otherwise we could solve the membership problem for A , using that $n \in A \iff f_n(n) = 1$.

Using some Gödel numbering, we can also state a version of Definition 1.1.2 for sequences of functions $(f_n)_{n \in \omega}$, where $\text{dom}(f_n) = \text{Sent}_L$ (or $\text{cdm}(f_n) = \text{Sent}_L$). One can then define a notion of *uniform* decidability for sequences of L -structures:

Definition 1.1.4. A sequence $(M_n)_{n \in \omega}$ of L -structures is *uniformly decidable* if the sequence of functions $(\chi_{M_n})_{n \in \omega}$ is uniformly recursive, i.e. if the function $\chi : \mathbb{N} \times \text{Sent}_L \rightarrow \{0, 1\} : (n, \phi) \mapsto \chi_{M_n}(\phi)$ is recursive.

Remark 1.1.5. If the sequence $\chi : \mathbb{N} \times \text{Sent}_L \rightarrow \{0, 1\} : (n, \phi) \mapsto \chi_{M_n}(\phi)$ is recursive when restricted to *existential* sentences, we naturally say that the sequence $(M_n)_{n \in \omega}$ is *uniformly existentially decidable*. Other syntactic variants may be defined analogously.

1.1.6 Interpretability

Our formalism follows closely §5.3 [Hod93], where details and proofs may be found.

Definition

Given a language L , an *unnested* atomic L -formula is one of the form $x = y$ or $x = c$ or $F(\bar{x}) = y$ or $R\bar{x}$, where x, y are variables, c is a constant symbol, F is a function symbol and R is a relation symbol of the language L .

Definition 1.1.7. An n -dimensional interpretation of an L -structure M in the L' -structure N is a triple consisting of:

1. An L' -formula $\partial_\Gamma(x_1, \dots, x_n)$.
2. A map $\phi \mapsto \phi_\Gamma$, that takes an unnested atomic L -formula $\phi(x_1, \dots, x_m)$ and sends it to an L' -formula $\phi_\Gamma(\bar{y}_1, \dots, \bar{y}_m)$, where each \bar{y}_i is an n -tuple of variables.
3. A surjective map $f_\Gamma : \partial_\Gamma(N^n) \twoheadrightarrow M$.

such that for all unnested atomic L -formulas $\phi(x_1, \dots, x_m)$ and all $\bar{a}_i \in \partial_\Gamma(N^n)$, we have

$$M \models \phi(f_\Gamma(\bar{a}_1), \dots, f_\Gamma(\bar{a}_m)) \iff N \models \phi_\Gamma(\bar{a}_1, \dots, \bar{a}_m)$$

An *interpretation* of an L -structure M in the L' -structure N is an n -dimensional interpretation, for some $n \in \mathbb{N}$. In that case, we also say that M is *interpretable* in N . The formulas ∂_Γ and ϕ_Γ (for all unnested atomic ϕ) are the *defining formulas* of Γ .

Interpretability is a *transitive* relation on the class of structures, i.e. if the L -structure M is interpretable in the L' -structure N and N is interpretable in the L'' -structure P , then there exists a *composite* interpretation of M in P (Exercise 2, pg. 218 [Hod93]).

If N is an L' -structure and $\bar{a} = (a_1, \dots, a_m) \in N^m$, we write (N, \bar{a}) for the expansion of N in the language $L(\bar{c})$, which is L together with an m -tuple of constant symbols (c_1, \dots, c_m) with $c_i^{(N, \bar{a})} = a_i$. If M is interpretable in (N, \bar{a}) , for some $\bar{a} \in N^m$, we say that M is interpretable in N *with parameters*.

Proposition 1.1.8 (Reduction Theorem 5.3.2 [Hod93]). Let Γ be an n -dimensional interpretation of an L -structure M in the L' -structure N . There exists a map $\phi \mapsto \phi_\Gamma$, extending the map of Definition 1.1.7(2), such that for every L -formula $\phi(x_1, \dots, x_m)$ and all $\bar{a}_i \in \partial_\Gamma(N^n)$, we have that

$$M \models \phi(f_\Gamma(\bar{a}_1), \dots, f_\Gamma(\bar{a}_m)) \iff N \models \phi_\Gamma(\bar{a}_1, \dots, \bar{a}_m)$$

Proof. We describe how $\phi \mapsto \phi_\Gamma$ is built, for completeness (omitting details). By Corollary 2.6.2 [Hod93], every L -formula is equivalent to one in which all atomic subformulas are unnested. One can then construct $\phi \mapsto \phi_\Gamma$ by induction on the complexity of formulas. The base case is handled by Definition 1.1.7(2). This definition extends inductively according to the following rules:

1. $(\neg\phi)_\Gamma = \neg(\phi)_\Gamma$.
2. $(\bigwedge_{i=1}^n \phi_i)_\Gamma = \bigwedge(\phi_i)_\Gamma$.
3. $(\forall\phi)_\Gamma = \forall x_1, \dots, x_n (\partial_\Gamma(x_1, \dots, x_n) \rightarrow \phi_\Gamma)$
4. $(\exists\phi)_\Gamma = \exists x_1, \dots, x_n (\partial_\Gamma(x_1, \dots, x_n) \wedge \phi_\Gamma)$

The resulting map satisfies the desired conditions of the Proposition. □

Definition 1.1.9. The map $\text{Form}_L \rightarrow \text{Form}_{L'} : \phi \mapsto \phi_\Gamma$ constructed in the proof of Proposition 1.1.8 is called the *reduction* map of the interpretation Γ .

Complexity of interpretations

The complexity of the defining formulas of an interpretation defines a measure of complexity of the interpretation itself:

Definition 1.1.10 (§5.4(a) [Hod93]). An interpretation Γ of an L -structure M in an L' -structure N is quantifier-free if the defining formulas of Γ are quantifier-free. Other syntactic variants are defined analogously (e.g., existential interpretation).

Remark 1.1.11. (a) The reduction map of a positive existential interpretation sends positive existential formulas to positive existential formulas.

(b) The reduction map of an existential interpretation sends *positive existential* formulas to existential formulas but does *not* necessarily send existential formulas to existential formulas.

Lemma 1.1.12. If the L -structure M is \exists^+ -interpretable in the L' -structure N and N is \exists^+ -interpretable in the L'' -structure P , then the composite interpretation of M in P is also an \exists^+ -interpretation.

Proof. Clear. □

Recursive interpretations

Definition 1.1.13 (Remark 4, pg. 215 [Hod93]). Suppose L is a recursive language. Let Γ be an interpretation of an L -structure M in the L' -structure N . We say that the interpretation Γ is *recursive* if the map $\phi \mapsto \phi_\Gamma$ on unnested atomic formulas is recursive.

Remark 1.1.14 (Remark 4, pg. 215 [Hod93]). If Γ is a recursive interpretation of an L -structure M in the L' -structure N , then the reduction map of Γ is also recursive.

Turing reducibility of theories

See Definition 14.3 [Pap94] for the formal definition of a Turing machine with an oracle.

Definition 1.1.15. A theory T is *Turing reducible* to a theory T' if there is a Turing machine which decides membership in T using an *oracle* for T' .

Remark 1.1.16. In particular, if T is Turing reducible to T' and T' is decidable, then so is T .

Example 1.1.17. If Γ is a recursive interpretation of an L -structure M in the L' -structure N , then $\text{Th}(M)$ is Turing reducible to $\text{Th}(N)$. Indeed, for any given $\phi \in \text{Sent}_L$, we have that $M \models \phi \iff N \models \phi_\Gamma$, where $\phi \mapsto \phi_\Gamma$ is the reduction map of Γ . This furnish us with an algorithm to decide whether $M \models \phi$ using an oracle for $\text{Th}(N)$.

Uniformly recursive interpretations

Definition 1.1.18. Suppose L and L' are languages. Let $(M_n)_{n \in \omega}$ be a sequence of L -structures and $(N_n)_{n \in \omega}$ be a sequence of L' -structure. For each $n \in \mathbb{N}$, let Γ_n be an interpretation of M_n in N_n . We say that the sequence of interpretations $(\Gamma_n)_{n \in \omega}$ is *uniformly recursive* if the sequence of reduction maps $(\phi \mapsto \phi_{\Gamma_n})_{n \in \omega}$ on unnested atomic formulas is uniformly recursive, i.e. if the map $(n, \phi) \mapsto \phi_{\Gamma_n}$ is recursive.

If an L -structure M is interpretable in the L' -structure N and the latter is decidable, then so is the former. It is not hard to prove the following uniform version:

Proposition 1.1.19. Suppose L is a recursive language, $(M_n)_{n \in \omega}$ a sequence of L -structures and N is an L' -structure. Suppose N is decidable, Γ_n is an interpretation of M_n in N and the sequence of interpretations $(\Gamma_n)_{n \in \omega}$ is uniformly recursive. Then the sequence $(M_n)_{n \in \omega}$ is uniformly decidable.

Proof. Rephrasing Proposition 1.1.8 for sentences, yields $\chi_{M_n}(\phi) = \chi_N(\phi_{\Gamma_n})$ for every $\phi \in \text{Sent}_L$. It follows that the map $(n, \phi) \mapsto \chi_{M_n}(\phi)$ is equal to the map $(n, \phi) \mapsto \phi_{\Gamma_n} \mapsto \chi_N(\phi_{\Gamma_n})$ and the latter is recursive as a composition of recursive functions. \square

If the interpretation of M in N is recursive and so is the interpretation of N in P , then the composite interpretation of M in P is recursive as well. Indeed, recursive functions are closed under composition. One also has a uniform version:

Lemma 1.1.20. Let $(M_n)_{n \in \omega}$ be a sequence of L -structures, $(N_n)_{n \in \omega}$ be a sequence of L' -structures and $(P_n)_{n \in \omega}$ be a sequence of L'' -structures. For each $n \in \mathbb{N}$, let Γ_n be an interpretation of M_n in N_n and Δ_n be an interpretation of N_n in P_n and suppose that the sequences of interpretations $(\Gamma_n)_{n \in \omega}$ and $(\Delta_n)_{n \in \omega}$ are uniformly recursive. Let E_n be the composite interpretation of M_n in P_n . Then the sequence of interpretations $(E_n)_{n \in \omega}$ is uniformly recursive.

Proof. Clear. \square

1.1.21 Ultraproducts

We now set our notation on ultraproducts. Fix an ultrafilter U on some index set I . Given a sequence $(M_i)_{i \in I}$ of L -structures, we write $\prod_{i \in I} M_i/U$ for the associated *ultraproduct*. The underlying set of $\prod_{i \in I} M_i/U$ is the quotient $\{(a_i)_{i \in I} : a_i \in M_i\} / \sim$, where $(a_i)_{i \in I} \sim (b_i)_{i \in I}$ if and only if $\{i \in I : a_i = b_i\} \in U$. We shall often write $\lim_{i \rightarrow U} a_i$ for the equivalence class of $(a_i)_{i \in I}$ in the ultraproduct. The ultraproduct readily becomes an L -structure. If the M_i 's are groups (resp. rings, fields, valued fields etc.), then so is the ultraproduct. More generally, Łoś' s Theorem ensures that any elementary property which holds for the majority of the M_i 's will also be inherited by the ultraproduct. More precisely, for any $\phi \in \text{Sent}_L$, one has that $\{i \in I : M_i \models \phi\} \in U \iff \prod_{i \in I} M_i/U \models \phi$. In case $M_i = M$, we write $M_U = \prod_{i \in I} M/U$ and call it the *ultrapower* of M along U (or simply the ultraproduct of M when the context is clear). A standard fact which will be used throughout is Corollary 2.2 [Kei70], saying that *non-principal* ultraproducts (resp. ultrapowers) are \aleph_1 -saturated.

1.2 Algebraic geometry

We now introduce the necessary background from algebraic geometry, related to desingularization. We also collect some basic facts from computational algebraic geometry in §???. We assume familiarity with the language of schemes throughout.

1.2.1 Regular schemes

Definition 1.2.2. Let (A, \mathfrak{m}) be a Noetherian local ring with residue field $k = A/\mathfrak{m}$. We say that A is regular if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

The concept of a "non-singular" scheme is formalized in the following:

Definition 1.2.3. Let X be a locally Noetherian scheme.

- (a) We say that X is regular at $x \in X$, or that x is a regular point of X , if $\mathcal{O}_{X,x}$ is a regular local ring.
- (b) We say that X is regular if it is regular at all $x \in X$.

If X is regular at $x \in X$, then a minimal set of generators for $\mathfrak{m}_{X,x}$ is said to be a *regular system* of parameters of X at x .

1.2.4 Normal Crossings

Definition

A divisor D is said to be *strict normal crossings* if Zariski locally D_{red} is the union of non-singular hypersurfaces crossing transversely. More formally:

Definition 1.2.5. (a) Let X be a locally Noetherian scheme and D be an effective Cartier divisor on X . We say that D has *strict normal crossings* at a point $x \in X$ if X is regular at x and there exists a regular system of parameters f_1, \dots, f_n of X at x , an integer $0 \leq m \leq n$ and integers $e_1, \dots, e_m \geq 1$ such that D is cut out by $f_1^{e_1} \cdot f_2^{e_2} \cdot \dots \cdot f_m^{e_m}$ in $\mathcal{O}_{X,x}$. If D has strict normal crossings at all points $x \in X$, then D is a strict normal crossings divisor.

(b) Let R be a DVR. We will simply say that $X \rightarrow \text{Spec}R$ has strict normal crossings if X is a regular scheme and $X_s = X \times_{\text{Spec}R} \text{Spec}(R/\mathfrak{m}_R)$ is a strict normal crossings divisor.

The integers e_i in Definition 1.2.5(a) are the *multiplicities* of the irreducible components of D passing through x . Some examples are given below:

Example 1.2.6. (a) Let $R = \mathbb{F}_p[[t]]$ and X be the affine R -scheme defined by $x^2y - t = 0$. Then $X \rightarrow \text{Spec}R$ has strict normal crossings; the irreducible components of $X_s = \text{Spec}(\mathbb{F}_p[x, y]/(x^2y))$ have multiplicities 2 and 1.

(b) Let $R = \mathbb{Z}_{(p)}$ and X be the affine R -scheme defined by $xy^p - p = 0$. Then $X \rightarrow \text{Spec}R$ has strict normal crossings; the irreducible components of $X_s = \text{Spec}(\mathbb{F}_p[x, y]/(x^py))$ have multiplicities p and 1.

Normal crossings vs strict normal crossings

Since the term "normal crossings" is used with subtly different meanings throughout the literature, we find it useful to clarify a few things. In addition to Definition 1.2.5, one has the following more general notion:

Definition 1.2.7. If X is a regular scheme and D is an effective Cartier divisor on X , we say that D has normal crossings if there exists an étale morphism $\pi : Z \rightarrow X$ such that the pullback π^*D has *strict normal crossings*.

However, sometimes authors use the term "normal crossings" for Definition 1.2.5 rather than the more general Definition 1.2.7 (see Remark 9.1.7). Thanks to the following well-known fact, this distinction is not going to be important for our purposes:

Remark 1.2.8 (see Proposition 2.2.2 [Nic13], [Con16]). If D has normal crossings, one can always find a blow-up $f : \tilde{X} \rightarrow X$ so that f^*D is a strict normal crossings divisor.

Remark 1.2.8 is proved in Proposition 2.2.2 [Nic13] when $\dim X = 2$ but the general case is similar. It is explained in detail in [Con16]. Note that in Conrad's terminology a strict normal crossings divisor is taken to be reduced (see also the last paragraph of §1.2.4). Nevertheless, the proof of [Con16] applies equally well in the non-reduced context. We illustrate Remark 1.2.8 with an example:

Example 1.2.9. Let $p \neq 2$, $R = \mathbb{F}_p[[t]]$ and X be the affine R -scheme defined by

$$x^2 - \alpha y^2 = t$$

where $\alpha \in \mathbb{F}_p^\times - (\mathbb{F}_p^\times)^2$. Let $\beta \in \overline{\mathbb{F}}_p$ be such that $\beta^2 = \alpha$. After a base change $S' \rightarrow S$, corresponding to $R \rightarrow \mathbb{F}_p(\beta)[[t]]$, the pullback of the divisor X_s has defining equation

$$(x - \beta y)(x + \beta y) = 0$$

and is thus a normal crossings divisor of $X_{S'} = X \times_S S'$. Blowing up the ideal (x, y, t) makes X_s into a strict normal crossings divisor, in accordance with Fact 1.2.8.

We finally warn the reader that in several places in the literature (e.g. [Con16] and 2.2.2 [Tem11]), the term strict normal crossings divisor means that the divisor is *reduced*. However, our Definition 1.2.5 allows non-reduced divisors as well. In fact, the reader should have the non-reduced case in mind as the typical case throughout the paper.

1.2.10 Resolution of singularities

Motivation

For the rest of the paper, unless otherwise stated, we will assume a certain form of resolution of singularities. Recall that a *resolution of singularities* of a scheme X is a proper and birational morphism $X' \rightarrow X$, with X' regular (Definition 8.3.39 [Liu06]). Before we state precisely the form of resolution that we shall use, let us first discuss the notion of an excellent DVR.

Excellent DVR

The concept of a quasi-excellent ring first appeared in §7.9 [Gro65]. Grothendieck showed in 7.9.5 [Gro65] that if R is a ring such that every integral scheme of finite type over R admits a resolution of singularities, then R is quasi-excellent.

We shall not dwell on the precise definition of quasi-excellence, which is rather technical. For the rest of the paper, our base ring R will always be a DVR, in which case R is *excellent* precisely when the field extension \widehat{K}/K is *separable*, where $K = \text{Frac}(R)$ (see Corollary 8.2.40(b) [Liu06]). The reader may take this as the definition of an excellent DVR, although this is a consequence of the actual definition. We record the following construction, due to F. K. Schmidt, which will appear again in Remark 5.3.8. It is a classical example of a non-excellent DVR:

Example 1.2.11 (see Examples 2.3.5 [Tem11], 3.1 [Kuh11]). Let $y = \sum_{i=0}^{\infty} \alpha_i x^i \in \mathbb{F}_p[[x^p]]$ be transcendental over $\mathbb{F}_p(x)$ and let $F = \mathbb{F}_p(x, y)$. Consider the embedding $F \hookrightarrow \mathbb{F}_p((x))$ and the restriction of the x -adic valuation on F . We have that $\widehat{F} = \mathbb{F}_p((x))$. It is clear that \widehat{F}/F is not separable as $y^{1/p} \in \widehat{F} - F$.

Resolution of singularities

Grothendieck conjectured that the converse of 7.9.5 [Gro65] is also true, i.e. every integral quasi-excellent scheme admits a resolution of singularities (see 7.9.6 [Gro65]). In practice, one often asks for stronger variants of resolution, some of which are discussed in §2.5 [Tem11]. These variants are mostly discussed in the characteristic 0 context, where we have proven results. However, in 2.3.2 [Tem11] Temkin explains that the stronger variants discussed in §2.5 are expected to be true for *general* quasi-excellent schemes. In this paper, we will assume the validity of the following statement:

Conjecture R (Log-Resolution). *Let X be a reduced, flat scheme of finite type over an excellent discrete valuation ring R . Then there exists a blow-up morphism $f : \tilde{X} \rightarrow X$ in a nowhere dense center $Z \subsetneq X$ such that*

1. \tilde{X} is a regular scheme.
2. $\tilde{X}_s = \tilde{X} \times_{\text{Spec} R} \text{Spec}(R/\mathfrak{m}_R)$ is a strict normal crossings divisor.

Condition (1) is the one predicted by Grothendieck's conjecture. A desingularization of X which also satisfies condition (2) is often called a desingularization of the pair (X, X_s) (see 2.5.3 [Tem11])¹ or a log-resolution of (X, X_s) . We emphasize that X_s will typically be non-reduced. In residue characteristic 0, one can also make X_s reduced, at the cost of replacing R with a finite extension (see 3.1.4 [Tem11]).

¹In the terminology of 2.5.3 [Tem11] a desingularization of (X, X_s) is only required to make X_s

Evidence for Conjecture R

We refer to §8.3.4 [Liu06], §3 [Tem11] for more information on established desingularization results. See also §3.6-§3.8 [Spi20] for a more up-to-date survey.

Fact 1.2.12 (Residue characteristic 0). Conjecture R is known when the residue field of R is of characteristic 0 by work of Hironaka (see Main Theorem I, pg. 132 and Corollary 3, pg.146 [Hir64]). These results are phrased for varieties over fields of characteristic 0 but on pg. 151 [Hir64] Hironaka explains that similar results hold more generally over quasi-excellent local rings of residue characteristic 0. This is also explained in the introduction of [Tem08a]. The notion of "quasi-excellence" does not actually appear in Hironaka's paper but was later introduced by Grothendieck.

To obtain Conjecture R in residue characteristic 0, one splits the desingularization of the pair (X, X_s) into an ordinary desingularization $X' \rightarrow X$ and an *embedded* desingularization of $X'_s \subsetneq X'$. Theorem 1.1 [Tem08a] proves a more general result for *general* quasi-excellent schemes. We note that the above cited results are stated for integral (rather than reduced) schemes but this is not very important (see Remark 2.3.8 [Tem08a]).

In positive characteristic, Conjecture R and other variants of resolution are widely open. We nevertheless have some partial results:

Fact 1.2.13 (Positive characteristic). (a) First, in a series of papers Abhyankar proved resolution of singularities for the case of varieties of dimension at most 3, over an algebraically closed field k of characteristic $p > 5$ (see [Abh66]). Cutkovsky gave a simplified and self-contained version of Abhyankar's proof in [Cut09]. In [CP08] and [CP09], Cossart and Piltant removed the restriction on the characteristic and generalized it for base fields k satisfying a very mild assumption (namely that $[k : k^p] < \infty$).

(b) Lipman showed that reduced, excellent, Noetherian schemes of dimension 2 can be desingularized, but his result does not include a divisor condition (see Theorem 8.3.44 [Liu06]). More recently, Cossart and Piltant [CP19] proved a strong desingularization result for general quasi-excellent schemes of dimension at most 3. Conjecture R for $\dim(X) = 3$ can be deduced from Theorem 1.1 [CP19].

into a normal crossings divisor but this is not really important in view of Fact 1.2.8. We warn the reader that in [Tem11] the term "monomial divisor" is used for what we called "normal crossings divisor" in Definition 1.2.7. In Temkin's terminology a normal crossings divisor is required to be reduced (see 2.2.2 [Tem11]). Note also that 2.5.3 [Tem11] asks for some control over the centers of the blow-ups but this is not going to be important for us.

Fact 1.2.14. A.J. de Jong proved a *weaker* statement than Conjecture R with alterations, in the case where R is a complete DVR (see Theorem 6.5 [Jon96]).

For a concrete and detailed calculation of a desingularization of $X = \text{Spec}(R[x, y]/(xy - a))$ with $a \in R$, which also makes X_s into a normal crossings divisor, see Example 8.3.53 [Liu06].

1.2.15 Computational algebraic geometry

Affine schemes as algorithmic inputs

Let R be a Noetherian and *computable* ring, i.e. one whose underlying set is (or may be identified with) a recursive subset of \mathbb{N} , so that the ring operations are (or are identified with) recursive functions (e.g., $R = \mathbb{F}_p[t]$ or $R = \mathbb{F}_p(t)$ or $R = \mathbb{F}_p$). Ideals $I \subseteq R[x_1, \dots, x_n]$ are construed as algorithmic inputs via some natural identification $R[x_1, \dots, x_n] \simeq \mathbb{N}$. For example, one can enumerate all finite sequences of elements from $R[x_1, \dots, x_n]$, which encode (non-faithfully) all ideals of $R[x_1, \dots, x_n]$ by Noetherianity.

Definition 1.2.16. (a) By saying that we are *explicitly given* (or simply given) an ideal $I \subseteq R[x_1, \dots, x_n]$, we mean that we are given a finite set of generators via the above identification.

(b) By saying that we are *explicitly given* (or simply given) a closed subscheme $X \subseteq \mathbb{A}_R^n$, we mean that $X = \text{Spec}(A)$ with $A = R[x_1, \dots, x_n]/I(X)$ and the ideal $I(X) \subseteq R[x_1, \dots, x_n]$ is given.

Some algorithms in algebraic geometry

Gröbner bases are at the basis of Facts 1.2.17 and 1.2.20. Background material on Gröbner bases can be found in Chapter 2 [CLO04]. Here, we simply focus on their applications in computational algebraic geometry, again using [CLO04] as our main reference.

The reader can assume throughout that $k = \mathbb{F}_p$ or $k = \mathbb{F}_p(t)$, which are the only cases used in §5.5. The results hold in much greater generality, although some care is needed in the choice of k in order to have effective primary decomposition of ideals (see also Remark 1.2.18).

Fact 1.2.17. There is an algorithm to compute:

- (a) The primary decomposition of a given ideal $I \subseteq k[x_1, \dots, x_n]$ as well as the associated primes.
- (b) The dimension $\dim V(I)$ of a given ideal $I \subseteq k[x_1, \dots, x_n]$.

Proof. (a) See pg. 213 [CLO04]. (b) See pg. 462 [CLO04]. \square

As was mentioned above, the choice of the base field k is delicate and merely assuming that k is computable does not guarantee the existence of an algorithm for computing primary decompositions. This is not discussed in [CLO04], but is discussed in [GTZ88] and [Sei78]. We shall not dwell on this but it is worth including a few comments:

Remark 1.2.18. (a) There are computable fields k , for which even the factorization of single-variable polynomials in $k[x]$ is not effectively computable (see pg. 273 [Sei78]). One can certainly add effectiveness of factorizations in $k[x]$ as an extra ad hoc condition on k . Still, primary decomposition in higher dimensions is not guaranteed (see pg. 274 [Sei78]).

(b) By [GTZ88], the situation is remedied by asking that k be *factorially closed*, meaning that factorization of single-variable polynomials in $k'[x]$ is effective, for any finitely generated extension k'/k . Prime fields have this property (see §1 [GTZ88]). We note that the main result of [GTZ88] is stated in much greater generality, covering the case where the base ring is a factorially closed algorithmic PID (not necessarily a field).

Remark 1.2.19. By the usual dictionary between algebra and geometry, Fact 1.2.17(a) says that we can compute the irreducible components of a given closed subscheme $X \subseteq \mathbb{A}_k^n$. Being able to compute the associated primes of the primary decomposition of a given ideal $I \subseteq k[x_1, \dots, x_n]$, in particular allows us to compute the reduced underlying scheme of any given closed subscheme $X \subseteq \mathbb{A}_k^n$.

Fact 1.2.20. There is an algorithm to check if $f \in I$, for any given $f \in k[x_1, \dots, x_n]$ and $I \subseteq k[x_1, \dots, x_n]$. In particular, there is an algorithm to check if $I = J$ for any two given ideals $I, J \subseteq k[x_1, \dots, x_n]$.

Proof. See Chapter 2, §8 [CLO04]. \square

1.3 Valued fields

We assume familiarity with the basics of valued fields. See [EP05] for a detailed treatment. Here we provide an overview of RV-structures which are of central importance in chapter 5 and also a synopsis of the various formalisms which are used in the model theory of valued fields.

1.3.1 RV-structures

We first provide an overview of the RV-structures associated to a valued field (K, v) .

History

The RV-structures (closely related to Krasner's hyperfields due to [Kra56]) first appeared in a model-theoretic context in the work of Scanlon [Sca99] (by the name *leading terms*) and were subsequently studied by his student Flenner in [Fle11]. These structures are simplified versions of Kuhlmann's earlier *amc-congruences*² (see [Kuh94]), which in turn were inspired by Basarab's foundational work in [Bas91]. Basarab [Bas91] introduced his *mixed structures* to obtain relative quantifier elimination and relative completeness results for *general* henselian valued fields of characteristic 0 (see Theorems A, B [Bas91]). However, his results require an *infinite* family of such residual structures, which in practice makes it difficult to use for decidability purposes.

Definition

Let (K, v) be a valued field with residue field k and value group Γ . Consider the following short exact sequence of abelian groups

$$0 \rightarrow k^\times \xrightarrow{\iota} K^\times / (1 + \mathfrak{m}) \xrightarrow{v} \Gamma \rightarrow 0$$

where $\iota(a)$ is the equivalence class in $K^\times / (1 + \mathfrak{m})$ of any lift $\tilde{a} \in \mathcal{O}_K$ of a . We introduce the structure $\text{RV}(K^\times) = K^\times / (1 + \mathfrak{m})$, and write $\text{rv} : K^\times \rightarrow K^\times / (1 + \mathfrak{m})$ for the natural map that sends $a \in K^\times$ to its equivalence class in $K^\times / (1 + \mathfrak{m})$. As with the value group, it will be convenient to include an element ∞ in $\text{RV}(K^\times)$, resulting in $\text{RV}(K) = \text{RV}(K^\times) \cup \{\infty\}$. We naturally extend $\text{rv} : K^\times \rightarrow \text{RV}(K^\times)$ to $\text{rv} : K \rightarrow \text{RV}(K)$ by requiring that $\text{rv}(0) = \infty$.

Apart from its natural multiplicative structure, inherited from K^\times , one can also equip $\text{RV}(K)$ with additional structure.

- We have a ternary relation \oplus for (multi-valued) addition among elements of $\text{RV}(K)$, so that $\oplus(a, b, c)$ holds if there are $x, y, z \in K$ such that $\text{rv}(x) = a, \text{rv}(y) = b, \text{rv}(z) = c$ and $x + y = z$. We shall simply write $a + b = c$ to mean that c is *unique* such that $\oplus(a, b, c)$.
- We equip $\text{RV}(K)$ with a binary relation $R(a, b) \iff vx \leq vy$, where $\text{rv}(x) = a$ and $\text{rv}(y) = b$.

²See the discussion on pg. 6 [Fle11] for a comparison between the two formalisms.

Since $\text{rv}(x) = \text{rv}(y) \implies vx = vy$, the relation R is well-defined. It will be harmless and also convenient to write $va \leq vb$ in place of $R(a, b)$. The "higher" RV-structures described in §2 [Fle11] will not be important for us.

Language

Definition 1.3.2. The language L_{RV} is a two-sorted language having the following sorts and symbols:

- (1) a VF-sort, which uses the language of rings $L_{\text{rings}} = \{+, -, \cdot, 0, 1\}$.
- (2) an RV-sort, which uses the group language $\{1, \cdot\}$, a constant symbol for ∞ , a ternary predicate for \oplus and a binary predicate intended for the relation $va \leq vb$ (see 1.3.1).
- (3) a function symbol $\text{rv} : \text{VF} \rightarrow \text{RV}$ for the natural map $\text{rv} : K \rightarrow \text{RV}(K)$. Recall from 1.3.1 that by convention $\text{rv}(0) = \infty$.

Lemma 1.3.3. Let (F, v) be a valued field. Then the natural inclusion map $\text{RV}(F) \hookrightarrow \text{RV}(F^h)$ (resp. $\text{RV}(F) \hookrightarrow \text{RV}(\widehat{F})$) is an isomorphism.

Proof. Since $(F^h, v)/(F, v)$ is immediate, we have the commutative diagram below, consisting of two short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \kappa^\times & \longrightarrow & \text{RV}(F^\times) & \longrightarrow & \Gamma & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \kappa^\times & \longrightarrow & \text{RV}((F^h)^\times) & \longrightarrow & \Gamma & \longrightarrow & 0 \end{array}$$

By the Five Lemma, it follows that the natural inclusion $\text{RV}(F^\times) \hookrightarrow \text{RV}((F^h)^\times)$ is an isomorphism of abelian groups. Finally, note that this isomorphism respects the additional structure. The same proof applies verbatim for the completion (\widehat{F}, v) . \square

Cross-section

A right inverse $s : \Gamma \rightarrow K^\times$ of $v : K^\times \rightarrow \Gamma$ is called a cross-section and makes the exact sequence of 1.3.1 split. Therefore, if such a cross-section exists, it allows us to identify the abelian group $\text{RV}(K^\times)$ with $k^\times \times \Gamma$. Explicitly, we identify $(a, \gamma) \in k^\times \times \Gamma$ with $\iota(a) \cdot s(\gamma) \in \text{RV}(K^\times)$.

We add to $k^\times \times \Gamma$ an additional symbol ∞ and equip $(k^\times \times \Gamma) \cup \{\infty\}$ with the following structure:

- A multiplication \cdot , which restricts to the natural group operation on $k^\times \times \Gamma$ and satisfies $\infty \cdot (a, \gamma) = (a, \gamma) \cdot \infty = \infty \cdot \infty = \infty$ for all $a \in k^\times, \gamma \in \Gamma$.

- A ternary relation \oplus for (multi-valued) addition, defined by $\oplus((a, \gamma), (b, \delta), (c, \epsilon))$ whenever (1) $\gamma < \delta$ and $c = a, \epsilon = \gamma$ or (2) $\delta < \gamma$ and $c = b, \epsilon = \delta$ or (3) $\gamma = \delta, a + b \neq 0$ and $c = a + b, \epsilon = \gamma$ or (4) $\gamma = \delta, a + b = 0$ and $\epsilon > \gamma$.
- A binary relation $R((a, \gamma), (b, \delta)) \iff \gamma \leq \delta$.

In the presence of a cross-section, the structure $(\text{RV}(K), \oplus, \cdot, \leq, 1, \infty)$ described in 1.3.1 is isomorphic to the structure $((k^\times \times \Gamma) \cup \{\infty\}, \oplus, \cdot, \leq, 1, \infty)$ via the identification described above.

Interpretability

In Fact 1.3.4, we record the observation that k and Γ are *quantifier-free* interpretable in $\text{RV}(K)$. We first introduce some notation and make some preliminary remarks, which will also be useful later on:

Consider a valuation ring \mathcal{O}_K with fraction field K . We consider the set $\text{RV}(\mathcal{O}_K) = \{x \in \text{RV}(K) : vx \geq 0\}$ and $\text{RV}(\mathcal{O}_K^\times) = \{x \in \text{RV}(K) : vx = 0\}$, equipped with the induced structure from $\text{RV}(K)$ and readily seen to be quantifier-free interpretable in the latter. We note that in [Den16] the structure $\text{RV}(\mathcal{O}_K)$ is denoted by $\text{MR}(\mathcal{O}_K)$ and is called the structure of multiplicative residues of \mathcal{O}_K .

The residue map $\text{res} : \text{RV}(\mathcal{O}_K) \rightarrow k$, mapping the equivalence class of $a \in \mathcal{O}_K$ in $\text{RV}(\mathcal{O}_K)$ to \bar{a} (and ∞ to 0), induces a multiplicative isomorphism $\text{RV}(\mathcal{O}_K^\times) \cong k^\times$, which is also a left inverse of $\iota : k^\times \rightarrow \text{RV}(\mathcal{O}_K^\times)$. The isomorphism $\text{res} : \text{RV}(\mathcal{O}_K^\times) \xrightarrow{\cong} k^\times$ extends to $\text{res} : \text{RV}(\mathcal{O}_K^\times) \cup \{\infty\} \xrightarrow{\cong} k$, by sending $\infty \mapsto 0$. It also respects addition in the following sense: For all for all $a, b, c \in \text{RV}(\mathcal{O}_K^\times) \cup \{\infty\}$, we have that $\oplus(a, b, c)$ if and only if $\text{res}(a) + \text{res}(b) = \text{res}(c)$. Moreover, if $\text{res}(c) \neq 0$, then $c \in \text{RV}(K)$ is unique such that $\oplus(a, b, c)$ and we may simply write $a + b = c$ (see §1.3.1).

Fact 1.3.4 (cf. Proposition 2.8 [Fle11]). The residue field $(k, +, \cdot, 0, 1)$ and the value group $(\Gamma, +, <, 0)$ are quantifier-free interpretable in $\text{RV}(K)$ with the structure described in Definition 1.3.2(2). Moreover, the interpretations are uniform in K .

Proof. We start with the residue field. We identify $(k, +, \cdot, 0, 1)$ with $\text{RV}(\mathcal{O}_K^\times) \cup \{\infty\} = \{x \in \text{RV}(K) : v(x) = 0\} \cup \{\infty\}$, which is quantifier-free definable in $\text{RV}(K)$ by the formula $vx = v1 \vee x = \infty$ (formally $vx = v1$ stands for $vx \leq v1 \wedge v1 \leq vx$). The identification is given by $\text{res} : \text{RV}(\mathcal{O}_K^\times) \cup \{\infty\} \rightarrow k$ as described above.

We next argue that the value group is quantifier-free interpretable in $\text{RV}(K)$. We introduce the following quantifier-free definable relation \sim on $\text{RV}(K)$: $x \sim y \iff vx = vy$. The value group is identified with the quotient $\text{RV}(K)/\sim$, addition in Γ is recovered by multiplication, the order $<$ is recovered by $vx < vy$ (viz. $vx \leq vy \wedge \neg vy \leq vx$) and $0 \in \Gamma$ is identified with $\infty \in \text{RV}(K)$.

□

1.3.5 Languages

We write $L_{\text{oag}} = \{0, +, <\}$ for the language of ordered abelian groups and $L_{\text{rings}} = \{0, 1, +, \cdot\}$ for the language of rings.

Valued fields language

Let L_{val} be the three-sorted language of valued fields, with sorts for the field, the value group and the residue field.

- The field sort \mathbf{K} is equipped with the language of rings L_{rings} .
- The value group sort $\mathbf{\Gamma}$ is equipped with L_{oag} , together with a constant symbol for ∞ .
- The residue field sort \mathbf{k} is equipped the language of rings L_{rings} .

We also have function symbols for the valuation map $v : \mathbf{K} \rightarrow \mathbf{\Gamma}$ and a residue map $\text{res} : \mathbf{K} \rightarrow \mathbf{k}$ (where $\text{res}(x) = 0$ when $vx < 0$ by convention).

Convention 1.3.6. We shall write $x \in \mathcal{O}$ as an abbreviation of the formula $x \in \mathbf{K} \wedge vx \geq 0$.

Ax-Kochen/Ershov language

Historically, the Ax-Kochen/Ershov formalism also included a function symbol for a cross-section, i.e. group homomorphism $s : \mathbf{\Gamma} \rightarrow K^\times$ satisfying $v \circ s = \text{id}_{\mathbf{\Gamma}}$ (see [AK65] and [Koc74]). One can also extend s by defining $s(\infty) = 0$. We write $L_{\text{val}, \times}$ for the language L_{val} enriched with such a cross-section symbol $s : \mathbf{\Gamma} \rightarrow \mathbf{K}$.

1.3.7 Ax-Kochen/Ershov with cross-section

Among other results, Ax-Kochen [AK65] and independently Ershov [Ers65] obtained the following:

Fact 1.3.8 (Ax-Kochen/Ershov). The field \mathbb{Q}_p , equipped with the normalized cross-section $s : n \mapsto p^n$, is decidable in $L_{\text{val}, \times}$.

Remark 1.3.9. (a) More generally, by Theorem 4 [Koc74], any unramified henselian field (K, v) , equipped with a *normalized* cross-section (viz., $s(1) = p$) and with perfect residue field k , is decidable in $L_{\text{val}, \times}$.

(b) Finite extensions of \mathbb{Q}_p are also decidable in $L_{\text{val}, \times}$, for a suitable choice of a cross-section (see §2.4).

Theorem 12 [AK66] in fact shows that \mathbb{Q}_p admits quantifier elimination in $L_{\text{val},\times}$ relative to the value group. The definable sets of the latter are perfectly understood by classical quantifier elimination results for Presburger arithmetic. Nevertheless, the following is worth noting:

Remark 1.3.10. The definable sets in the Ax-Kochen/Ershov language are complicated and are generally not definable in the valued field language. For instance, the image of the cross-section is not definable without the cross-section (see Example, pg. 609 [Mac86]). For this reason, at least for the purpose of studying definable subsets of the p -adics, the Ax-Kochen/Ershov formalism was superseded by Macintyre's language (see pg. 606 [Mac86]).

Despite Remark 1.3.10, the Ax-Kochen/Ershov formalism has remained relevant. The fact that p -adic fields are decidable in such an expressive language is a strong result. This is in stark contrast with the fact that positive characteristic local fields are undecidable in $L_{\text{val},\times}$ ([Phe87]).

2

Ax-Kochen/Ershov principles

Introduction

The theory of the p -adic numbers \mathbb{Q}_p was determined by Ax-Kochen [AK65] and independently by Ershov [Ers65]. Since then, several questions about the model theory of local fields and their extensions have been raised and answered:

- In mixed characteristic, Kochen [Koc74] determined the theory of \mathbb{Q}_p^{ur} , the maximal unramified extension of \mathbb{Q}_p in the language $L_{\text{val}} = \{0, 1, +, \cdot, \mathcal{O}\}$ (see notation). More generally, by work of [Zie72], [Ers65], [Bas78], [Bél99] and more recently [AJ19], [Lee20] and [LL21], we have a good understanding of the model theory of unramified and finitely ramified mixed characteristic henselian fields. In a somewhat different spirit, in Theorem 17 [Ax68], Ax determined the theory of $\{\mathbb{Q}_p : p \in \mathbb{P}\}$ and also that the theory of $\{\mathbb{Q}_p(\zeta_n) : p \nmid n\}$, namely the collection of all finite *unramified* extensions of \mathbb{Q}_p . To summarize the above, theories of mixed characteristic henselian fields and collections thereof are by and large well understood model-theoretically, as long as the ramification is bounded.
- In positive characteristic, by work of Denef-Schoutens [DS03], we know that $\mathbb{F}_p((t))$ is *existentially* decidable in $L_t = \{0, 1, t, +, \cdot, \mathcal{O}\}$ (see notation), modulo resolution of singularities.¹ In fact, Theorem 4.3 [DS03] applies to show that (assuming resolution) any finitely ramified extension of $\mathbb{F}_p((t))$ can be

¹The formalism of [DS03] does not include a unary predicate \mathcal{O} for the valuation ring. This does not make a difference because of the equivalences $x \in \mathcal{O} \leftrightarrow \exists y(y^2 = 1 + t \cdot x^2)$ (for $p > 2$; replace squares with cubes for $p = 2$) and $x \notin \mathcal{O} \leftrightarrow x^{-1} \in t \cdot \mathcal{O}$.

recursively axiomatized relative to its residue field. Anscombe-Fehm [AF16] showed *unconditionally* that the existential theory of equal characteristic henselian valued fields is determined by the existential theory of the residue field. The caveat is that their result is stated in the weaker language L_{val} , which does not include a constant symbol for t . Finally, we mention Kuhlmann's theory of tame fields [Kuh16a], which provides many decidability results for positive characteristic tame fields. This will be discussed further in §3.3.

From the above (incomplete) list, we see that our understanding is less clear in the positive characteristic case. Notably, the theory of $\mathbb{F}_p((t))$ still remains unknown and is perhaps the prominent open problem in the area. On the other hand, certain *infinitely ramified* mixed characteristic fields have also been difficult to understand. The theories of collections of p -adic extensions where the ramification is *unbounded* (e.g., the collection of all finite extensions of \mathbb{Q}_p) have also been mysterious from a model-theoretic standpoint. It has long been suspected that the two unknown domains mentioned above, i.e., positive characteristic and infinite/unbounded ramification, are interlinked. In chapters 3 and 6, we shall justify this premonition by establishing precise connections.

In all cases which have been understood thus far, the theories are ultimately understood via so-called Ax-Kochen/Ershov principles. The essence of Ax-Kochen/Ershov principles is to understand the theory of a henselian valued field in terms of more primitive structures associated to it. If, in turn, the theories of those primitive structures can be recursively axiomatized, this furnish us with a recursive axiomatization of the valued field itself. In equal characteristic 0 or *unramified* mixed characteristic, these primitive structures are simply the value group and the residue field, so that the theory of the valued field is simply determined by the theories of the residue field and the value group (and the fact that it is henselian). Already in the presence of finite ramification, this naive reduction may fail and one needs more algebraic information. Still, the situation is tractable.

The goal of the present chapter is to collect certain Ax-Kochen/Ershov principles that are perhaps less known.

- The first one is an unpublished result by van den Dries. It is an Ax-Kochen/Ershov principles for general mixed characteristic henselian valued fields (i.e., with no restriction on the ramification). However, it requires knowledge of *infinitely* many residual structures (see §2.1). On top of requiring an infinite number of them, each individual one can be quite complicated in general. For those two reasons, this Ax-Kochen/Ershov principle has been

rather difficult to use in practice, e.g., for decidability problems. Nevertheless, the result by van den Dries will be instrumental for our perfectoid transfer theorem in chapter 3. In order to get an *existential* perfectoid transfer, we will also prove an existential version of van den Dries' theorem.

- The Ax-Kochen/Ershov principle proved in §2.4.6 yields a recursive axiomatization for each finite extension of \mathbb{Q}_p in the language with a cross-section. This will not be used directly. Its main purpose is rather to illustrate that while each individual finite extension of \mathbb{Q}_p is decidable with a (suitable) cross-section, the theory of all of them is less well-behaved, thereby putting the main result of chapter 6 in context.

A general *existential* Ax-Kochen/Ershov principle will be established in chapter 5, by putting some constraints on the *type* of ramification in play (but *not* on its size) and assuming a certain form of resolution of singularities. This result requires entirely different techniques and we shall therefore defer its treatment to chapter 5.

2.1 A result by van den Dries

2.1.1 A result by van den Dries

Introduction

We start with a general Ax-Kochen/Ershov style result in mixed characteristic, due to van den Dries (unpublished), which is briefly discussed on pg. 144 in [Dri12]. We shall sketch the proof (due to van den Dries), which does not seem to appear anywhere in the published literature. Some references, which use a similar *coarsening* argument, include pg. 2 [Dri99], the proof of Corollary 12.3 [AJ19] and the proof of Proposition 9.6 [Sca99]. For background material on coarsenings of valuations, see §7.4 [Dri12].

Inverse systems

The formalism of multi-sorted structures, used in this section, is spelled out in §3 [Sca99]. The proof of van den Dries' Theorem 2.1.3 requires a technical lemma for inverse systems, which we now discuss.

Let $\mathcal{R} = (R_n)_{n \in \omega}$ be a sequence of rings, viewed as a multi-sorted structure with sorts $(\mathbf{R}_n)_{n \in \omega}$, each equipped with the language of rings L_{rings} , and for each $n \in \mathbb{N}$ we have a map $f_n : \mathbf{R}_{n+1} \rightarrow \mathbf{R}_n$. Let T be the theory that requires of $\mathcal{R} = (R_n)_{n \in \omega}$ that the map $f_n : R_{n+1} \rightarrow R_n$ be surjective with $\text{Ker}(f_n) = p^n R_{n+1}$,

i.e. $R_n \cong R_{n+1}/p^n R_{n+1}$. If $\mathcal{R} = (R_n)_{n \in \omega}$ and $\mathcal{S} = (S_n)_{n \in \omega}$ are two models of T , they are isomorphic precisely when there is a *compatible* system $(\phi_n)_{n \in \omega}$ of isomorphisms $\phi_n : R_n \xrightarrow{\cong} S_n$, i.e. such that the diagram commutes

$$\begin{array}{ccc} R_{n+1} & \longrightarrow & S_{n+1} \\ \downarrow & & \downarrow \\ R_n & \longrightarrow & S_n \end{array}$$

for each $n \in \mathbb{N}$. Compatibility of the ϕ_n 's is essential as there are examples where $R_n \cong S_n$ for each $n \in \mathbb{N}$ but $\mathcal{R} \not\cong \mathcal{S}$ (see e.g., Remark 3.1.37). Somewhat surprisingly, compatibility comes for free in a saturated setting:

Lemma 2.1.2. Assume CH. Let $\mathcal{R} = (R_n)_{n \in \omega}$ and $\mathcal{S} = (S_n)_{n \in \omega}$ be two models of T . Suppose that for each $n \in \mathbb{N}$, we have that $R_n \cong S_n$ and the rings R_n, S_n are saturated with $|R_n| = |S_n| \leq \aleph_1$. Then $\mathcal{R} \cong \mathcal{S}$.

Proof. Let U be a non-principal ultrafilter on $\mathcal{P}(\mathbb{N})$ and consider the ultraproducts $R_U = \prod_{n \in \omega} R_n/U$ and $S_U = \prod_{n \in \omega} S_n/U$.

Claim 1: $R_U \cong S_U$.

Proof. Since $R_n \cong S_n$ for each $n \in \mathbb{N}$, we get that $R_U \cong S_U$ by Łoś's Theorem. For each $n \in \mathbb{N}$, let $\mathcal{F}_m = \{n \in \mathbb{N} : |R_n| \geq m\}$. If there exists $m \in \mathbb{N}$ with $\mathcal{F}_m \notin U$, then R_U and S_U are both finite and thus $R_U \cong S_U$. If on the other hand $\mathcal{F}_m \in U$ for all $m \in \mathbb{N}$, then R_U and S_U are both infinite. Since $|R_n| \leq \aleph_1$, we get that $|R_U| \leq \aleph_1^{\aleph_0} = 2^{\aleph_0^2} = 2^{\aleph_0} = \aleph_1$, using the continuum hypothesis (similarly $|S_U| \leq \aleph_1$). Moreover, the ultraproducts R_U and S_U are \aleph_1 -saturated (Exercise 4.5.37 [Mar02]) and thus saturated of size \aleph_1 . Since R_U and S_U are elementary equivalent and both saturated of size \aleph_1 , we conclude that $R_U \cong S_U$ (Theorem 4.3.20 [Mar02]). $\square_{\text{Claim 1}}$

Next we prove:

Claim 2: For each $n \in \mathbb{N}$, we have $R_U/p^n R_U \cong R_n$.

Proof. Fix $n \in \mathbb{N}$. Since $R_m/p^n R_m \cong R_n$ for $m > n$, we get that $R_U/p^n R_U \cong R_n$ in L_{rings} by Łoś Theorem. If R_n is finite, then $R_U/p^n R_U \cong R_n$ and we are done. Otherwise, we will have that R_n is saturated of size \aleph_1 . The same is true for $R_U/p^n R_U$, being interpretable in the structure R_U , which is saturated of size \aleph_1 by CH (see the proof of Claim 1). We conclude that $R_U/p^n R_U \cong R_n$ (Theorem 4.3.20 [Mar02]). $\square_{\text{Claim 2}}$

Similarly, for each $n \in \mathbb{N}$, we have $S_U/p^n S_U \cong S_n$. By Claim 1, we obtain $\phi : R_U \xrightarrow{\cong} S_U$. Note that $\phi(p^n R_U) = p^n S_U$, for each $n \in \mathbb{N}$. By Claim 2, this gives rise to a *compatible* system $(\phi_n)_{n \in \omega}$ of isomorphisms $\phi_n : R_n \xrightarrow{\cong} S_n$, which yields $\mathcal{R} \xrightarrow{\cong} \mathcal{S}$. \square

Statement and proof

The following result will be of fundamental importance in chapter 3:

Theorem 2.1.3 (van den Dries). *Let $(K, v), (K', v')$ be two henselian valued fields of mixed characteristic. Then $(K, v) \equiv (K', v')$ in L_{val} if and only if $\mathcal{O}_v/p^n\mathcal{O}_v \equiv \mathcal{O}_{v'}/p^n\mathcal{O}_{v'}$ in L_{rings} for all $n \in \mathbb{N}$ and $(\Gamma_v, vp) \equiv (\Gamma_{v'}, v'p)$ in L_{oag} together with a constant for vp .*

Proof. " \Rightarrow ": Clear.

" \Leftarrow ": As the statement at hand is *absolute*, we may assume the continuum hypothesis (see §8 [Sca99] or pg. 122 [Dri12]). We may therefore assume that both (K, v) and (K', v') are saturated of size \aleph_1 (Corollary 4.3.13 [Mar02]). By our assumption, we have an isomorphism of ordered abelian groups $(\Gamma_v, vp) \cong (\Gamma_{v'}, v'p)$ and a ring isomorphism $\mathcal{O}_{v'}/p^n\mathcal{O}_{v'} \cong \mathcal{O}_v/p^n\mathcal{O}_v$ for each $n \in \mathbb{N}$. We shall argue that $(K, v) \cong (K', v')$.

Consider the finest coarsening w of v for which the associated residue field k_w has characteristic 0. The corresponding valuation ring is $\mathcal{O}_w = \mathcal{O}_v[\frac{1}{p}]$ and the corresponding value group is $\Gamma_w = \Gamma_v/\text{Conv}(\mathbb{Z}vp)$. Let \bar{v} be the induced valuation from v on the residue field k_w . We then have that $\mathcal{O}_{\bar{v}} = \mathcal{O}_v/\bigcap_{n \in \omega} p^n\mathcal{O}_v$. We also consider the analogous objects for K' .

Claim 1: We have a ring isomorphism $\mathcal{O}_{\bar{v}} \cong \varprojlim \mathcal{O}_v/p^n\mathcal{O}_v$.

Proof. Consider the ring homomorphism $f : \mathcal{O}_{\bar{v}} \rightarrow \varprojlim \mathcal{O}_v/p^n\mathcal{O}_v : x + \bigcap_{n \in \omega} p^n\mathcal{O}_v \mapsto (x + p^n\mathcal{O}_v)_{n \in \omega}$, which is clearly injective. We shall argue that it is also surjective. For a given $(x_n + p^n\mathcal{O}_v)_{n \in \omega} \in \varprojlim \mathcal{O}_v/p^n\mathcal{O}_v$, we may find $x \in \mathcal{O}_v$ with $x \equiv x_n \pmod{p^n\mathcal{O}_v}$, using that \mathcal{O}_v is \aleph_1 -saturated. It follows that $f(x) = (x_n + p^n\mathcal{O}_v)_{n \in \omega}$. $\square_{\text{Claim 1}}$

Similarly, one obtains an isomorphism $\mathcal{O}_{\bar{v}'} \cong \varprojlim \mathcal{O}_{v'}/p^n\mathcal{O}_{v'}$.

Claim 2: We have an isomorphism of valued fields $\phi : (k_w, \bar{v}) \cong (k'_w, \bar{v}')$.

Proof. Let $\mathcal{R} = (\mathcal{O}_v/p^n\mathcal{O}_v)_{n \in \omega}$ and $\mathcal{S} = (\mathcal{O}_{v'}/p^n\mathcal{O}_{v'})_{n \in \omega}$. Then $\mathcal{R} \cong \mathcal{S}$ by Lemma 2.1.2. This yields $\varprojlim \mathcal{O}_v/p^n\mathcal{O}_v \cong \varprojlim \mathcal{O}_{v'}/p^n\mathcal{O}_{v'}$ and thus $\mathcal{O}_{\bar{v}} \cong \mathcal{O}_{\bar{v}'}$ by Claim 1. $\square_{\text{Claim 2}}$

We also have that $\Gamma_w \cong \Gamma_{w'}$, since the isomorphism $(\Gamma_v, vp) \cong (\Gamma_{v'}, v'p)$ descends to the quotients $\Gamma_v/\text{Conv}(\mathbb{Z}vp) \cong \Gamma_{v'}/\text{Conv}(\mathbb{Z}v'p)$. By Lemma 7.13 [Dri12], the coarsened valued fields (K, w) and (K', w') are henselian too. By the Ax-Kochen/Ershov principle in pure characteristic 0 (see e.g., Corollary 5.22 [Dri12]), we get that $(K, w) \equiv (K', w')$ in L_{val} .

Passing once again to elementary extensions, in a suitable language that includes unary predicates for both \mathcal{O}_v and \mathcal{O}_w , we may even assume that $(K, w) \cong (K', w')$ to begin with. By stable embeddedness of residue fields for henselian valued fields of pure characteristic 0 (see Corollary 5.25 [Dri12]), there is even an isomorphism $\Phi : (K, w) \xrightarrow{\cong} (K', w')$ inducing $\phi : k_w \xrightarrow{\cong} k'_{w'}$.

Claim 3: The map Φ is an isomorphism of the valued fields (K, v) and (K', v') .

Proof. Given $x \in K$, we need to show that $vx \geq 0 \iff v'(\Phi(x)) \geq 0$. If $vx \geq 0$, then either (i) $wx > 0$ or (ii) $wx = 0$ and $\bar{v}\bar{x} \geq 0$, where $\bar{x} \in k_v$ is the image of x via $\text{res}_w : \mathcal{O}_w \rightarrow k_w$. In the first case, we get that $w'(\Phi(x)) > 0$ as $\Phi : (K, w) \rightarrow (K', w')$ is a valued field homomorphism and therefore $v'(\Phi(x)) > 0$ as $\mathfrak{m}_w \subset \mathfrak{m}_{w'}$. Suppose now that $wx = 0$ and $\bar{v}\bar{x} \geq 0$. Then we also get that $wx = 0$ and $\bar{v}'(\phi(\bar{x})) \geq 0$ as $\phi : (k_w, \bar{v}) \rightarrow (k'_{w'}, \bar{v}')$ is a valued field homomorphism. Since Φ induces ϕ , we get that $\bar{v}'(\Phi(x)) = \bar{v}'(\phi(\bar{x})) \geq 0$ and conclude that $v'(\Phi(x)) \geq 0$. $\square_{\text{Claim 3}}$

Claim 3 finishes the proof. \square

2.2 Existential AKE in mixed characteristic

In this section we prove an existential version of Theorem 2.1.3. We first review some known AKE results in the equal characteristic setting.

2.2.1 Comparison with the equal characteristic case

For equal characteristic henselian valued fields one has the following simple Ax-Kochen/Ershov principles due to Anscombe-Fehm:

Theorem 2.2.2 (Corollary 1.2 [AF16]). *Let $(K, v), (K', v')$ be two equal characteristic non-trivially valued henselian fields. Then $(K, v) \equiv_{\exists} (K', v')$ in L_{val} if and only if $k \equiv_{\exists} k'$ in L_{rings} .*

Theorem 2.2.3 (Corollary 7.5 [AF16]). *Let (K, v) be an equal characteristic henselian valued field. Then $\text{Th}_{\exists}(K, v)$ is decidable in L_{val} if and only if $\text{Th}_{\exists}(k)$ is decidable in L_{rings} .*

Remark 2.2.4. In residue characteristic 0, Theorems 2.2.2 and 2.2.3 were essentially known by work of Ax-Kochen/Ershov prior to the work of Anscombe-Fehm (see Remark 7.3 [AF16]).

2.2.5 Existential AKE in mixed characteristic

In mixed characteristic, one can easily construct counterexamples of Theorems 2.2.2 and 2.2.3 (see Remark 7.6 [AF16]). It is then natural to ask what an existential AKE principle in mixed characteristic would look like. This will be Theorem 2.2.7 below, whose proof follows closely the proof of Theorem 2.1.3. The role of stable embeddedness in the proof of Theorem 2.1.3 will now be played by the relative embedding property in equal characteristic 0. More generally Kuhlmann shows the following:

Fact 2.2.6 (Theorem 7.1 in [Kuh16a]). Let $(K, v), (K', v')$ be tame fields with a common defectless subfield (K_0, v_0) . Suppose that we have embeddings $\rho : k_v \rightarrow k_{v'}$ over k_{v_0} and $\sigma : \Gamma_v \rightarrow \Gamma_{v'}$ over Γ_{v_0} . Suppose further that k_v/k_{v_0} is separable and that Γ_v/Γ_{v_0} is torsion-free. Then there exists a valued field embedding $\iota : (K, v) \rightarrow (K', v')$ inducing ρ and σ .

We write \mathfrak{m}_n for the maximal ideal of $\mathcal{O}_v/p^n\mathcal{O}_v$ and $(\mathcal{O}_v/p^n\mathcal{O}_v, \mathfrak{m}_n)$ for the local ring, viewed as an L_{lc} -structure (see notation).

Theorem 2.2.7. *Let $(K, v), (K', v')$ be two henselian valued fields of mixed characteristic. Then we have that (a) $(K, v) \equiv_{\exists} (K', v')$ in L_{val} if and only if $\mathcal{O}_v/p^n\mathcal{O}_v \equiv_{\exists} \mathcal{O}_{v'}/p^n\mathcal{O}_{v'}$ in L_{rings} for all $n \in \mathbb{N}$, and (b) $(K, v) \equiv_{\exists} (K', v')$ in L_{val} if and only if $(\mathcal{O}_v/p^n\mathcal{O}_v, \mathfrak{m}_n) \equiv_{\exists^+} (\mathcal{O}_{v'}/p^n\mathcal{O}_{v'}, \mathfrak{m}'_n)$ in L_{lc} for all $n \in \mathbb{N}$.*

Proof. (a) " \Rightarrow ": Clear.

" \Leftarrow ": By symmetry, it will suffice to show that $(K, v) \models \text{Th}_{\exists}(K', v')$. We may further assume (K', v') is countable by downward Löwenheim-Skolem and that (K, v) is \aleph_1 -saturated.

We again consider the valuations w, \bar{v} (resp. w', \bar{v}') that were introduced in the proof of Theorem 2.1.3. By our assumption, we have for each $n \in \mathbb{N}$ an embedding of rings $\mathcal{O}_{v'}/p^n\mathcal{O}_{v'} \hookrightarrow \mathcal{O}_v/p^n\mathcal{O}_v$.

Claim: There is an injective ring embedding $\phi : \mathcal{O}_{\bar{v}'} \hookrightarrow \mathcal{O}_{\bar{v}}$.

Proof. Let us fix an enumeration of $\mathcal{O}_{v'}$, say $\mathcal{O}_{v'} = (a_i)_{i \in \mathbb{N}}$. Consider the set of formulas in countably many variables $x = (x_n)_{n \in \omega}$ of the form

$$\Sigma(x) = \{x_i \diamond x_j = x_v(p^n), x_m \square x_\rho(p^n) : a_i \in \mathcal{O}_{v'}, a_i \diamond a_j = a_v(p^n), a_m \square a_\rho(p^n)\}$$

where \diamond is either $+$ or \cdot and \square is either $=$ or \neq . Since for each $n \in \mathbb{N}$ we have an embedding of rings $\mathcal{O}_{v'}/p^n\mathcal{O}_{v'} \hookrightarrow \mathcal{O}_v/p^n\mathcal{O}_v$, we get that $\Sigma(x)$ is finitely satisfiable. The ring \mathcal{O}_v is \aleph_1 -saturated and we thus have $b = (b_n)_{n \in \omega}$ with $b \models \Sigma(x)$. Using that $\mathcal{O}_{\bar{v}} = \mathcal{O}_v / \bigcap_{n \in \omega} p^n\mathcal{O}_v$ (resp. $\mathcal{O}_{\bar{v}'} = \mathcal{O}_{v'} / \bigcap_{n \in \omega} p^n\mathcal{O}_{v'}$), one readily checks that the map $\mathcal{O}_{v'} \rightarrow \mathcal{O}_v : a_i \mapsto b_i$ descends to a ring embedding $\mathcal{O}_{\bar{v}'} \hookrightarrow \mathcal{O}_{\bar{v}}$. \square_{Claim}

Claim 1 provides us with a valued field embedding $\phi : (k'_{w'}, \bar{v}') \hookrightarrow (k_w, \bar{v})$. By the existential Ax-Kochen/Ershov principle in pure characteristic 0 (see Theorem 2.2.2 and Remark 2.2.4), we get that $(K, w) \models \text{Th}_{\exists}(K', w')$. Replacing K with an \aleph_1 -saturated extension in a suitable language that includes unary predicates for both \mathcal{O}_v and \mathcal{O}_w , we will also have an embedding $(K', w') \hookrightarrow (K, w)$.

By Fact 2.2.6 for equal characteristic 0 henselian valued fields, we can even find $\Phi : (K', w') \hookrightarrow (K, w)$ that induces $\phi : (k'_{w'}, \bar{v}') \hookrightarrow (k_w, \bar{v})$. Finally, we get that the map $\Phi : (K', v') \hookrightarrow (K, v)$ is an embedding of valued fields, as in the proof of Claim 2, Theorem 2.1.3.

(b) For $f(x_1, \dots, x_m) \in \mathbb{Z}[x_1, \dots, x_m]$ and $(a_1, \dots, a_m) \in \mathcal{O}_v^m$, note that $f(a_1, \dots, a_m) \neq 0 \pmod{p^n \mathcal{O}_v}$ if and only if there exists $y \in \mathfrak{m}_v$ such that $f(a_1, \dots, a_m) \cdot y = p^n \pmod{p^{n+1} \mathcal{O}_v}$ (similarly for $\mathcal{O}_{v'}$). Consequently, for each $n \in \mathbb{N}$, we see that if $(\mathcal{O}_v/p^{n+1} \mathcal{O}_v, \mathfrak{m}_{n+1}) \equiv_{\exists+} (\mathcal{O}_{v'}/p^{n+1} \mathcal{O}_{v'}, \mathfrak{m}'_{n+1})$ in L_{lcr} , then $\mathcal{O}_v/p^n \mathcal{O}_v \equiv_{\exists} \mathcal{O}_{v'}/p^n \mathcal{O}_{v'}$ in L_{rings} . The conclusion follows from part (a). \square

2.3 Decidability

We now harvest the consequences of Theorem 2.1.3 and 2.2.7 in relation to decidability. Since the countable union of recursive sets is not guaranteed to be recursive, we need to ask not only that each individual $\mathcal{O}_K/(p^n)$ be decidable in L_{rings} but also that the sequence $(\mathcal{O}_K/(p^n))_{n \in \omega}$ be *uniformly decidable* in L_{rings} :

Corollary 2.3.1. *Let (K, v) be a henselian valued field of mixed characteristic. Suppose that (Γ_v, vp) is decidable in L_{oag} with a constant for vp and that the sequence $(\mathcal{O}_K/(p^n))_{n \in \omega}$ is uniformly decidable in L_{rings} . Then (K, v) is decidable in L_{val} .*

Proof. The identification $\Gamma_v = K^\times/\mathcal{O}^\times$ furnish us with a recursive interpretation Γ of (Γ_v, vp) in the valued field (K, v) . Let also Γ_n be the natural interpretation of $\mathcal{O}_K/(p^n)$ in the L_{val} -structure (K, v) , for each $n \in \mathbb{N}$. If $g_n : \text{Sent}_{L_{\text{rings}}} \rightarrow \text{Sent}_{L_{\text{val}}}$ denotes the reduction map of Γ_n , then one can see that the sequence $(g_n)_{n \in \omega}$ is uniformly recursive (using that Γ_n is uniform in $n \in \mathbb{N}$). Let

$$\Sigma := \text{Hen}_{(0,p)} \cup \left(\bigcup_{n \in \omega} \{ \phi_{\Gamma_n} : \mathcal{O}_K/(p^n) \models \phi \} \right) \cup \{ \phi_{\Gamma} : (\Gamma_v, vp) \models \phi \}$$

where $\text{Hen}_{(0,p)}$ is a first-order axiom schema capturing Hensel's lemma (see e.g., pg.21 [Kuh16a]), together with a set of sentences capturing that the valued field has mixed characteristic $(0, p)$.

Claim: The axiomatization Σ is r.e.

Proof. The set $\text{Hen}_{(0,p)}$ is clearly r.e. The set $\{\phi_\Gamma : (\Gamma_v, vp) \models \phi\} \subseteq L_{\text{val}}$ is r.e., being the image of a recursive set via the recursive reduction map of Γ . Since recursively enumerable sets are closed under finite unions, it remains to show that $\bigcup_{n \in \omega} \{\phi_{\Gamma_n} : \mathcal{O}_K/(p^n) \models \phi\}$ is r.e.

Let $\chi : \mathbb{N} \times \text{Sent}_{L_{\text{rings}}} \rightarrow \mathbb{N}$ be the recursive function associated to the uniformly decidable sequence $(\mathcal{O}_K/(p^n))_{n \in \omega}$. We construct the partial recursive function $\chi' : \mathbb{N} \times \text{Sent}_{L_{\text{rings}}} \rightarrow \mathbb{N} \times \text{Sent}_{L_{\text{rings}}}$ which maps $(n, \phi) \mapsto (n, \phi)$ if $\chi(n, \phi) = 1$ and is undefined if $\chi(n, \phi) = 0$. Consider also the recursive function $g : \mathbb{N} \times \text{Sent}_{L_{\text{rings}}} \rightarrow \text{Sent}_{L_{\text{rings}}} : (n, m) \mapsto g_n(m)$ associated to the uniformly recursive sequence $(g_n)_{n \in \omega}$. Observe that $\bigcup_{n \in \omega} \{\phi_{\Gamma_n} : \mathcal{O}_K/(p^n) \models \phi\} = \text{Im}(F)$ where F is the (partial) recursive function $F = g \circ \chi'$. It follows that $\bigcup_{n \in \omega} \{\phi_{\Gamma_n} : \mathcal{O}_K/(p^n) \models \phi\}$ is r.e. \square *Claim*

If $(K', v') \models \Sigma$, then $(K', v') \equiv (K, v)$ in L_{val} by Theorem 2.1.3. We therefore get that Σ is a complete axiomatization of (K, v) . We conclude that the L_{val} -theory of (K, v) admits a r.e. and complete axiomatization, whence (K, v) is decidable. \square

Corollary 2.3.2. *Let (K, v) be a henselian valued field of mixed characteristic. Suppose that the sequence $((\mathcal{O}_K/(p^n), \mathfrak{m}_n)_{n \in \omega}$ is uniformly \exists^+ -decidable in L_{lcr} . Then (K, v) is \exists -decidable in L_{val} .*

Proof. Similar to Corollary 2.3.1, ultimately using Theorem 2.2.7(b). \square

2.4 Ax-Kochen/Ershov with cross-section

2.4.1 Motivation

The main result of chapter 6 is Theorem 6.0.1, which states that the asymptotic theory of finite extensions of \mathbb{Q}_p is undecidable in L_{AKE} . This should be contrasted with the fact that each *individual* finite extension F of \mathbb{Q}_p is decidable in L_{AKE} (see Remark 1.3.9), at least for a suitable choice of a cross-section (see Example 2.4.4). It is true that F is decidable in L_{val} —being interpretable (without parameters) in \mathbb{Q}_p itself—but it is not immediately obvious that F is also decidable in L_{AKE} . For certain extensions F/\mathbb{Q}_p (e.g., unramified extensions or extensions of the form $\mathbb{Q}_p(p^{1/n})$), this again follows by interpreting (without parameters) the L_{AKE} -structure F in the L_{AKE} -structure \mathbb{Q}_p , equipped with the normalized cross-section. The general case does not seem to be amenable to such interpretability methods and we are not aware of a reference. This result is important to put Theorem 6.0.1 in context and we shall therefore spell out the details below.

2.4.2 Setting

Suppose F/\mathbb{Q}_p is a finite extension with residue field \mathbb{F}_q and write $\mathcal{O}_F = W(\mathbb{F}_q)[x]/(E(x))$, where $E(x) \in \mathbb{Z}[\zeta_{q-1}][x]$ is Eisenstein of degree $e = e(K/\mathbb{Q}_p)$ and $W(\mathbb{F}_q)$ is the ring of Witt vectors over \mathbb{F}_q . We can indeed assume that $E(x)$ has coefficients in $\mathbb{Z}[\zeta_{q-1}] \subsetneq W(\mathbb{F}_q)$ by Krasner's Lemma. We shall argue that F , equipped with the cross-section $s : \mathbb{Z} \rightarrow F^\times : n \mapsto x^n$, is decidable in L_{AKE} . We shall provide a complete and recursive axiomatization.

2.4.3 Axiomatization

Let T be the theory of henselian valued fields (K, v, s) such that:

1. $k_v = \mathbb{F}_q$.
2. $(\Gamma, vp) \equiv (\mathbb{Z}, e)$ in L_{oag} .
3. π is a root of $E(x)$, where $\pi = s(1)$.

Note that (3) is first-order expressible because $E(x) \in \mathbb{Z}[\zeta_{q-1}][x]$. Theorem 2.4.6 shows that T is complete, by demonstrating that any two saturated models are isomorphic. The decidability of F will then follow from the *recursiveness* of the above set of axioms. Note that we have been careful with the choice of the cross-section. Indeed, not every choice will lead to a decidable L_{AKE} -theory:

Example 2.4.4. Let $a = \sum_{n=0}^{\infty} \alpha_n \cdot p^n \in \mathbb{Z}_p$, with $\alpha_n \in \{0, \dots, p-1\}$, be an *uncomputable* p -adic integer with $v_p a = 1$. The former condition means that the function $d_a : \mathbb{N} \rightarrow \{0, \dots, p-1\} : n \mapsto \alpha_n$ is non-recursive. Let $s_a : \mathbb{Z} \rightarrow \mathbb{Q}_p^\times : 1 \mapsto a$ be the associated cross-section. One sees that \mathbb{Q}_p , equipped with the cross-section s_a , is undecidable in L_{AKE} .

2.4.5 Statement and proof

Following the notation of [Dri12], given an embedding $\Phi : (K, v) \rightarrow (K', v')$ of valued fields, we write $\Phi_r : k \rightarrow k'$ and $\Phi_v : \Gamma \rightarrow \Gamma'$ for the induced residue field and value group embeddings respectively. When K and K' are equipped with cross-sections s and s' respectively, an embedding $(K, v, s) \rightarrow (K', v', s')$ must also satisfy $\Phi \circ s = s' \circ \Phi_v$. If in addition K and K' are equipped with liftings of their residue fields, say ι and ι' respectively, an embedding $(K, v, s, \iota) \rightarrow (K', v', s', \iota')$ is also required to satisfy $\Phi \circ \iota = \iota' \circ \Phi_r$.

The proof of Theorem 2.4.6 is a standard *coarsening* argument (see §7.4 [Dri12]), ultimately reducing the problem to equal characteristic 0, and follows closely the proof of Theorem 1 [DM16]. The only nuance is that we need to equip the coarsened valued fields with suitable cross-sections.

Theorem 2.4.6. *The theory T is complete. In particular, F is decidable in L_{AKE} .*

Proof. As the statement at hand is absolute, we may assume the continuum hypothesis (see pg. 122 [Dri12]). We may therefore assume throughout the existence of saturated models of uncountable cardinality (Corollary 4.3.13 [Mar02]). Let (K, v, s) and (K', v', s') be two saturated models of T of the same size. By our assumption, we have an isomorphism of ordered abelian groups $\Gamma_v \cong \Gamma_{v'}$ and a ring isomorphism $k_v \cong k_{v'}$. We shall argue that there is an isomorphism $(K, v, s) \cong (K', v', s')$.

Consider the finest coarsening w of v of residue characteristic 0. By Lemma 7.13 [Dri12], the coarsened valued field (K, w) is henselian. The value group is $\Gamma_w = \Gamma_v/\Delta$, where $\Delta = \mathbb{Z}v\pi$ and the residue field is \mathbb{F}_q . Let \bar{v} be the induced valuation from v on the residue field k_w . Let also $\bar{s} = \text{res}_w \circ s|_{\Delta}$ be the induced cross-section on (k_w, \bar{v}) . We also have a lifting $\iota : k_w \rightarrow \mathcal{O}_w$ of $\text{res}_w : \mathcal{O}_w \rightarrow k_w$ (Theorem 2.9 [Dri12]). We consider analogous objects for K' .

Claim 1: There are isomorphisms $\phi : (k_w, \bar{v}, \bar{s}) \cong (k_{w'}, \bar{v}', \bar{s}')$ and $\Gamma_w \cong \Gamma_{w'}$.

Proof. By Hensel's Lemma, we have that $\zeta_{q-1} \in K$. The subfield $\mathbb{Q}(\zeta_{q-1}) \subseteq \mathcal{O}_w$ maps isomorphically into k_w via $\text{res}_w : \mathcal{O}_w \rightarrow k_w$ and we still write $E(x)$ for the image of $E(x)$ in k_w . The element $\pi = s(1)$ maps to a root $\bar{\pi} \in k_w$ of $E(x)$. By \aleph_1 -saturation, the valued field (k_w, \bar{v}) is a complete discrete valued field (see Lemma 7.14 [Dri12]), of absolute ramification e and with residue field \mathbb{F}_q . By the structure theorem for mixed characteristic complete DVRs, we deduce that $\mathcal{O}_{\bar{v}} \cong W(\mathbb{F}_q)[x]/(E(x))$ via an isomorphism which identifies $\bar{\pi}$ with x . Moreover, we have that $\bar{s}(1) = x$, via the above identification. A similar analysis holds for K' and we get an isomorphism $\phi : (k_w, \bar{v}, \bar{s}) \cong (k_{w'}, \bar{v}', \bar{s}')$. We also have that $\Gamma_w \cong \Gamma_{w'}$, since the isomorphism $\Gamma_v \cong \Gamma_{v'}$ descends to the quotients $\Gamma_v/\Delta \cong \Gamma_{v'}/\Delta'$. $\square_{\text{Claim 1}}$

Note that $\iota \circ \bar{s} = s|_{\Delta}$ (resp. $\iota' \circ \bar{s}' = s'|_{\Delta'}$). Indeed, since $\Delta \cong \mathbb{Z}$, it suffices to check that $\iota(\bar{\pi}) = \pi$. Now $\iota(\bar{\pi})$ is a root of $E(x)$ which maps to $\bar{\pi}$ via res_w . By the uniqueness part of Hensel's Lemma, we get that $\iota(\bar{\pi}) = \pi$.

Claim 2: There exist saturated elementary extensions $(K_*, v_*, w_*, s_*, t_*)$ (resp.

$(K'_*, v'_*, w'_*, s'_*, t'_*)$ of (K, v, w, s, t) (resp. (K', v', w', s', t')) such that the exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Delta_* & \longrightarrow & \Gamma_* & \xrightarrow{\quad} & \Gamma_*/\Delta_* \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & \swarrow \text{---} \chi & \downarrow \cong \\ 0 & \longrightarrow & \Delta'_* & \longrightarrow & \Gamma'_* & \xrightarrow{\quad} & \Gamma'_*/\Delta'_* \longrightarrow 0 \\ & & & & & \swarrow \text{---} \chi' & \end{array}$$

split and the diagram commutes.

Proof. Set $(K_0, v_0, w_0, s_0, t_0) = (K, v, w, s, t)$ and for each $n \in \mathbb{N}$ let $(K_{n+1}, v_{n+1}, w_{n+1}, s_{n+1}, t_{n+1})$ be a $|K_n|^+$ -saturated elementary extension of $(K_n, v_n, w_n, s_n, t_n)$. For each $n \in \mathbb{N}$, we shall argue that Γ_n/Δ_n embeds in Γ_{n+1} :

Since $(\Gamma_v, v\pi) \equiv (\mathbb{Z}, 1)$, we will have that $\Gamma_v/\mathbb{Z}v\pi$ is divisible. Since $(\Gamma_n, \Delta_n) \equiv (\Gamma_v, \mathbb{Z}v\pi)$, it follows that Γ_n/Δ_n is also divisible. We may now view Γ_n/Δ_n as an ordered \mathbb{Q} -vector space. Let $\mathcal{B} = \{\gamma_i + \Delta_n\}_{i \in I}$ be a \mathbb{Q} -basis of Γ_n/Δ_n . Since Γ_n/Δ_n is divisible, we have that for each $m \in \mathbb{N}$ and each equivalence class $\gamma_i + \Delta_n$, there is $\gamma'_i \in \Gamma_n$ such that $\gamma_i + \Delta_n = \gamma'_i + \Delta_n$ with γ'_i being m -divisible in Γ_n . By $|K_n|^+$ -saturation of Γ_{n+1} , for each equivalence class $\gamma_i + \Delta_n$, there is $\gamma'_i \in \Gamma_{n+1}$ such that $\gamma_i + \Delta_{n+1} = \gamma'_i + \Delta_{n+1}$, with γ'_i divisible in Γ_{n+1} . We can now embed Γ_n/Δ_n in Γ_{n+1} via $\chi_n : \gamma_i + \Delta_n \mapsto \gamma'_i$ and extending it \mathbb{Q} -linearly. Note that χ_n is order preserving because

$$\sum_{i \in I_0} q_i(\gamma_i + \Delta_n) > 0 \Rightarrow \sum_{i \in I_0} q_i \gamma_i > \Delta_n \Rightarrow \sum_{i \in I_0} q_i \gamma_i > \Delta_{n+1} \Rightarrow \sum_{i \in I_0} q_i \gamma'_i > 0$$

for any finite $I_0 \subseteq I$ and $q_i \in \mathbb{Q}$.

The structure $K_\infty := \bigcup_{n \in \mathbb{N}} K_n$ is also an elementary extension of K by Tarski-Vaught and the quotient map $\Gamma_\infty \rightarrow \Gamma_\infty/\Delta_\infty$ admits a section $\chi : \Gamma_\infty/\Delta_\infty \rightarrow \Gamma_\infty$, namely $\chi = \bigcup_{n=1}^\infty \chi_n$. Having assumed the continuum hypothesis, we may pass to a saturated elementary extension $(K_*, v_*, w_*, s_*, t_*, \chi_*)$ of $(K_\infty, v_\infty, w_\infty, s_\infty, t_\infty, \chi_\infty)$. We also pass to a saturated elementary extension $(K'_*, v'_*, w'_*, s'_*, t'_*)$ of (K', v', w', s', t') of the same size. We then have an isomorphism $\Gamma'_* \cong \Gamma_*$, which restricts to an isomorphism $\Delta'_* \cong \Delta_*$. In particular, we may choose a section $\chi' : \Gamma'_*/\Delta'_* \rightarrow \Gamma'_*$ such that the above diagram commutes. □*Claim 2*

It is now harmless to replace K (resp. K') with K_* (resp. K'_*). Claim 2 allows us to write $\Gamma_v = \Delta \oplus Q$ (resp. $\Gamma_{v'} = \Delta' \oplus Q'$), where $Q = \text{Im}(\chi)$ (resp. $Q' = \text{Im}(\chi')$). Let $s|_Q : Q \rightarrow K$ (resp. $s'|_{Q'} : Q' \rightarrow K'$) be the cross-section of (K, w) obtained by restricting s on Q (resp. Q').

By Claim 1, we have isomorphisms $\phi : (k_w, \bar{v}, \bar{s}) \cong (k'_{w'}, \bar{v}', \bar{s}')$ and $\Gamma_w \cong \Gamma_{w'}$. By the Ax-Kochen/Ershov principle in equal characteristic 0 (see Theorem 5.11

[Dri12]) and saturation, we therefore get that $(K, w, s|_Q, \iota_w) \cong (K', w', s'|_{Q'}, \iota_{w'})$. Now recall that residue fields are stably embedded in henselian valued fields of equal characteristic 0, even in the enriched language with a cross-section and a lifting of the residue field (see Corollary 5.15 [Dri12]). Therefore, by one of the equivalent conditions for stable embeddedness (see Lemma 1(6) [CH99]) there is even an isomorphism $\Phi : (K, w, s|_Q, \iota_w) \xrightarrow{\cong} (K', w', s'|_{Q'}, \iota_{w'})$ inducing $\phi : (k_w, \bar{v}, \bar{s}) \xrightarrow{\cong} (k'_{w'}, \bar{v}', \bar{s}')$, i.e., with $\Phi_{\mathfrak{r}} = \phi$.

Claim 3: The map Φ is an isomorphism between (K, v, s) and (K', v', s') .

Proof. First we check that Φ is a valued field isomorphism. Given $x \in K$, we need to show that $vx \geq 0 \iff v'(\Phi(x)) \geq 0$. If $vx \geq 0$, then either (i) $wx > 0$ or (ii) $wx = 0$ and $\bar{v}\bar{x} \geq 0$, where $\bar{x} \in k_v$ is the image of x via $\text{res}_w : \mathcal{O}_w \rightarrow k_w$. In the first case, we get that $w'(\Phi(x)) > 0$ as $\Phi : (K, w) \rightarrow (K', w')$ is a valued field homomorphism and therefore $v'(\Phi(x)) > 0$ as $\mathfrak{m}_{w'} \subset \mathfrak{m}_{v'}$. Suppose now that $wx = 0$ and $\bar{v}\bar{x} \geq 0$. Then we also get that $wx = 0$ and $\bar{v}'(\phi(\bar{x})) \geq 0$ as $\phi : (k_w, \bar{v}) \rightarrow (k'_{w'}, \bar{v}')$ is a valued field homomorphism. Since Φ induces ϕ , we get that $\bar{v}'(\overline{\Phi(x)}) = \bar{v}'(\phi(\bar{x})) \geq 0$ and conclude that $v'(\Phi(x)) \geq 0$.

We also need to check that Φ respects the cross-sections, i.e. $\Phi(s(\gamma)) = s'(\Phi_v(\gamma))$. Write $\gamma = \gamma_{\Delta} + \gamma_Q$. We have that $\Phi(s(\gamma_Q)) = s'(\Phi_v(\gamma_Q))$ and also

$$\Phi(s(\gamma_{\Delta})) = \Phi(\iota(\bar{s}(\gamma_{\Delta}))) = \iota'(\phi(\bar{s}(\gamma_{\Delta}))) = \iota'(\bar{s}'(\phi_v(\gamma_{\Delta}))) = s'(\phi_v(\gamma_{\Delta})) = s'(\Phi_v(\gamma_{\Delta}))$$

using that $s = \iota \circ \bar{s}$ (resp. $s' = \iota' \circ \bar{s}'$), $\phi \circ \bar{s} = \bar{s}' \circ \phi_v$ and $\Phi \circ \iota = \iota' \circ \Phi_{\mathfrak{r}} = \iota' \circ \phi$. We now compute

$$\begin{aligned} \Phi(s(\gamma)) &= \Phi(s(\gamma_{\Delta} + \gamma_Q)) = \Phi(s(\gamma_{\Delta}) \cdot s(\gamma_Q)) = \Phi(s(\gamma_{\Delta})) \cdot \Phi(s(\gamma_Q)) \\ &= s'(\Phi_v(\gamma_{\Delta})) \cdot s'(\Phi_v(\gamma_Q)) = s'(\Phi_v(\gamma_{\Delta} + \gamma_Q)) = s'(\Phi_v(\gamma)) \end{aligned}$$

□ *Claim 3*

□

3

Perfectoid fields

Introduction

The material presented in this chapter is based on [Kar20]. We shall investigate the model theory of perfectoid fields, focusing on certain extensions of \mathbb{Q}_p and $\mathbb{F}_p((t))$. For the purposes of the introduction, the reader is welcome to think of extensions of \mathbb{Q}_p or $\mathbb{F}_p((t))$ containing lots of p -power roots of elements in the field. Some examples which have been of historical interest to model-theorists are listed below. (The fields in the list below are not quite perfectoid, as they are not complete, but we will soon explain that this is of minor importance for any model-theoretic purposes.)

- In mixed characteristic, these include \mathbb{Q}_p^{ab} , the maximal abelian extension of \mathbb{Q}_p and the totally ramified extension $\mathbb{Q}_p(\zeta_{p^\infty})$, obtained by adjoining all p^n -th roots of unity. These extensions had already been discussed in Macintyre's survey on pg.140 [Mac86] and a conjectural axiomatization of \mathbb{Q}_p^{ab} was given by Koenigsmann on pg.55 [Koe18]. Another interesting extension is $\mathbb{Q}_p(p^{1/p^\infty})$, a totally ramified extension of \mathbb{Q}_p obtained by adjoining a compatible system of p -power roots of p .
- In positive characteristic, two very natural infinitely ramified fields are the perfect hulls of $\mathbb{F}_p((t))$ and $\overline{\mathbb{F}}_p((t))$. These are *semitame* fields, using the terminology of Kuhlmann-Rzepka [KR21], and are both expected to be decidable (see pg. 4 [KR21]). The ultimate goal of [KR21] is to extend Kuhlmann's results for tame fields to the more general framework of semitame fields.

The above fields will be the main objects of interest throughout the chapter. Their p -adic (resp. t -adic) completions are typical examples of *perfectoid fields* in the sense of Scholze [Sch12a] (see §3.1). For any such field K , one can define its tilt K^\flat (see §3.1.5), which intuitively is its *local* function field analogue. This can be thought of as a strengthening of the classical (local) function field analogy between \mathbb{Q}_p and $\mathbb{F}_p((t))$. The perfectoid framework often allows one to reduce arithmetic problems about K to arithmetic problems about K^\flat . This kind of transfer principle, which works for a fixed p , should be contrasted with the Ax-Kochen principle which achieves such a reduction only *asymptotically*, i.e. with the residue characteristic $p \rightarrow \infty$. This is explained in detail in Scholze's ICM report (see pg.2 [Sch14]).

Our main goal is to set the stage for incorporating ideas from perfectoid geometry in the model theory of henselian fields. In the present chapter we focus on decidability, although it is conceivable that our methods can be used in different model-theoretic contexts. We will prove the following relative decidability result for the fields discussed above:

Corollary 3.0.1. (a) Assume $\mathbb{F}_p((t))^{1/p^\infty}$ is decidable (resp. \exists -decidable) in L_t . Then $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p(\zeta_{p^\infty})$ are decidable (resp. \exists -decidable) in L_{val} .
 (b) Assume $\overline{\mathbb{F}_p}((t))^{1/p^\infty}$ is decidable (resp. \exists -decidable) in L_t . Then \mathbb{Q}_p^{ab} is decidable (resp. \exists -decidable) in L_{val} .

As usual, we write $\mathbb{F}_p((t))^{1/p^\infty}$ (resp. $\overline{\mathbb{F}_p}((t))^{1/p^\infty}$) for the perfect hull of $\mathbb{F}_p((t))$ (resp. $\overline{\mathbb{F}_p}((t))$), L_{val} for the language of valued fields and L_t for the language L_{val} enriched with a constant symbol for t (see notation). Corollary 3.2.20 is essentially a special case of a general relative decidability result for perfectoid fields, which is discussed next. As was noted above, the fields in Corollary 3.0.1 are not quite perfectoid (they are not complete) but one can readily adapt the proof of Theorem 3.0.2 to obtain Corollary 3.0.1. We also note that the above mixed characteristic fields have the same theory as their completions (see Remark 3.2.21). The same is true for the characteristic p fields listed above. This will not be discussed here but will appear in upcoming joint work with F. Jahnke.

Let F be a perfectoid field of characteristic p (e.g., $F = \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$). An *untilt* of F is a mixed characteristic perfectoid field K together with an isomorphism $\iota : K^\flat \xrightarrow{\cong} F$. In general, there will be many non-isomorphic untilts of F . In fact, there will be too many untilts even up to elementary equivalence (see Proposition 3.1.36(b)), thus shattering the naive guess of K being decidable relative to its tilt K^\flat that one might initially have. On a more elementary level, ones needs

to assume decidability with *parameters* on the positive characteristic side (see Example 3.2.13). Nevertheless, a relative decidability result for perfectoids can be salvaged by asking that K be an R_0 -computable untilt of F , a notion which is briefly explained below (see §3.2.2 for details).

Write $W(\mathcal{O}_F)$ for the ring of Witt vectors over \mathcal{O}_F (see §3.1.13). An element $\xi \in W(\mathcal{O}_F)$ is identified with its associated Witt vector, which is an infinite sequence (ξ_0, ξ_1, \dots) with $\xi_n \in \mathcal{O}_F$. Whenever there is a *computable* subring $R_0 \subseteq \mathcal{O}_F$ such that $\xi_n \in R_0$ and the function $\mathbb{N} \rightarrow R_0 : n \mapsto \xi_n$ is *recursive*, we say that ξ is R_0 -computable. As usual, a computable ring R_0 is one whose underlying set is (or may be identified with) a recursive subset of \mathbb{N} , so that the ring operations are (or are identified with) recursive functions (e.g., $\mathbb{F}_p[t]$ is a computable ring). An important result by Fargues-Fontaine gives a 1-1 correspondence between untilts of F and certain principal ideals of $W(\mathcal{O}_F)$ (Theorem 3.1.29). An untilt K of F is then said to be an R_0 -computable untilt, whenever there exists an R_0 -computable generator $\xi \in W(\mathcal{O}_F)$ of the ideal corresponding to K . We then have the following relative decidability result (the name is due to Scanlon):

Theorem 3.0.2 (Perfectoid transfer). *Suppose K is an R_0 -computable untilt of F . If F is decidable in $L_{\text{val}}(R_0)$, then K is decidable in L_{val} .*

Here $L_{\text{val}}(R_0)$ is the language L_{val} enriched with constant symbols for the elements of R_0 . We also obtain an *existential* version of Theorem 3.0.2 in §3.2.8. The condition on K being an R_0 -computable untilt of F is virtually true in all cases of interest (although generically false), even for a natural choice of R_0 . Corollary 3.2.20 is an immediate consequence of Theorem 3.0.2, by considering the t -adic completion of $\mathbb{F}_p((t))^{1/p^\infty}$ (resp. $\overline{\mathbb{F}_p}((t))^{1/p^\infty}$ for part (b)) and $R_0 = \mathbb{F}_p[t]$. The two key ingredients in the proof of Theorem 3.0.2 are:

1. An (unpublished) Ax-Kochen/Ershov style result by van den Dries, presented in §2.1.
2. The Fargues-Fontaine correspondence described above, presented in §3.1.22.

The result by van den Dries will enable us to reduce the decidability of K to the *uniform* decidability of the sequence of residue rings $(\mathcal{O}_K/(p^n))_{n \in \omega}$ and also the decidability of (Γ_v, vp) . The Fargues-Fontaine correspondence then gives us a way to interpret each individual residue ring $\mathcal{O}_K/(p^n)$ inside the tilt F , with parameters from R_0 , via an interpretation Γ_n . The assumption on K being R_0 -computable will then imply that the sequence of interpretations $(\Gamma_n)_{n \in \omega}$ is *uniformly recursive*

(see §1.1.6). Having assumed the decidability of F in $L_{\text{val}}(R_0)$, this will then give us that the sequence $(\mathcal{O}_K/(p^n))_{n \in \omega}$ is uniformly decidable. The pointed value group (Γ_v, vp) is easily seen to be interpretable in F , with parameters from R_0 and the decidability of K follows.

It should be mentioned that—prior to this work—Rideau-Scanlon had made a formative observation, namely that the Fargues-Fontaine correspondence already provides us with an interpretation of the valuation ring \mathcal{O}_K in \mathcal{O}_F (with parameters) in the sense of continuous logic. This however does not yield an interpretation in the sense of ordinary first-order logic (see §3.2.1). On the other hand, there do exist honest first-order interpretations of the *truncated* residue rings $\mathcal{O}_K/(p^n)$ in \mathcal{O}_F , which is why it is convenient to work with them instead.

Theorem 3.0.2 allows us to reduce decidability problems from the mixed characteristic to the positive characteristic world. Although our model-theoretic understanding of the latter is notoriously limited, this method still yields several applications. In sections §3.3 and §3.4 we obtain new *unconditional* decidability results in mixed characteristic via reduction to characteristic p . In §3.3, we prove the following:

Corollary 3.0.3. The valued field $(\mathbb{Q}_p(p^{1/p^\infty}), v_p)$ (resp. $(\mathbb{Q}_p(\zeta_{p^\infty}), v_p)$) admits a maximal immediate extension which is decidable in L_{val} .

Note that the fields $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p(\zeta_{p^\infty})$ are *not* Kaplansky and have many non-isomorphic maximal immediate extensions, all of which are *tame* in the sense of Kuhlmann [Kuh16a] (see Example 3.3.5). Although the work of Kuhlmann yields several decidability results for equal characteristic tame fields, Corollary 3.0.3 is, to my knowledge, the first decidability result for tame fields of mixed characteristic (see §3.3). The proof uses a recent decidability result of Lisinski [Lis21] for the Hahn field $\mathbb{F}_p((t^\Gamma))$ with $\Gamma = \frac{1}{p^\infty}\mathbb{Z}$ in the language L_t , strengthening Kuhlmann’s earlier result for $\mathbb{F}_p((t^\Gamma))$ in the language L_{val} (see Theorem 1.6 [Kuh16a]). Corollary 3.0.3 then follows from Theorem 3.0.2 and basic properties of the tilting equivalence (see §3.1.22). It is worth remarking that most maximal immediate extensions of $\mathbb{Q}_p(p^{1/p^\infty})$ (resp. $\mathbb{Q}_p(\zeta_{p^\infty})$) are *undecidable* (see Remark 3.3.14).

Without making essential use of the perfectoid machinery but only the philosophy thereof, we show in §3.4 the following:

Theorem 3.0.4. Let K be any of the valued fields $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{Q}_p(\zeta_{p^\infty})$ or \mathbb{Q}_p^{ab} . The existential theory of $\mathcal{O}_K/p\mathcal{O}_K$ in the language of rings L_{rings} is decidable.

The proof is again via reduction to characteristic p , using a recent existential decidability result for equal characteristic henselian valued fields in the language L_{val} , due to Anscombe-Fehm [AF16]. However, as Corollary 3.0.1 suggests, if we aim to understand the theories of $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{Q}_p(\zeta_{p^\infty})$ and \mathbb{Q}_p^{ab} via reduction to positive characteristic, we will need stronger decidability results for $\mathbb{F}_p((t))^{1/p^\infty}$ and $\overline{\mathbb{F}_p}((t))^{1/p^\infty}$ in the language L_t on the characteristic p side. This is also supported by Proposition 3.5.5 which shows that the Diophantine problem for $\mathbb{F}_p((t))^{1/p^\infty}$ in L_t is Turing reducible to the $\forall^1\exists$ -theory of $\mathbb{Q}_p(p^{1/p^\infty})$.

An application of a different flavor, which yields an *undecidability* result in mixed characteristic via reduction to characteristic p , will be presented in Chapter 6. Theorem 6.0.1 shows that the asymptotic theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ in the language L_{val} with a cross-section is undecidable.

3.1 Perfectoid fields

3.1.1 Introduction

Motivation

The theory of perfectoid fields (and spaces), introduced by Scholze in [Sch12a], was initially designed as a means of transferring results available in positive characteristic to mixed characteristic (see §1 [Sch14]). It formalizes the earlier Krasner-Kazhdan-Deligne philosophy (due to [Kra56], [Kaz86] and [Del84]), of approximating a *highly ramified* mixed characteristic field with a positive characteristic field. Within the framework of perfectoid fields, this kind of approximation becomes precise and robust with the use of the tilting functor (see §3.1.5).

All this is substantially different from the Ax-Kochen method (see e.g., 2.20 [Dri12]), which achieves a model-theoretic transfer principle *asymptotically*, i.e. with the residue characteristic $p \rightarrow \infty$. The theory of perfectoid fields will allow us to transport decidability information for a fixed residue characteristic (but with high ramification), setting the stage for a different type of model-theoretic transfer principle.

Definition

Definition 3.1.2. A perfectoid field is a complete valued field (K, v) of residue characteristic $p > 0$ such that Γ_v is a dense subgroup of \mathbb{R} and the Frobenius map $\Phi : \mathcal{O}_K/(p) \rightarrow \mathcal{O}_K/(p) : x \mapsto x^p$ is surjective.

Example 3.1.3. (a) The p -adic completions of $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{Q}_p(\zeta_{p^\infty})$ and \mathbb{Q}_p^{ab} are mixed characteristic perfectoid fields.

(b) The t -adic completions of $\mathbb{F}_p((t))^{1/p^\infty}$ and $\overline{\mathbb{F}}_p((t))^{1/p^\infty}$ are perfectoid fields of characteristic p .

Remark 3.1.4. In characteristic p , a perfectoid field is simply a perfect, complete non-archimedean valued field.

3.1.5 Tilting

Introduction

A construction, originally due to Fontaine, provides us with a tilting functor that takes any perfectoid field K and transforms it into a perfectoid field K^\flat of characteristic p . We shall now describe this tilting functor. For more details, see §3 [Sch12a].

Definition

Given a perfectoid field (K, v) , we shall now define its tilt (K^\flat, v^\flat) . Let $\varprojlim_{x \mapsto x^p} K$ be the limit of the inverse system

$$\dots \xrightarrow{x \mapsto x^p} K \xrightarrow{x \mapsto x^p} K \xrightarrow{x \mapsto x^p} K$$

which is identified as $\varprojlim_{x \mapsto x^p} K = \{(x_n)_{n \in \omega} : x_{n+1}^p = x_n\}$, viewed as a multiplicative monoid via $(x_n)_{n \in \omega} \cdot (y_n)_{n \in \omega} = (x_n \cdot y_n)_{n \in \omega}$. Similarly, one can define the multiplicative monoid $\varprojlim_{x \mapsto x^p} \mathcal{O}_K$.

Let $\varpi \in \mathcal{O}_K$ be such that $0 < v\varpi \leq vp$ (e.g., $\varpi = p$ when $\text{char}(K) = 0$ and $\varpi = 0$ when $\text{char}(K) = p$) and consider the ring $\varprojlim_{\Phi} \mathcal{O}_K/(\varpi)$ which is the limit of the inverse system of rings

$$\dots \xrightarrow{\Phi} \mathcal{O}_K/(\varpi) \xrightarrow{\Phi} \mathcal{O}_K/(\varpi) \xrightarrow{\Phi} \mathcal{O}_K/(\varpi)$$

where $\Phi : \mathcal{O}_K/(\varpi) \rightarrow \mathcal{O}_K/(\varpi) : x \mapsto x^p$ is the Frobenius homomorphism.

Lemma 3.1.6 (Lemma 3.4 (i), (ii) [Sch12a]). (a) The ring $\varprojlim_{\Phi} \mathcal{O}_K/(\varpi)$ is independent of the choice of ϖ and there is a multiplicative isomorphism $\varprojlim_{x \mapsto x^p} \mathcal{O}_K \xrightarrow{\cong} \varprojlim_{\Phi} \mathcal{O}_K/(\varpi)$. Moreover, we get a multiplicative morphism $\sharp : \varprojlim_{\Phi} \mathcal{O}_K/(\varpi) \rightarrow \mathcal{O}_K : x \mapsto x^\sharp$ such that if $x = (x_n + (\varpi))_{n \in \omega}$, then $x^\sharp \equiv x_0 \pmod{(\varpi)}$.

(b) There is an element $\varpi^\flat \in \varprojlim_{\Phi} \mathcal{O}_K/(\varpi)$ with $v(\varpi^\flat)^\sharp = v\varpi$.

The definition of \sharp goes as follows: Let $x = (x_n)_{n \in \omega} \in \varprojlim_{\Phi} \mathcal{O}_K/(\varpi)$, i.e. $x_n \in \mathcal{O}_K/(\varpi)$ and $x_{n+1}^p = x_n$. Let $\tilde{x}_n \in \mathcal{O}_K$ be an arbitrary lift of $x_n \in \mathcal{O}_K/(\varpi)$. Then the limit $\lim_{n \rightarrow \infty} \tilde{x}_n^{p^n}$ exists and is independent of the choice of the \tilde{x}_n 's (see Lemma 3.4(i) [Sch12a]). We define $x^\sharp := \lim_{n \rightarrow \infty} \tilde{x}_n^{p^n}$.

We now introduce $K^\flat := \varprojlim_{\Phi} \mathcal{O}_K/(\varpi)[(\varpi^\flat)^{-1}]$. For now this is merely a ring. It can be made into a valued field according to the following:

Lemma 3.1.7 (Lemma 3.4 (iii) [Sch12a]). (a) There is a morphism of multiplicative monoids $K^\flat \rightarrow K : x \mapsto x^\sharp$ (extending the one of Lemma 3.1.6(a)), which induces an isomorphism of multiplicative monoids $K^\flat \xrightarrow{\cong} \varprojlim_{x \mapsto x^p} K : x \mapsto (x^\sharp, (x^\sharp)^{1/p}, \dots)$. The map $v^\flat : K^\flat \rightarrow \Gamma_v \cup \{\infty\} : x \mapsto vx^\sharp$ is a valuation on K^\flat , which makes (K^\flat, v^\flat) into a perfectoid field of characteristic p . If \mathcal{O}_{K^\flat} is the valuation ring of K^\flat , then we have a ring isomorphism $\mathcal{O}_{K^\flat} \cong \varprojlim_{\Phi} \mathcal{O}_K/(\varpi)$.

(b) We have an isomorphism of ordered abelian groups $(\Gamma_v, v\varpi) \cong (v^\flat K^\flat, v^\flat \varpi^\flat)$ and a field isomorphism $k_v \cong k_{v^\flat}$. Moreover, we have a ring isomorphism $\mathcal{O}_K/(\varpi) \cong \mathcal{O}_{K^\flat}/(\varpi^\flat)$.

Remark 3.1.8. Lemma 3.1.7(a) allows us to identify the multiplicative underlying monoid of K^\flat with $\varprojlim_{x \mapsto x^p} K$. It is not hard to see that, via this identification, addition is described by $(x_n)_{n \in \omega} + (y_n)_{n \in \omega} = (z_n)_{n \in \omega}$, where $z_n = \lim_{m \rightarrow \infty} (x_{n+m} + y_{n+m})^{p^m}$.

Definition 3.1.9. We say that the valued field (K^\flat, v^\flat) constructed in Lemma 3.1.7(a) is the tilt of the perfectoid field (K, v) .

Remark 3.1.10 (Lemma 3.4 (iv) [Sch12a]). If (K, v) is a perfectoid field of characteristic p , then $(K^\flat, v^\flat) \cong (K, v)$.

Example 3.1.11 (see Corollary 3.2.17). In the examples below, \widehat{K} stands for the p -adic (resp. t -adic) completion of the field K depending on whether its characteristic is 0 or p .

- (a) $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}^\flat \cong \widehat{\mathbb{F}_p((t))}^{1/p^\infty}$ and $t^\sharp = p$.
- (b) $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}^\flat \cong \widehat{\mathbb{F}_p((t))}^{1/p^\infty}$ and $(t+1)^\sharp = \zeta_p$.
- (c) $\widehat{\mathbb{Q}_p^{ab}}^\flat \cong \widehat{\mathbb{F}_p((t))}^{1/p^\infty}$ and $(t+1)^\sharp = \zeta_p$.

Remark 3.1.12. The tilting construction makes sense for non-perfectoid fields as well. However, in the absence of infinite wild ramification, it is too lossy for it to be useful (e.g., $\mathbb{Q}_p^\flat = \mathbb{F}_p$).

3.1.13 Witt vectors

We review the basics of Witt vectors. Details and proofs can be found in §5,6 [Ser80] and §6 [Dri12].

p -rings

Definition 3.1.14. A ring R of characteristic p is called perfect if the Frobenius homomorphism $\Phi : R \rightarrow R : x \mapsto x^p$ is bijective.

Definition 3.1.15. (a) A p -ring is a commutative ring A provided with a filtration $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \dots$ such that $\mathfrak{a}_n \mathfrak{a}_m \subset \mathfrak{a}_{n+m}$ and so that A is Hausdorff and complete with respect to the topology induced by the filtration and A/\mathfrak{a}_1 is a perfect ring of characteristic p .

(b) If in addition $\mathfrak{a}_n = p^n A$ and p is not a zero-divisor, then we say that A is a strict p -ring.

If A is a p -ring, we call the quotient ring A/pA the *residue ring* of the p -ring A and write $\text{res} : A \rightarrow A/pA$ for the quotient map. A system of *multiplicative representatives* is a multiplicative homomorphism $f_A : A/pA \rightarrow A$ such that $f_A(\text{res}(x)) = x$. A p -ring A always has a system of representatives $f_A : A/pA \rightarrow A$ and when A is strict every element $a \in A$ can be written uniquely in the form $a = \sum_{i=0}^{\infty} f_A(\alpha_i) \cdot p^i$ (pg. 37 [Ser80]).

Proposition 3.1.16 (Proposition 10, pg. 38 [Ser80]). Let A and A' be two p -rings with residue rings R and R' respectively, multiplicative representatives $f_A : R \rightarrow A$ and $f_{A'} : R' \rightarrow A'$ respectively and suppose that A is strict. For every ring homomorphism $\phi : R \rightarrow R'$ there exists a unique homomorphism $g : A \rightarrow A'$ making the diagram below commute

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ R & \longrightarrow & R' \end{array}$$

More precisely, $g(\sum_{i=0}^{\infty} f_A(\alpha_i) \cdot p^i) = \sum_{i=0}^{\infty} f_{A'}(\phi(\alpha_i)) \cdot p^i$.

Theorem 3.1.17 (Corollary pg.39 [Ser80]). *For every perfect ring R , there exists a unique strict p -ring, denoted by $W(R)$, with residue ring $W(R)/pW(R) \cong R$.*

The ring $W(R)$ is said to be the *ring of Witt vectors* over the ring R .

Teichmüller representatives

When R is perfect, one has a system of *Teichmüller representatives* of R in $W(R)$. This is the (unique) multiplicative homomorphism $[\] : R \rightarrow W(R)$ with the property that $\text{res}([x]) = x$ for all $x \in R$. Explicitly, for $x \in R$ and $n \in \mathbb{N}$, let $x_n \in R$ be such that $x_n^{p^n} = x$ and $\tilde{x}_n \in W(R)$ be an arbitrary lift of x_n . The sequence $(\tilde{x}_n^{p^n})_{n \in \mathbb{N}}$ is a Cauchy sequence, whose limit is independent of the chosen lifts. We let $[x] := \lim_{n \rightarrow \infty} \tilde{x}_n^{p^n}$ (see Proposition 8 pg. 35 [Ser80]).

It is easy to see that any element $x \in W(R)$ can be written *uniquely* in the form $x = \sum_{n=0}^{\infty} [x_n] \cdot p^n$, for some $x_i \in R$. The vector $(x_0, x_1, \dots) \in R^\omega$ is called the *Teichmüller vector* of x .

Witt vectors

The advantage of Witt vectors over Teichmüller vectors, comes from the fact that the ring operations in $W(R)$ have nicer coordinatewise descriptions when using the former (see §3.1.13). Write $x \in W(R)$ in the form $x = \sum_{n=0}^{\infty} [x_n^{p^{-n}}] \cdot p^n$, for some $x_i \in R$. The vector $(x_0, x_1, \dots) \in R^\omega$ is called the *Witt vector* of x .

Ring operations

By the discussion above, the ring $W(R)$ can be thought of as the p -adic analogue of formal power series with coefficients in R . By identifying x with its Witt vector, we see that $W(R)$ has R^ω as its underlying set. By Lemma 6.5 [Dri12], the ring operations are given by

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) = (S_0(a_0, b_0), S_1(a_0, a_1, b_0, b_1), \dots)$$

and

$$(a_0, a_1, \dots) \cdot (b_0, b_1, \dots) = (P_0(a_0, b_0), P_1(a_0, a_1, b_0, b_1), \dots)$$

for suitable polynomials $S_i, P_i \in \mathbb{Z}[x_0, \dots, x_i, y_0, \dots, y_i]$ which are universal, in the sense that they do not depend on R .

Observation 3.1.18. The polynomials S_i (resp. P_i) are *computable*, i.e. the function $\mathbb{N} \rightarrow \mathbb{Z}[x_0, y_0, \dots] : n \mapsto S_n$ (resp. $\mathbb{N} \rightarrow \mathbb{Z}[x_0, y_0, \dots] : n \mapsto P_n$) is recursive.

Proof. For $n \in \mathbb{N}$, we introduce the n -th Witt polynomial $W_n(x_0, \dots, x_n) = x_0^{p^n} + px_1^{p^{n-1}} + \dots + p^n \cdot x_n \in \mathbb{Z}[x_0, \dots, x_n]$ (pg. 135 [Dri12]). The proof of Lemma 6.5 [Dri12] shows that $S_0(x_0, y_0) = x_0 + y_0$ and $W_{n-1}(S_0^p, \dots, S_{n-1}^p) + p^n \cdot S_n = W_{n-1}(S_0(x_0^p, y_0^p), \dots, S_{n-1}(x_0^p, \dots, x_{n-1}^p, y_0^p, \dots, y_{n-1}^p)) + p^n(x_n + y_n)$, whence the polynomial S_n may be computed recursively from S_0, \dots, S_{n-1} . The proof is similar for the P_n 's. \square

3.1.19 Truncated Witt vectors

Definition

In this chapter, we will mostly be working with truncated Witt vectors. These can be thought of as p -adic analogues of truncated power series, i.e. elements of the ring $R[[t]]/(t^n) \cong R[t]/(t^n)$ (over some base ring R). More formally:

Definition 3.1.20. Let R be a perfect ring. Given $n \in \mathbb{N}$, the ring of n -truncated Witt vectors over R is defined as $W_n(R) := W(R)/p^n W(R)$.

Language

For a perfect ring R , the pair $(W(R), R)$ (resp. $(W_n(R), R)$) is viewed as a two-sorted structure with sorts \mathbf{W} for the Witt ring $W(R)$ (resp. $W_n(R)$) and \mathbf{R} for the residue ring R . The sort \mathbf{W} is equipped with the language of rings L_{rings} , while the sort \mathbf{R} may be equipped with any $L \supseteq L_{\text{rings}}$. We also have a function symbol for the Teichmüller map $[] : \mathbf{R} \rightarrow \mathbf{W}$. For each choice of a language L for the \mathbf{R} -sort, the resulting language will be denoted by $\langle L_{\text{rings}}, L \rangle$.

Interpretability in R

Lemma 3.1.21. Let R be a perfect ring, viewed as an L -structure with $L \supseteq L_{\text{rings}}$. For each $n \in \mathbb{N}$, there exists a quantifier-free interpretation Γ_n of the $\langle L_{\text{rings}}, L \rangle$ -structure $(W_n(R), R)$ in the L -structure R such that the sequence of interpretations $(\Gamma_n)_{n \in \omega}$ is uniformly recursive.

Proof. By §3.1.13, for $n \in \mathbb{N}$ the underlying set of $W_n(R)$ can be identified with R^n , so we take $\partial_{\Gamma_n}(x_1, \dots, x_n)$ to be $\bigwedge_{i=1}^n x_i = x_i$ and the coordinate map $f_{\Gamma_n} : R^n \rightarrow W_n(R)$ as the identity map. The ring operations of $W_n(R)$ are given by $(a_0, \dots, a_{n-1}) + (b_0, \dots, b_{n-1}) = (S_0(a_0, b_0), \dots, S_{n-1}(a_0, b_0, \dots, a_{n-1}, b_{n-1}))$ and $(a_0, \dots, a_{n-1}) \cdot (b_0, \dots, b_{n-1}) = (P_0(a_0, b_0), \dots, P_{n-1}(a_0, b_0, \dots, a_{n-1}, b_{n-1}))$, for certain polynomials $S_i, P_i \in \mathbb{Z}[x_0, \dots, x_i, y_0, \dots, y_i]$, for $i = 0, \dots, n-1$. We now need to describe the map $\phi \mapsto \phi_{\Gamma_n}$ on *unnested* atomic L_{rings} -formulas:

1. If $\phi(x, y, z)$ is the formula $x + y = z$ (here $x, y, z \in \mathbf{W}$), we may take $\phi_{\Gamma_n}(\bar{x}, \bar{y}, \bar{z})$ to be the L_{rings} -formula $\bigwedge_{i=0}^{n-1} z_i = S_i(x_0, y_0, \dots, x_{n-1}, y_{n-1})$.
2. If $\phi(x, y, z)$ is the formula $x \cdot y = z$ (here $x, y, z \in \mathbf{W}$), we may take $\phi_{\Gamma_n}(\bar{x}, \bar{y}, \bar{z})$ to be the L_{rings} -formula $\bigwedge_{i=0}^{n-1} z_i = P_i(x_0, y_0, \dots, x_{n-1}, y_{n-1})$.
3. If $\phi(x, y)$ is the formula $[x] = y$ (here $x \in \mathbf{R}$ and $y \in \mathbf{W}$), we may take $\phi_{\Gamma_n}(x, \bar{y})$ to be the formula $\bigwedge_{i=1}^{n-1} y_i = 0 \wedge y_0 = x$.

4. If $\phi(\bar{x})$ is an unnested atomic L -formula with variables from the sort \mathbf{R} , then we may take $\phi_{\Gamma_n}(\bar{x}) := \phi(\bar{x})$ (here $x_i \in \mathbf{R}$ for the latter formula).

The above data defines a quantifier-free interpretation Γ_n of the $\langle L_{\text{rings}}, L \rangle$ -structure $(W_n(R), R)$ in the L -structure R . Moreover, since the polynomials S_i, P_i are computable (Observation 3.1.18), the sequence of interpretations $(\Gamma_n)_{n \in \omega}$ is uniformly recursive. \square

3.1.22 Untilting

Expository notes on the material of this section may be found either in §5 of the Bourbaki seminar given by Morrow [Mor17] or in the lecture series notes by Lurie (see Lecture 2 [Lur18]).

Overview

The functor $K \mapsto K^\flat$ is far from being faithful, i.e. there will be several non-isomorphic mixed characteristic perfectoid fields K that tilt to the same perfectoid field of characteristic p . For example, the p -adic completions of $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p(\zeta_{p^\infty})$ both tilt to the t -adic completion of $\mathbb{F}_p((t))^{1/p^\infty}$. For a perfectoid field F of characteristic p , an *untilt* of F is a pair (K, ι) , where (K, v) is a perfectoid field and $\iota : (F, w) \xrightarrow{\cong} (K^\flat, v^\flat)$ is a valued field isomorphism. Fargues-Fontaine give a description of all possible untilts of F in an *intrinsic* fashion, i.e. in a way that uses only arithmetic from F itself (see Theorem 3.1.29). This result will be of vital importance for Theorem 3.2.11.

The ring \mathbf{A}_{inf}

Fix any perfectoid field (F, w) of positive characteristic. We introduce $\mathbf{A}_{\text{inf}} := W(\mathcal{O}_F)$, called the *infinitesimal* period ring, which is the ring of Witt vectors over the valuation ring of F .

Definition 3.1.23. An element $\xi \in \mathbf{A}_{\text{inf}}$ is said to be *distinguished* if it is of the form $\xi = \sum_{n=0}^{\infty} [c_n] p^n$, with $c_0 \in \mathfrak{m}_F$ and $c_1 \in \mathcal{O}_F^\times$.

In other words, distinguished elements are those of the form $\xi = [\pi] - up$ where $w\pi > 0$ and $u \in \mathbf{A}_{\text{inf}}$ is a unit.

Remark 3.1.24. Let $\text{res} : \mathcal{O}_F \rightarrow \mathcal{O}_F/\mathfrak{m}_F$ be the residue map and $W(\text{res}) : \mathbf{A}_{\text{inf}} \rightarrow W(\mathcal{O}_F/\mathfrak{m}_F)$ be the (unique) induced ring homomorphism provided by Proposition 3.1.16 that maps $\sum_{n=0}^{\infty} [c_n] p^n \mapsto \sum_{n=0}^{\infty} [\text{res}(c_n)] \cdot p^n$. The element $\xi \in \mathbf{A}_{\text{inf}}$ is distinguished precisely when $W(\text{res})(\xi)$ is a unit multiple of p in $W(\mathcal{O}_F/\mathfrak{m}_F)$.

Distinguished elements and untilts

We now explain the relationship between distinguished elements and untilts of F . Let (K, ι) be an untilt of F , i.e. we have $\iota : (F, w) \xrightarrow{\cong} (K^\flat, v^\flat)$. By Lemma 3.1.6(a), we have an isomorphism of multiplicative monoids $\sharp : \mathcal{O}_{K^\flat} \rightarrow \mathcal{O}_K$. We also write $\sharp : \mathcal{O}_F \rightarrow \mathcal{O}_K$, for the map obtained by precomposing with ι , which in turn induces a surjective ring homomorphism $\mathcal{O}_F \rightarrow \mathcal{O}_K/(p)$. By Proposition 3.1.16, the latter lifts uniquely to a ring homomorphism $\theta : \mathbf{A}_{\text{inf}} \rightarrow \mathcal{O}_K$ sending $\sum_{n=0}^{\infty} [c_n] \cdot p^n \mapsto \sum_{n=0}^{\infty} c_n^\sharp \cdot p^n$.

By Lemma 3.1.6(b), there exists an element $\pi \in \mathcal{O}_F$ such that $v^\flat \pi = vp$. It follows that $v\pi^\sharp = vp$ and therefore $\pi^\sharp = u \cdot p$ for some $u \in \mathcal{O}_K^\times$. By Lemma 3.1.7(b), we have $\mathcal{O}_K/(p) \cong \mathcal{O}_F/(\pi)$. Let $\bar{u} \in \mathcal{O}_F/(\pi)$ be the image of u via $\mathcal{O}_K \rightarrow \mathcal{O}_K/(p) \xrightarrow{\cong} \mathcal{O}_F/(\pi)$. Note that $[\pi] - \bar{u} \cdot p \in \mathbf{A}_{\text{inf}}$ is a distinguished element and that $[\pi] - \bar{u} \cdot p \in \text{Ker}(\theta)$. In fact, the following is true:

Proposition 3.1.25 (Corollary 17 [Lur18]). Let (F, w) be a perfectoid field of characteristic p and (K, ι) be an untilt. Let $\theta : \mathbf{A}_{\text{inf}} \rightarrow \mathcal{O}_K$ be as above. Then $\text{Ker}(\theta)$ is a principal ideal generated by *any* distinguished element $\xi \in \text{Ker}(\theta)$.

Starting with an untilt (K, ι) , we have thus produced a distinguished element $\xi_K \in \mathbf{A}_{\text{inf}}$ (unique only up to a unit in \mathbf{A}_{inf}).

Conversely, starting with $(\xi) \subseteq \mathbf{A}_{\text{inf}}$, we may produce an untilt (K, ι) of F as follows. Write $\theta : \mathbf{A}_{\text{inf}} \rightarrow \mathbf{A}_{\text{inf}}/(\xi)$ for the quotient map. We then have:

Lemma 3.1.26 (Lecture 3, pg. 4, Claim (a) [Lur18]). For every $y \in \mathbf{A}_{\text{inf}}/(\xi)$, there exists $x \in \mathcal{O}_F$ such that $(y) = (\theta([x]))$.

Proposition 3.1.27 (Proposition 16 [Lur18]). Let (F, w) be a perfectoid field of characteristic p and $\xi \in \mathbf{A}_{\text{inf}}$ a distinguished element. Then the quotient ring $\mathbf{A}_{\text{inf}}/(\xi)$ is the valuation ring \mathcal{O}_K of a perfectoid field (K, v) such that $(K^\flat, v^\flat) \cong (F, w)$. The valuation v is such that if $y \in \mathbf{A}_{\text{inf}}/(\xi)$, then $vy := wx$ with $x \in \mathcal{O}_F$ so that $(y) = (\theta([x]))$ in $\mathbf{A}_{\text{inf}}/(\xi)$.

The isomorphism $\iota : (F, w) \rightarrow (K^\flat, v^\flat)$ of Proposition 3.1.27 is described as follows. If $\xi = [\pi] - up$, then $\mathcal{O}_K/(p) \cong W(\mathcal{O}_F)/(p, [\pi] - up) \cong \mathcal{O}_F/(\pi)$. Passing to inverse limits, this induces a ring isomorphism $\mathcal{O}_{K^\flat} \cong \mathcal{O}_F$, which in turn yields $\iota : F \xrightarrow{\cong} K^\flat$ by passing to fraction fields. Starting with $(\xi) \subseteq \mathbf{A}_{\text{inf}}$, we have thus produced an untilt (K, ι) .

Definition 3.1.28. Two untilts (K, ι) and (K', ι') are isomorphic when there exists a valued field isomorphism $\phi : K \xrightarrow{\cong} K'$ inducing a commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & K^\flat \\ \downarrow = & & \downarrow \\ F & \longrightarrow & K'^\flat \end{array}$$

where $\phi^\flat : (x, x^{1/p}, \dots) \mapsto (\phi(x), \phi(x^{1/p}), \dots)$. Let Y_F denote the set of characteristic 0 untilts of (F, w) , up to isomorphism.

We write 0 for the isomorphism class of the unique characteristic p untilt of (F, ι) , represented by F itself together with the natural isomorphism $\iota : F \xrightarrow{\cong} F^\flat : x \rightarrow (x, x^{1/p}, \dots)$, and set $\overline{Y}_F = Y_F \cup \{0\}$ for the set of all untilts of (F, w) , up to isomorphism.

Theorem 3.1.29 (Fargues-Fontaine). *Let (F, w) be a perfectoid field of characteristic p . The map $\xi \mapsto \text{Frac}(\mathcal{O}_F/(\xi))$ defines a bijective correspondence between the set of ideals $(\xi) \subseteq \mathbf{A}_{\text{inf}}$ generated by a distinguished element and the set \overline{Y}_F .*

Proof. See Proposition 5.1 [Mor17] or Lecture 2, Corollary 18 [Lur18]. \square

Remark 3.1.30. Let (K, ι) be an untilt of (F, w) and $(\xi) = ([\pi] - up)$ be its associated ideal. Note that $(p) = (\theta([\pi]))$ in $\mathbf{A}_{\text{inf}}/(\xi)$ and therefore $vp = w\pi$.

Tilting equivalence

We emphasized in §3.1.22 that untilting is ambiguous, in the sense that there are many ways to untilt a perfectoid field of positive characteristic. However, the ambiguity is eliminated by *fixing a base* perfectoid field K and its associated tilt K^\flat . This leads to an equivalence of categories of perfectoid extensions, known as the *tilting equivalence*:

Theorem 3.1.31 (Tilting equivalence). *The categories of perfectoid field extensions of K and perfectoid field extensions of K^\flat are equivalent.*

Proof. See Theorem 2.8 [Sch12b]. Theorem 5.2 [Sch12a] shows a more general result about perfectoid algebras; the case of perfectoid fields follows as a special case by Lemma 5.21 [Sch12a]. \square

In the discussion after Theorem 2.8 [Sch12b], Scholze explains that there are two proofs of Theorem 3.1.31:

1. His original proof in [Sch12a], using Faltings' almost mathematics (see §4 [Sch12a]).
2. An alternative proof which describes the functor \sharp inverse to \flat as $L \mapsto W(\mathcal{O}_L) \otimes_{W(\mathcal{O}_{K^\flat})} K$ (see Remark 5.19 [Sch12a]).

Let us elaborate more on the second approach, which is in the spirit of §3.1.22 and will be more suitable for us. If $(\xi) \subset W(\mathcal{O}_{K^\flat})$ is the ideal associated to K , then one computes that $W(\mathcal{O}_L) \otimes_{W(\mathcal{O}_{K^\flat})} \mathcal{O}_K = W(\mathcal{O}_L) \otimes_{W(\mathcal{O}_{K^\flat})} W(\mathcal{O}_{K^\flat})/(\xi) = W(\mathcal{O}_L)/(\xi)$. In other words, if L is a perfectoid field extending K^\flat , then L^\sharp is simply the untilt of K whose associated ideal in $W(\mathcal{O}_L)$ is (ξ) .

3.1.32 Space of untilts

In §3.1.32 we exhibit an appealing model-theoretic property of the space Y_F of untilts of (F, w) . This will not be used in the rest of the chapter. We then study in §3.1.32 the size of the space of untilts up to elementary equivalence. We will see that the cardinality is often too big, so that one cannot possibly expect K to be decidable simply relative to K^\flat .

Metric structure on Y_F

Fargues-Fontaine equip \overline{Y}_F with a metric topology, which allows us to view the space of untilts geometrically. Suppose $x = (K_x, v_x)$ and $y = (K_y, v_y)$ are two "points" of \overline{Y}_F , corresponding to the ideals (ξ_x) and (ξ_y) respectively, provided by Theorem 3.1.29. We choose an embedding $\Gamma_w \hookrightarrow \mathbb{R}$, which determines embeddings $\Gamma_{v_y} \hookrightarrow \mathbb{R}$ for all $y \in Y_F$ via the canonical identification $\Gamma_{v_y} \cong \Gamma_w$. One then defines $d(x, y) := |\theta_y(\xi_x)|_y$, where $\theta_y : \mathbf{A}_{\text{inf}} \rightarrow \mathbf{A}_{\text{inf}}/(\xi_y) = \mathcal{O}_{K_y}$ is the quotient map and as usual $|a|_y = \frac{1}{p^{v_y(a)}}$.

Proposition 3.1.33 (Proposition 2.3.2(1) [FF19]). Let $d : \overline{Y}_F \times \overline{Y}_F \rightarrow \mathbb{R}$ be as above. The pair (\overline{Y}_F, d) is a complete ultrametric space.

Proof. See Propositions 2.3.2, 2.3.4 [FF19] or Lecture 14, Propositions 6, 7 [Lur18]. □

Recall that $0 \in \overline{Y}_F$ is the isomorphism class of the untilt corresponding to (F, w) itself together with the natural isomorphism $\iota : F \xrightarrow{\cong} F^\flat : x \rightarrow (x, x^{1/p}, \dots)$. We can define a radius function $r(y) := d(0, y)$ for $y \in \overline{Y}_F$, which allows us to think of \overline{Y}_F intuitively as the unit disc with center 0.

A model-theoretic property of Y_F

We now show that limits in the punctured unit disc Y_F , with respect to the Fargues-Fontaine metric, agree with limits in the model-theoretic sense:

Proposition 3.1.34. Let $(x_n)_{n \in \omega}$ be a sequence in Y_F such that $x_n \xrightarrow{d} x$ and $x \neq 0$. Let (K_n, v_n) be the untilt associated to x_n and (K, v) the untilt associated to x . We set

$$(K^*, v^*) := \prod_{n \in \omega} (K_n, v_n) / U$$

for a non-principal ultrafilter U on $\mathcal{P}(\mathbb{N})$. Then $(K^*, v^*) \equiv (K, v)$ in L_{val} .

Proof. Let $(\xi_n) = ([\pi_n] - u_n p)$ be the ideal in \mathbf{A}_{inf} corresponding to (K_n, v_n) and $(\xi) = ([\pi] - u p)$ be the ideal corresponding to (K, v) . Note that the set $\{v_n p : n \in \mathbb{N}\} \subseteq \Gamma_w$ is bounded from above; otherwise, there would be a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ with $x_{n_k} \rightarrow 0$.

Fix $m \in \mathbb{N}$ and let $(\bar{\xi}_n)$ and $(\bar{\xi})$ be the images of the ideals (ξ_n) and (ξ) in $W_m(\mathcal{O}_F)$ via $\mathbf{A}_{\text{inf}} \rightarrow \mathbf{A}_{\text{inf}}/(p^m) = W_m(\mathcal{O}_F)$.

Claim: We have that $(\bar{\xi}_n) = (\bar{\xi})$, for sufficiently large n .

Proof. We let $\theta : \mathbf{A}_{\text{inf}} \rightarrow \mathbf{A}_{\text{inf}}/(\xi) = \mathcal{O}_K$ be the quotient map. Since $d(x_n, x) \rightarrow 0$, we get that $v(\theta(\xi_n)) \rightarrow \infty$. We will thus have that $\xi_n \equiv p^m \cdot \alpha_n \pmod{(\xi)}$, for some $\alpha_n \in \mathbf{A}_{\text{inf}}$ and for all $n \gg 0$. Consequently, one gets that $(\bar{\xi}_n) \subseteq (\bar{\xi})$ for $n \gg 0$. Similarly, since $v_n(\theta_n(\xi)) \rightarrow \infty$ and $\{v_n p : n \in \mathbb{N}\}$ is bounded, we get that $v_n(\theta_n(\xi)) \geq m v_n p$ for $n \gg 0$. It follows that there exists $\beta_n \in \mathbf{A}_{\text{inf}}$ such that $\xi \equiv p^m \cdot \beta_n \pmod{(\xi_n)}$, for $n \gg 0$. We conclude that $(\bar{\xi}_n) = (\bar{\xi})$, for $n \gg 0$. \square_{Claim}

We have in particular that (ξ_n) and (ξ) have the same image in $\mathbf{A}_{\text{inf}}/(p) \cong \mathcal{O}_F$, for $n \gg 0$ and therefore $(\pi_n) = (\pi)$ for $n \gg 0$. By Lemma 3.1.7(b), we get that $(\Gamma_{v_n}, v_n p) \cong (\Gamma_w, w \pi_n) = (\Gamma_w, w \pi) \cong (\Gamma_v, v p)$, for all sufficiently large $n \in \mathbb{N}$. The conclusion follows from Theorem 2.1.3 and Łoś' s Theorem. \square

The space Z_F

It is natural to consider the set Y_F up to elementary equivalence. More precisely:

Definition 3.1.35. Let (F, w) be a perfectoid field of characteristic p . For $x = (K_x, \iota_x), y = (K_y, \iota_y) \in Y_F$ define the equivalence relation $x \sim y \iff (K_x, v_x) \equiv (K_y, v_y)$ in L_{val} . We define $Z_F := Y_F / \sim$.

Note that the definition of Z_F only takes the underlying valued fields (K, v) into account and not the map ι . We determine the size of Z_F in a few cases. The construction in part (b) is an elaborated version of Scholze's answer [Sch] to a closely related mathoverflow question asked by the author.

Proposition 3.1.36. (a) Let $(F, w) = (\widehat{\overline{\mathbb{F}_p((t))}}, v_t)$, the t -adic completion of an algebraic closure of $\mathbb{F}_p((t))$. Then $|Z_F| = 1$.

(b) Set $(F, w) = (\overline{\mathbb{F}_p((t))}^{1/p^\infty}, v_t)$. Then $|Z_F| = \mathfrak{c}$.

Proof. (a) **First proof:** Let (K, v) be such that $(K^b, v^b) \cong (\widehat{\overline{\mathbb{F}_p((t))}}, v_t)$. By Proposition 4.3 [Sch12a], we get that K is algebraically closed. By Robinson's completeness of $\text{ACVF}_{(0,p)}$ (see Corollary 3.34 [Dri12]), we get that $\text{Th}(K, v) = \text{ACVF}_{(0,p)}$ and hence $|Z_F| = 1$.

Second proof: Alternatively, one can use the fact that we have a ring isomorphism $\mathcal{O}_K/(p^n) \cong \mathcal{O}_{\mathbb{C}_p}/(p^n)$ for each $n \in \mathbb{N}$ (Remark 2.24 [FF14]) together with Theorem 2.1.3. One also has to check that $(\Gamma_v, vp) \equiv (v_p \mathbb{C}_p, v_p p)$ in L_{oag} with a constant for vp . First note that $\Gamma_v \cong \Gamma_w$ is divisible as F is algebraically closed. The conclusion now is an application of quantifier elimination for the theory ODAG of ordered divisible abelian groups (Corollary 3.1.17 [Mar02]).

(b) We assume that p is odd; the case $p = 2$ is similar and is left to the reader. For each $\alpha \in 2^\omega$, we define a algebraic extension K_α of \mathbb{Q}_p as follows. Set $\pi_{\alpha(0)} = p$ and $K_{\alpha(0)} = \mathbb{Q}_p$. We now define inductively:

- (1) $K_{\alpha(n)} = K_{\alpha(n-1)}(\pi_{\alpha(n-1)}^{1/p})$ and $\pi_{\alpha(n)} = \pi_{\alpha(n-1)}^{1/p}$ if $\alpha(n) = 0$ or
- (2) $K_{\alpha(n)} = K_{\alpha(n-1)}(((1+p) \cdot \pi_{\alpha(n-1)})^{1/p})$ and $\pi_{\alpha(n)} = ((1+p) \cdot \pi_{\alpha(n-1)})^{1/p}$ if $\alpha(n) = 1$.

We let $K_\alpha = \bigcup_{n \in \omega} K_{\alpha(n)}$. Set $\bar{\alpha}_n = \sum_{k=0}^{n-1} \alpha(k) \cdot p^k$, note that $X^{p^n} - (1+p)^{\bar{\alpha}_n} p \in \mathbb{Z}_p[X]$ is an Eisenstein (hence irreducible) polynomial and that we have $K_{\alpha(n)} = \mathbb{Q}_p(((1+p)^{\bar{\alpha}_n} p)^{1/p^n})$ for all $n \in \mathbb{N}$. We have that $\zeta_p \notin K_\alpha$ since $e(K_{\alpha(n)}/\mathbb{Q}_p) = p^n$ while $e(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p) = p - 1$.

Claim 1: For each $\alpha \in 2^\omega$, we have $(1+p)^{1/p} \notin K_\alpha$.

Proof. Suppose that $(1+p)^{1/p} \in K_{\alpha(n)}$, for some $n \in \mathbb{N}$. By the unique subfield property for radical extensions (see Theorem 2.1 [OV82]),¹ we would then have that $\mathbb{Q}_p((1+p)^{1/p}) = \mathbb{Q}_p(((1+p)^{\alpha(0)} \cdot p)^{1/p})$. Whether $\alpha(0) = 0$ or 1, this implies that $(1+p)^{1/p} \in \mathbb{Q}_p(p^{1/p})$. Now suppose $a \in \mathbb{Q}_p(p^{1/p})$ is such that $a^p = 1+p$. Write

¹A finite extension L/K has the *unique subfield property* if for every $d|[L:K]$, there is a unique subextension F/K such that $[F:K] = d$. We state Theorem 2.1 [OV82] here for the convenience of the reader: Let K be a field, $n \in \mathbb{N}$ be such that $\text{char}(K) \nmid n$, $a \in K$ such that $X^n - a \in K[x]$ is irreducible and set $L = K(a^{1/n})$. Suppose that for every odd prime $p|n$, we have that $\zeta_p \notin L - K$ and in case $4|n$ we have $\zeta_4 \notin L - K$. Then L/K has the unique subfield property.

$a \equiv c_0 + c_1 \cdot p^{1/p} \pmod{p^{2/p}\mathbb{Z}_p[p^{1/p}]}$, for some $c_0, c_1 \in \{0, \dots, p-1\}$. We then compute $a^p \equiv c_0^p + c_1^p \cdot p + c_0 \cdot c_1 \cdot p^{1+1/p} \not\equiv 1+p \pmod{p^{1+2/p}}$, for any choice of c_0 and c_1 . We conclude that $(1+p)^{1/p} \notin K_\alpha$. $\square_{\text{Claim 1}}$

Using this, we can show:

Claim 2: If $\alpha \neq \beta$, then $K_\alpha \not\equiv_{\exists} K_\beta$ in L_{rings} .

Proof. Let $n \in \omega$ be least such that $\alpha(n) \neq \beta(n)$ and say $\alpha(n) = 1$. Now if $K_\alpha \equiv_{\exists} K_\beta$, there would be $a, b \in K_\alpha$ such that $a^{p^n} = (1+p)^{\bar{\alpha}_n} p$ and $b^{p^n} = (1+p)^{\bar{\beta}_n} p$. Set $c := \frac{a}{b}$ and note that $c^{p^n} = \frac{(1+p)^{\bar{\alpha}_n} p}{(1+p)^{\bar{\beta}_n} p} = (1+p)^{p^{n-1}}$. This implies that $(\frac{c^p}{1+p})^{p^{n-1}} = 1$. Since $\zeta_p \notin K_\alpha$, we get that $c^p = 1+p$, which is impossible by Claim 1. $\square_{\text{Claim 2}}$

Next we prove:

Claim 3: For every $\alpha \in 2^\omega$, we have that $\widehat{K_\alpha^b} = \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$.

Proof. For every $n \in \mathbb{N}$, we have that $K_{\alpha(n+1)}/K_{\alpha(n)}$ is totally ramified and $\mathcal{O}_{K_{\alpha(n+1)}} = \mathcal{O}_{K_{\alpha(n)}}[\pi_{\alpha(n+1)}]$ (Corollary pg.19 [Ser80]). It follows that $\mathcal{O}_{K_\alpha} = \mathbb{Z}_p[\{\pi_{\alpha(n)} : n \in \mathbb{N}\}]$. We compute

$$\begin{aligned} \mathcal{O}_{K_\alpha}/(p) &= \mathbb{Z}_p[x_1, x_2, \dots]/(p, x_1^p - (1+p)^{\alpha(0)}p, x_2^p - (1+p)^{\alpha(1)}x_1, \dots) \cong \\ &\cong \mathbb{F}_p[x_1, x_2, \dots]/(x_1^p, x_2^p - x_1, \dots) \cong \mathbb{F}_p[x_1^{1/p^\infty}]/(x_1^p) \stackrel{t=x_1^p}{=} \mathbb{F}_p[t^{1/p^\infty}]/(t) \end{aligned}$$

We thus get that $\varprojlim_{\Phi} \mathcal{O}_{K_\alpha}/(p) \cong \widehat{\mathbb{F}_p[[t]]^{1/p^\infty}}$ and the conclusion follows. $\square_{\text{Claim 3}}$

Claims 2,3 show that $|Z_F| \geq \mathfrak{c}$. On the other hand, the bound $|Z_F| \leq \mathfrak{c}$ is automatic, as any first-order L_{val} -theory can be identified with a subset of $\text{Sent}_{L_{\text{val}}} \simeq \mathbb{N}$. \square

Remark 3.1.37. Set $(F, w) = (\widehat{\mathbb{F}_p((t))}, v_t)$. In Remark 2.24 [FF14], Fargues-Fontaine ask whether every untilt (K, v) of (F, w) is isomorphic to (\mathbb{C}_p, v_p) . They note in Remark 2.24 that for any untilt (K, v) of (F, w) one has $\mathcal{O}_K/(p^n) \cong \mathcal{O}_{\mathbb{C}_p}/(p^n)$ for each $n \in \mathbb{N}$. However, these isomorphisms are obtained in a non-canonical way and do not necessarily yield a valued field isomorphism between K and \mathbb{C}_p . In fact, an example of such a K with $K \not\cong \mathbb{C}_p$ was provided by Kedlaya-Temkin in Theorem 1.3 [KT18]. This should be contrasted with Proposition 3.1.36(a), saying that all untilts of (F, w) are elementary equivalent.

Remark 3.1.38. In Remark 7.6 [AF16], the authors write:

"At present, we do not know of an example of a mixed characteristic henselian valued field (K, v) for which $\text{Th}_{\exists}(k)$ and $\text{Th}_{\exists}(\Gamma_v, vp)$ are decidable but $\text{Th}_{\exists}(K, v)$ is undecidable."

Let us provide such an example. By Proposition 3.1.36(b) and the fact that there are countably many Turing machines, there must exist an undecidable perfectoid (thus henselian) field (K, v) with $K^{\flat} \cong \widehat{\mathbb{F}_p((t))}^{1/p^\infty}$. By Lemma 3.1.7(b), we have that $k_v \cong \mathbb{F}_p$ and $(\Gamma_v, vp) \cong (\frac{1}{p^\infty}\mathbb{Z}, 1)$, both of which are decidable—the latter being an easy application of Robinson-Zakon [RZ60].

Remark 3.1.39. For future use in Proposition 3.5.2, we record here that Claim 2 gives $t = (\pi_{\alpha(0)} + (p), \pi_{\alpha(1)} + (p), \dots)$, via the identifications $\widehat{\mathbb{F}_p[[t]]}^{1/p^\infty} \cong \varprojlim_{\Phi} \mathcal{O}_{K_\alpha}/(p)$.

Question 3.1.40. Is there a perfectoid field (F, w) of characteristic p with $1 < |Z_F| < \mathfrak{c}$?

3.2 Relative decidability for perfectoid fields

3.2.1 Introduction

We shall now use the results of §3.1 to prove Theorem 3.0.2 and Corollary 3.2.20.

Work of Rideau-Scanlon

It is my understanding that there is ongoing work by Rideau-Scanlon, which aims at giving a model-theoretic account of many of the concepts and facts that were discussed in §3.1, in the context of *continuous* logic. In particular, they obtain a bi-interpretability result between \mathcal{O}_K and \mathcal{O}_{K^\flat} in the sense of continuous logic. It should be noted that their bi-interpretability result was conceived prior to the present work and in fact influenced the material that is presented here.

Plan of action

Without any adjustments, the interpretation of Rideau-Scanlon does not quite yield an interpretation in the sense of ordinary first-order logic. The problem is that when one converts statements about \mathcal{O}_K to statements about \mathcal{O}_{K^\flat} via $\mathcal{O}_K \cong W(\mathcal{O}_{K^\flat})/(\xi)$, one ends up with infinitely many variables, coming from the Witt vector coordinates. This problem disappears by interpreting one residue ring $\mathcal{O}_K/(p^n)$ at a time, via $\mathcal{O}_K/(p^n) \cong W_n(\mathcal{O}_{K^\flat})/(\xi \bmod (p^n))$. This approach is facilitated by Corollary 2.3.1. In §2.3, we emphasized that *uniform* decidability of the residue rings is key. As we will see, this eventually comes down to the computability of the distinguished element ξ itself.

3.2.2 Computable untilts

Computable Witt vectors

In analogy with a computable real number, a computable p -adic integer is one for which there is an algorithm which outputs the sequence of its p -adic digits. More precisely, a p -adic integer $a \in \mathbb{Z}_p$ of the form $a = \sum_{n=0}^{\infty} [\alpha_n] \cdot p^n$ is said to be *computable* precisely when the function $\mathbb{N} \rightarrow \mathbb{Z}/p\mathbb{Z} : n \mapsto \alpha_n$ is recursive. The notion of a computable p -adic integer extends naturally to the more general concept of a *computable Witt vector*.

First recall that a *computable* ring R_0 is one whose underlying set is (or is identified with) a *recursive* subset of \mathbb{N} , via which the operations $+$: $R_0 \times R_0 \rightarrow R_0$ and \cdot : $R_0 \times R_0 \rightarrow R_0$ are identified with *recursive* functions.

Definition 3.2.3. Fix a perfect ring R and let $R_0 \subseteq R$ a computable subring. Consider the ring $W(R)$, the ring of Witt vectors over R . An element $\xi \in W(R)$ with Witt vector coordinates $(\xi_0, \xi_1, \dots) \in R^\omega$ is said to be R_0 -computable if:

1. For each $n \in \mathbb{N}$, we have that $\xi_n \in R_0$.
2. The function $\mathbb{N} \rightarrow R_0 : n \mapsto \xi_n$ is recursive.

Example 3.2.4. (a) Let $R_0 = R = \mathbb{F}_p$. Then the computable elements of $\mathbb{Z}_p = W(\mathbb{F}_p)$ are the usual computable p -adic integers.

(b) A *non-example*: Let R be any perfect ring, $R_0 \subseteq R$ any computable subring and $S \subseteq \mathbb{N}$ be a non-recursive set. The element $\xi = \sum_{n \in S} p^n \in W(R)$ is not computable.

Computable untilts

Let (F, w) be a perfectoid field of characteristic p and $\mathbf{A}_{\text{inf}} = W(\mathcal{O}_F)$. Let also $R_0 \subseteq \mathcal{O}_F$ be a computable subring.

Remark 3.2.5. The reader is welcome to assume that $F = \widehat{\mathbb{F}_p((t))}^{1/p^\infty}$ or $\widehat{\overline{\mathbb{F}_p}((t))}^{1/p^\infty}$ and $R_0 = \mathbb{F}_p[t]$. This case is enough for the applications presented here, i.e. Corollaries 3.2.20 and 3.3.11.

We shall now define what it means for an untilt K of F to be R_0 -computable:

Definition 3.2.6. Let (K, ι) be an untilt of (F, w) and $R_0 \subseteq \mathcal{O}_F$ a computable subring. We say that (K, v) is an R_0 -computable untilt of (F, w) if there is an R_0 -computable distinguished element $\xi_K \in \mathbf{A}_{\text{inf}}$ with $\mathcal{O}_K \cong \mathbf{A}_{\text{inf}}/(\xi_K)$.

Example 3.2.7. In Corollary 3.2.19 we will see that $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ and $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ are computable untilts of $\widehat{\mathbb{F}_p((t))}^{1/p^\infty}$. Also, that $\widehat{\mathbb{Q}_p^{ab}}$ is a computable untilt of $\widehat{\overline{\mathbb{F}_p}((t))}^{1/p^\infty}$.

3.2.8 Relative decidability

Let (K, v) be an untilt of (F, w) and $\xi_K \in \mathbf{A}_{\text{inf}}$ be such that $\mathcal{O}_K = \mathbf{A}_{\text{inf}}/(\xi_K)$ (Theorem 3.1.29). We write $\bar{\xi}_{K,n}$ for the image of ξ_K in $W_n(\mathcal{O}_F)$ via $\mathbf{A}_{\text{inf}} \twoheadrightarrow \mathbf{A}_{\text{inf}}/(p^n) \cong W_n(\mathcal{O}_F)$.

Interpretability

All the necessary background material related to interpretability is presented in §1.1.6.

Lemma 3.2.9. For each $n \in \mathbb{N}$, there is a \exists^+ -interpretation A_n of the local ring $(\mathcal{O}_K/(p^n), \mathfrak{m}_n)$ in the $\langle L_{\text{rings}}, L_{\text{lcr}} \rangle$ -structure $((W_n(\mathcal{O}_F), \mathcal{O}_F), \bar{\xi}_{K,n})$ such that the sequence of interpretations $(A_n)_{n \in \omega}$ is uniformly recursive.

Proof. Fix $n \in \mathbb{N}$. We have that $\mathcal{O}_K/(p^n) \cong \mathbf{A}_{\text{inf}}/(p^n, \xi_K) \cong W_n(\mathcal{O}_F)/(\bar{\xi}_{K,n})$. By Proposition 3.1.27, we get that $\mathfrak{m}_n = (\{\theta_n([x]) : x \in \mathfrak{m}_F\})$, where θ_n is the composite map $\mathbf{A}_{\text{inf}} \xrightarrow{\theta} \mathbf{A}_{\text{inf}}/(\xi_K) \xrightarrow{\text{mod } p^n} W_n(\mathcal{O}_F)/(\bar{\xi}_{K,n}) \xrightarrow{\cong} \mathcal{O}_K/(p^n)$. We take $\partial_{A_n}(x)$ to be $x = x$. Let c be a constant symbol with $c^{(W_n(\mathcal{O}_F), \mathcal{O}_F)} = \bar{\xi}_{K,n}$. The reduction map $\text{Form}_{L_{\text{lcr}}} \rightarrow \langle L_{\text{rings}}, L_{\text{lcr}} \rangle \cup \{c\} : \phi \mapsto \phi_{A_n}$ on unnested atomic formulas is described as follows:

1. If $\phi(x, y)$ is $x = y$, then we take ϕ_{A_n} to be the formula $x, y \in \mathbf{W} \wedge \exists z \in \mathbf{W}(x = y + z \cdot c)$. As usual, the conjunct $x, y \in \mathbf{W}$ is an informal way of saying that the variables x, y are taken from the sort \mathbf{W} . The cases of $x = 0$ and $x = 1$ are similar.
2. If $\phi(x, y, z)$ is $x \diamond y = z$, then we take ϕ_{A_n} to be the formula $x, y, z \in \mathbf{W} \wedge \exists w \in \mathbf{W}(x \diamond y = z + w \cdot c)$, where \diamond is either \cdot or $+$.
3. If $\phi(x)$ is $x \in \mathfrak{m}$, then we take $\phi_{A_n}(x)$ to be the formula $x \in \mathbf{W} \wedge \exists z, w \in \mathbf{W}, y \in \mathbf{R}(y \in \mathfrak{m} \wedge x = [y] \cdot z + w \cdot c)$.

The coordinate map $f_{A_n} : W_n(\mathcal{O}_F) \twoheadrightarrow \mathcal{O}_K/(p^n)$ is the one induced by the map θ_n above. The above data defines a recursive \exists^+ -interpretation A_n of $(\mathcal{O}_K/(p^n), \mathfrak{m}_n)$ in $((W_n(\mathcal{O}_F), \mathcal{O}_F), \bar{\xi}_{K,n})$. The sequence of interpretations $(A_n)_{n \in \omega}$ is uniform in n and thus also (trivially) uniformly recursive. \square

Proposition 3.2.10. Suppose (K, v) is an untilt of (F, w) and that $\xi_K \in \mathbf{A}_{\text{inf}}$ is such that $\mathcal{O}_K = \mathbf{A}_{\text{inf}}/(\xi_K)$. Let $(\xi_0, \xi_1, \dots) \in \mathcal{O}_F^\omega$ be the Witt vector coordinates of ξ_K . Then:

- (a) The value group (Γ_v, vp) is interpretable in the valued field $((F, w), \xi_0)$.

(b) For each $n \in \mathbb{N}$, there exists a \exists^+ -interpretation Γ_n of the local ring $(\mathcal{O}_K/(p^n), \mathfrak{m}_n)$ in the local ring $((\mathcal{O}_F, \mathfrak{m}_F), \xi_0, \dots, \xi_{n-1})$ such that the sequence of interpretations $(\Gamma_n)_{n \in \omega}$ is uniformly recursive.

Proof. (a) By Lemma 3.1.7(b), we will have that $(\Gamma_v, vp) \cong (\Gamma_w, w\xi_0)$ and the value group $(\Gamma_w, w\xi_0)$ is interpretable in the $L_{\text{val}} \cup \{c\}$ -structure (F, w) with $c^{(F,w)} = \xi_0$. (b) Lemma 3.2.9 provides us with a \exists^+ -interpretation A_n of the local ring $(\mathcal{O}_K/(p^n), \mathfrak{m}_n)$ in the $\langle L_{\text{rings}}, L_{\text{lcr}} \rangle \cup \{c\}$ -structure $((W_n(\mathcal{O}_F), \mathcal{O}_F), \bar{\xi}_{K,n})$ such that the sequence of interpretations $(A_n)_{n \in \omega}$ is uniformly recursive. Lemma 3.1.21 provides us with a quantifier-free interpretation B_n of $((W_n(\mathcal{O}_F), \mathcal{O}_F), \bar{\xi}_{K,n})$ in the local ring $((\mathcal{O}_F, \mathfrak{m}_F), \xi_0, \dots, \xi_{n-1})$, such that the sequence of interpretations $(B_n)_{n \in \omega}$ is uniformly recursive. Let Γ_n be the composite interpretation of $(\mathcal{O}_K/(p^n), \mathfrak{m}_n)$ in $((\mathcal{O}_F, \mathfrak{m}_F), \xi_0, \dots, \xi_{n-1})$. This is also a \exists^+ -interpretation by Lemma 1.1.12 and the sequence of interpretations $(\Gamma_n)_{n \in \omega}$ is uniformly recursive by Lemma 1.1.20. \square

Proof of Theorem 3.2.11

Given a computable subring $R_0 \subseteq \mathcal{O}_F$, we write $L_{\text{val}}(R_0)$ for the language L_{val} enriched with a constant c_a for each $a \in R_0$ (see notation). The valued field (F, w) can then be updated to an $L_{\text{val}}(R_0)$ -structure with $c_a^{(F,w)} = a$.

Theorem 3.2.11. *Let (F, w) be a perfectoid field of characteristic p . Suppose $R_0 \subseteq \mathcal{O}_F$ is a computable subring and (K, v) is an R_0 -computable untilt of (F, w) .*

(a) *If (F, w) is decidable in $L_{\text{val}}(R_0)$, then (K, v) is decidable in L_{val} .*

(b) *If (F, w) is \exists -decidable in $L_{\text{val}}(R_0)$, then (K, v) is \exists -decidable in L_{val} .*

Proof. (a) By Proposition 3.2.10(a), we get that (Γ_v, vp) is decidable in L_{oag} with a constant for vp . By Proposition 3.2.10(b), for each $n \in \mathbb{N}$, there exists an interpretation Γ_n of the L_{lcr} -structure $(\mathcal{O}_K/(p^n), \mathfrak{m}_n)$ in the $L_{\text{lcr}} \cup \{c_0, \dots, c_{n-1}\}$ -structure $((\mathcal{O}_F, \mathfrak{m}_F), \xi_0, \dots, \xi_{n-1})$ with $c_m^{(\mathcal{O}_F, \mathfrak{m}_F)} = \xi_m$ for $m = 0, \dots, n-1$. Moreover, the sequence of interpretations $(\Gamma_n)_{n \in \omega}$ is uniformly recursive.

Claim: There exists an interpretation Δ_n of the $L_{\text{lcr}} \cup \{c_0, \dots, c_{n-1}\}$ -structure $((\mathcal{O}_F, \mathfrak{m}_F), \xi_0, \dots, \xi_{n-1})$ in the $L_{\text{val}}(R_0)$ -structure (F, w) such that the sequence of interpretations $(\Delta_n)_{n \in \omega}$ is uniformly recursive.

Proof. Fix $n \in \mathbb{N}$. We take $\partial_{\Delta_n}(x)$ to be the formula $x \in \mathcal{O}$. The reduction map on unnested atomic formulas is described as follows:

1. If $\phi(x)$ is the formula $x = c_m$, for some $m = 0, \dots, n-1$, we take $\phi_{\Delta_n}(x)$ to be the formula $x = c_{\xi_m}$.

2. If $\phi(x)$ is the formula $x \in \mathfrak{m}$, we take $\phi_{\Delta_n}(x)$ to be the formula $\exists y(xy = 1 \wedge y \notin \mathcal{O})$.
3. If $\phi(x, y, z)$ is $x \diamond y = z$, then we take ϕ_{Δ_n} to be the formula $x, y, z \in \mathcal{O} \wedge \phi(x, y, z)$, where \diamond is either \cdot or $+$.

The coordinate map $f_{\Delta_n} : \partial_{\Delta_n}(F) \rightarrow \mathcal{O}_F$ is the identity. Since $\mathbb{N} \rightarrow R_0 : m \mapsto \xi_m$ is recursive, the reduction map $\mathbb{N} \times \text{Form}_{L_{\text{lcr}} \cup \{c_0, \dots, c_{n-1}\}} \rightarrow \text{Form}_{L_{\text{val}}(R_0)} : (n, \phi) \mapsto \phi_{\Delta_n}$ restricted to unnested atomic formulas is recursive. By definition, this means that the sequence of interpretations $(\Delta_n)_{n \in \omega}$ is uniformly recursive. \square *Claim*

For each $n \in \mathbb{N}$, let E_n be the composite interpretation of Γ_n and Δ_n . The sequence $(E_n)_{n \in \omega}$ is uniformly recursive by Proposition 1.1.20. By Proposition 1.1.19, we get that the sequence of rings $(\mathcal{O}_K/(p^n))_{n \in \omega}$ is *uniformly* decidable in L_{rings} . The conclusion follows from Corollary 2.3.1.

(b) For the existential version, one needs to keep track of the complexity of formulas. Since Γ_n is a \exists^+ -interpretation and Δ_n is an \exists -interpretation, we see that the reduction map $L_{\text{lcr}} \rightarrow L_{\text{val}}(R_0) : \phi \mapsto \phi_{E_n}$ sends \exists^+ -formulas to \exists -formulas (see Remark 1.1.11). It follows that the sequence of local rings $((\mathcal{O}_K/(p^n), \mathfrak{m}_n)_{n \in \omega}$ is uniformly \exists^+ -decidable in L_{lcr} . We conclude by Corollary 2.3.2. \square

The assumption of completeness in Theorem 3.2.11 can be easily relaxed to henselianity:

Corollary 3.2.12. *Let (F, w) be a perfect non-trivially valued field with a rank 1 value group and $R_0 \subseteq \mathcal{O}_F$ be a computable subring. Suppose (K, v) is a mixed characteristic henselian valued field such that $(\widehat{K}, \widehat{v})$ is an R_0 -computable untillt of $(\widehat{F}, \widehat{w})$. Then:*

- (a) *If (F, w) is decidable in $L_{\text{val}}(R_0)$, then (K, v) is decidable in L_{val} .*
- (b) *If (F, w) is \exists -decidable in $L_{\text{val}}(R_0)$, then (K, v) is \exists -decidable in L_{val} .*

Proof. Suppose $\widehat{\mathcal{O}}_K = W(\widehat{\mathcal{O}}_F)/(\xi)$, where $\xi = [\pi] - up$, with $\pi \in \mathfrak{m}_F \cap R_0$ and $u \in W(\mathcal{O}_F)^\times$. Note that $W(\widehat{\mathcal{O}}_F)/([\pi] - up) \cong W(\mathcal{O}_F)/([\pi] - up)$; indeed, $W(\mathcal{O}_F)$ is p -adically complete and $[\pi]$ and p are associates in the quotient $W(\mathcal{O}_F)/([\pi] - up)$. For each $n \in \mathbb{N}$, one has that

$$\mathcal{O}_K/(p^n) \cong \widehat{\mathcal{O}}_K/(p^n) \cong W_n(\widehat{\mathcal{O}}_F)/([\pi] - up) \cong W_n(\mathcal{O}_F)/([\pi] - up)$$

We now proceed as in the proof of Theorem 3.0.2. \square

Remarks on Theorem 3.2.11

Note that decidability of K in L_{val} relative to its tilt K^{\flat} in L_{val} , i.e. without taking parameters into account, is false. Essentially, the point is that p is named in L_{val} , while t is not:

Example 3.2.13. Let $(\Gamma, +, <) \subseteq (\mathbb{R}, +, <)$ be any p -divisible ordered abelian group, which is decidable in L_{oag} but undecidable in $L_{\text{oag}} \cup \{1\}$, where 1 a distinguished element of Γ . For example, let $S \subsetneq \mathbb{P}$ be a non-recursive set of primes and $S' := \mathbb{P} - S$ be its complement. Denote by $\frac{1}{S}\mathbb{Z}$ the group generated by $\{\frac{1}{s} : s \in S\}$ and consider the ordered abelian group

$$\Gamma = \frac{1}{p^\infty} \left(\frac{1}{S}\mathbb{Z} \oplus \frac{1}{S'}\mathbb{Z}\sqrt{2} \right)$$

equipped with the order induced from the natural embedding $\Gamma \hookrightarrow \mathbb{R}$. This is a regularly dense group, in the sense of Robinson-Zakon [RZ60], with prime invariants $[\Gamma : p\Gamma] = 1$ and $[\Gamma : q\Gamma] = q$ for $q \neq p$. By Robinson-Zakon [RZ60], this group is decidable in L_{oag} but is clearly undecidable in $L_{\text{oag}} \cup \{1\}$. Since Γ is p -divisible, we may form the tame valued field $(F, w) = (\mathbb{F}_p((t^\Gamma)), v_t)$, which is decidable in L_{val} by Theorem 1.4 [Kuh16a]. However, the untilt (K, v) , whose associated Witt vector is $[t] - p$, is undecidable in L_{val} . Indeed, even (Γ_v, vp) is undecidable in L_{oag} with a constant for vp .

Remark 3.2.14. Theorem 3.0.2 holds also for untilts K of F , which have an associated distinguished element ξ_K satisfying a more relaxed condition than the one of Definition 3.2.3. Namely, suppose that $\xi_K = (\xi_0, \xi_1, \dots) \in \mathcal{O}_F^\omega$ and each ξ_n is definable in the valued field (F, w) by a formula $\phi_n(x) \in L_{\text{val}}(R_0)$ with parameters from R_0 . Suppose furthermore that the function $\mathbb{N} \rightarrow \text{Form}_{L_{\text{val}}(R_0)} : n \mapsto \phi_n(x)$ is recursive. Then the conclusion of Theorem 3.0.2 is still valid, i.e. (K, v) is decidable in L_{val} relative to (F, w) in $L_{\text{val}}(R_0)$.

It is clear that when each $\phi_n(x)$ is a quantifier-free formula, the configuration described in Remark 3.2.14 specializes to the notion of a computable untilt. We have no particular application in mind that requires the level of generality described in Remark 3.2.14, which is one reason we have restricted ourselves to the quantifier-free case (the other being clarity of exposition).

3.2.15 Corollary 3.0.1

In order to prove Corollary 3.2.20, we need to calculate the tilts of our fields of interest and compute the associated distinguished elements.

Computing the distinguished elements

All computations below are well-known to experts (see e.g., Example 2.22 [FF14]). For lack of a detailed reference, we shall spell out the details.

Lemma 3.2.16. There exist ring isomorphisms:

- (a) $\mathbb{Z}_p[p^{1/p^\infty}]/(p) \cong \mathbb{F}_p[t^{1/p^\infty}]/(t)$ mapping $p^{1/p^n} + (p) \mapsto t^{1/p^n} + (t)$.
- (b) $\mathbb{Z}_p[\zeta_{p^\infty}]/(p) \cong \mathbb{F}_p[t^{1/p^\infty}]/(t^{p-1})$ mapping $\zeta_p + (p) \mapsto t + 1 + (t^{p-1})$.
- (c) $\mathbb{Z}_p^{ab}/(p) \cong \overline{\mathbb{F}}_p[t^{1/p^\infty}]/(t^{p-1})$ mapping $\zeta_p + (p) \mapsto t + 1 + (t^{p-1})$.

Proof. (a) For each $n \in \mathbb{N}$, the irreducible polynomial of $p^{1/p^{n+1}}$ over $\mathbb{Q}_p(p^{1/p^n})$ is the Eisenstein polynomial $x^p - p^{1/p^n} \in \mathbb{Z}_p[p^{1/p^n}][x]$. We therefore compute

$$\begin{aligned} \mathbb{Z}_p[p^{1/p^\infty}]/(p) &\cong \mathbb{Z}_p[x_1, x_2, \dots]/(p, x_1^p - p, x_2^p - x_1, \dots) \\ &\cong \mathbb{F}_p[x_1, x_2, \dots]/(x_1^p, x_2^p - x_1, \dots) \stackrel{x_n \mapsto t^{1/p^n}}{\cong} \mathbb{F}_p[t^{1/p^\infty}]/(t) \end{aligned}$$

(b) The irreducible polynomial of ζ_p over \mathbb{Q}_p is the cyclotomic polynomial $\Phi_p(x) = x^{p-1} + \dots + 1$. Moreover, given $n > 1$, the irreducible polynomial of ζ_{p^n} over $\mathbb{Q}_p(\zeta_{p^{n-1}})$ is $x^p - \zeta_{p^{n-1}} \in \mathbb{Z}[\zeta_{p^n}][x]$. We now proceed as in (a) and compute

$$\mathbb{Z}_p[\zeta_{p^\infty}]/(p) \cong \mathbb{Z}_p[x_1, x_2, \dots]/(p, \Phi_p(x_1), x_2^p - x_1, \dots) \stackrel{x_{n+1} \mapsto x^{1/p^n}}{\cong} \mathbb{F}_p[x^{1/p^\infty}]/(\overline{\Phi}_p(x))$$

Note that $\overline{\Phi}_p(x) = \frac{x^p-1}{x-1} = \frac{(x-1)^p}{x-1} = (x-1)^{p-1}$ and thus

$$\mathbb{Z}_p[\zeta_{p^\infty}]/(p) \cong \mathbb{F}_p[x^{1/p^\infty}]/(x-1)^{p-1} \stackrel{t=x-1}{\cong} \mathbb{F}_p[t^{1/p^\infty}]/(t^{p-1})$$

(c) By local Kronecker-Weber (see Theorem 14.2 in [Was97]) and Proposition 17 in [Ser80], we get that $\mathbb{Z}_p^{ab} = \mathbb{Z}_p^{ur}[\zeta_{p^\infty}]$. We now proceed as in (b). \square

Corollary 3.2.17. (a) $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}^b \cong \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$ and $t^\sharp = p$.

(b) $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}^b \cong \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$ and $(t+1)^\sharp = \zeta_p$.

(c) $\widehat{\mathbb{Q}_p^{ab}}^b \cong \widehat{\overline{\mathbb{F}}_p((t))^{1/p^\infty}}$ and $(t+1)^\sharp = \zeta_p$.

Proof. (a) By Lemma 3.2.16(a), we have that $\mathbb{Z}_p[p^{1/p^\infty}]/(p) \cong \mathbb{F}_p[t^{1/p^\infty}]/(t)$ via an isomorphism mapping $p^{1/p^n} + (p) \mapsto t^{1/p^n} + (t)$. One can verify directly that $\widehat{\mathbb{F}_p[[t]]^{1/p^\infty}} \rightarrow \varprojlim_{\Phi} \mathbb{F}_p[t^{1/p^\infty}]/(t) : x \mapsto (x \bmod (t), x^{1/p} \bmod (t), \dots)$ is a ring isomorphism. It follows that $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}^b \cong \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$ and by definition $t^\sharp = \lim_{n \rightarrow \infty} (p^{1/p^n})^{p^n} = p$.

(b) By Lemma 3.2.16(b), we have that $\mathbb{Z}_p[\zeta_{p^\infty}]/(p) \cong \mathbb{F}_p[t^{1/p^\infty}]/(t^{p-1})$ via an isomorphism mapping $\zeta_p + (p) \mapsto t + 1 + (t^{p-1})$. One again verifies that $\widehat{\mathbb{F}_p[[t]]^{1/p^\infty}} \rightarrow$

$\varprojlim_{\Phi} \mathbb{F}_p[t^{1/p^\infty}]/(t^{p-1}) : x \mapsto (x \bmod (t^{p-1}), x^{1/p} \bmod (t^{p-1}), \dots)$ is a ring isomorphism. It follows that $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}^b \cong \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$ and by definition $(t+1)^\sharp = \lim_{n \rightarrow \infty} \zeta_{p^{n+1}}^{p^n} = \zeta_p$.

(c) By Lemma 3.2.16(c), we have that $\mathbb{Z}_p^{ab}/(p) \cong \overline{\mathbb{F}_p[t^{1/p^\infty}]/(t^{p-1})}$ mapping $\zeta_p + (p) \mapsto t + 1 + (t^{p-1})..$ One verifies that $\widehat{\mathbb{F}_p[[t]]^{1/p^\infty}} \rightarrow \varprojlim_{\Phi} \overline{\mathbb{F}_p[t^{1/p^\infty}]/(t^{p-1})} : x \mapsto (x \bmod (t^{p-1}), x^{1/p} \bmod (t^{p-1}), \dots)$ is a ring isomorphism. It follows that $\widehat{\mathbb{Q}_p^{ab}} \cong \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$ and by definition $(t+1)^\sharp = \lim_{n \rightarrow \infty} \zeta_{p^{n+1}}^{p^n} = \zeta_p$. \square

Proposition 3.2.18 (cf. Example 2.22 [FF14]). We have the following isomorphisms:

- (a) $\widehat{\mathbb{Z}_p[p^{1/p^\infty}]} \cong W(\widehat{\mathbb{F}_p[[t]]^{1/p^\infty}})/([t] - p)$.
- (b) $\widehat{\mathbb{Z}_p[\zeta_{p^\infty}]} \cong W(\widehat{\mathbb{F}_p[[t]]^{1/p^\infty}})/([t+1]^{p-1} + [t+1]^{p-2} + \dots + 1)$.
- (c) $\widehat{\mathbb{Z}_p^{ab}} \cong W(\widehat{\mathbb{F}_p[[t]]^{1/p^\infty}})/([t+1]^{p-1} + [t+1]^{p-2} + \dots + 1)$.

Proof. (a) By Corollary 3.2.17(a), we have that $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}^b \cong \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$. Consider the ring homomorphism $\theta : \mathbf{A}_{\text{inf}} \rightarrow \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$ inducing $\sharp : \widehat{\mathbb{F}_p[[t]]^{1/p^\infty}} \rightarrow \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$. Since $t^\sharp = p$, we get that $\theta([t]) = p$ and therefore $[t] - p \in \text{Ker}(\theta)$. By Proposition 3.1.25, it follows that $\text{Ker}(\theta) = ([t] - p)$.

(b) By Corollary 3.2.17(b), we have that $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}^b \cong \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$. mapping $1+t \mapsto \zeta_p$. Consider the ring homomorphism $\theta : \mathbf{A}_{\text{inf}} \rightarrow \widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}$ inducing $\sharp : \widehat{\mathbb{F}_p[[t]]^{1/p^\infty}} \rightarrow \widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}$. Since $(t+1)^\sharp = \zeta_p$, we get that $\theta([t+1]) = \zeta_p$ and therefore $[t+1]^{p-1} + \dots + [t+1] + 1 \in \text{Ker}(\theta)$. By Proposition 3.1.25, it suffices to show that $\xi = [t+1]^{p-1} + \dots + [t+1] + 1$ is a distinguished element of \mathbf{A}_{inf} . Set $\text{res} : \mathcal{O}_F \rightarrow \mathcal{O}_F/\mathfrak{m}_F$ for the residue map and $W(\text{res})$ for the unique induced homomorphism $W(\text{res}) : W(R) \rightarrow W(R/\mathfrak{m}_R)$ of Proposition 3.1.16. We compute that $W(\text{res})(\xi) = 1 + 1 + \dots + 1 = p$, whence ξ is a distinguished element of \mathbf{A}_{inf} (Remark 3.1.24).

(c) Similar to (b). \square

Proof of Corollary 3.2.20

Corollary 3.2.19. *We have the following:*

- (a) $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ and $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ are $\mathbb{F}_p[t]$ -computable untilts of $\widehat{\mathbb{F}_p((t))^{1/p^\infty}}$.
- (b) $\widehat{\mathbb{Q}_p^{ab}}$ is an $\mathbb{F}_p[t]$ -computable untilt of $\widehat{\mathbb{F}_p((t))^{1/p^\infty}}$.

Proof. The case of $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ is clear. For the other two cases, note that $\xi := [t+1]^{p-1} + [t+1]^{p-2} + \dots + 1$ is $\mathbb{F}_p[t]$ -computable, using the *computable* polynomials S_0, \dots, S_n for Witt vector addition (Observation 3.1.18). \square

Corollary 3.2.20. (a) Assume $\mathbb{F}_p((t))^{1/p^\infty}$ is (existentially) decidable in L_t . Then $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p(\zeta_{p^\infty})$ are (existentially) decidable in L_{val} .

(b) Assume $\overline{\mathbb{F}}_p((t))^{1/p^\infty}$ is (existentially) decidable in L_t . Then \mathbb{Q}_p^{ab} is (existentially) decidable in L_{val} .

Proof. (a) By Corollary 3.2.19 and Corollary 3.2.12.

(b) Similar to (a). □

Remark 3.2.21. It is worth mentioning that van den Dries' Theorem 2.1.3 implies that for any mixed characteristic henselian field (K, v) , we have that $(K, v) \equiv (\widehat{K}, \widehat{v})$ in L_{val} . Indeed, for every $n \in \mathbb{N}$ we have canonical isomorphisms $\mathcal{O}_v/(p^n) \cong \mathcal{O}_{\widehat{v}}/(p^n)$ and $(\Gamma_v, vp) \cong (\Gamma_{\widehat{v}}, \widehat{v}p)$. The former is the natural inclusion $\mathcal{O}_v/(p^n) \hookrightarrow \mathcal{O}_{\widehat{v}}/(p^n)$, which is also surjective as for any $x \in \mathcal{O}_{\widehat{v}}$, there is $x' \in \mathcal{O}_v$ such that $x - x' \in p^n \mathcal{O}_{\widehat{v}}$. The conclusion follows from Theorem 2.1.3. For instance, we have that $\mathbb{Q}_p(p^{1/p^\infty}) \equiv \overline{\mathbb{Q}_p(p^{1/p^\infty})}$ in L_{val} .

3.3 Applications: Tame fields of mixed characteristic

3.3.1 Introduction

Motivation

In this section we prove Corollary 3.3.11, which to my knowledge is the first decidability result for tame fields of mixed characteristic. Corollary 3.3.11 is proved by transposing a decidability result for tame fields in positive characteristic and ultimately using Theorem 3.2.11.

Preliminaries

The algebra and model theory of tame fields was introduced and studied by Kuhlmann in [Kuh16a]. Recall the definition:

Definition 3.3.2. Let (K, v) be a henselian valued field. A finite valued field extension $(L, w)/(K, v)$ is said to be tame if

1. $p \nmid [\Delta : \Gamma]$, where p is the *characteristic exponent* of k , i.e. $p = \text{char}(k)$ if this is positive and $p = 1$ if $\text{char}(k) = 0$.
2. The residue field extension l/k is separable.

3. The extension $(L, w)/(K, v)$ is defectless, meaning that we have the fundamental equality $[L : K] = [\Delta : \Gamma] \cdot [l : k]$.

An algebraic valued field extension is said to be tame if every finite subextension is tame.

Definition 3.3.3. A henselian valued field (K, v) is said to be tame if every finite valued field extension $(L, w)/(K, v)$ is tame.

In practice, one often needs a more intrinsic description of tame fields. This is provided by the following:

Proposition 3.3.4 (Theorem 3.2 [Kuh16a]). Let (K, v) be henselian valued field. Then the following are equivalent:

1. (K, v) is a tame field.
2. (K, v) is algebraically maximal, Γ is p -divisible and k is perfect.

Example 3.3.5. Using Proposition 3.3.4, one can verify the following:

- (a) Any (algebraically) maximal immediate extension of $\mathbb{Q}_p(p^{1/p^\infty})$ or $\mathbb{Q}_p(\zeta_{p^\infty})$ is tame.
- (b) Any (algebraically) maximal immediate extension of $\mathbb{F}_p((t))^{1/p^\infty}$ is tame. In particular, the Hahn field $(\mathbb{F}_p((t^\Gamma)), v_t)$ with value group $\Gamma = \frac{1}{p^\infty}\mathbb{Z}$ and residue field \mathbb{F}_p is tame.

Equal characteristic

Kuhlmann obtained the following Ax-Kochen/Ershov principle for tame fields of equal characteristic:

Theorem 3.3.6 (Theorem 1.4 [Kuh16a]). *Let (K, v) and (K', v') be two equal characteristic tame fields. Then $(K, v) \equiv (K', v')$ in L_{val} if and only if $k \equiv k'$ in L_{rings} and $\Gamma \equiv \Gamma'$ in L_{oag} .*

Kuhlmann then deduces the following decidability result (among others):

Corollary 3.3.7 (Theorem 1.6 [Kuh16a]). *Set $\Gamma = \frac{1}{p^\infty}\mathbb{Z}$, i.e. for the p -divisible hull of \mathbb{Z} . The Hahn field $(\mathbb{F}_p((t^\Gamma)), v_t)$ is decidable in L_{val} .*

In recent work, Lisinski combined results of Kuhlmann [Kuh16a] with work of Kedlaya [Ked06] and obtained the following strengthening of Corollary 3.3.7:

Theorem 3.3.8 (Theorem 1 [Lis21]). *Set $\Gamma = \frac{1}{p^\infty}\mathbb{Z}$. The Hahn field $(\mathbb{F}_p((t^\Gamma)), v_t)$ is decidable in L_t .*

Mixed characteristic

Kuhlmann's Theorem 3.3.6 fails as such for mixed characteristic tame fields. A counterexample is given in Theorem 1.5(c) [AK16]. The lack of such a principle has been a fundamental obstacle in obtaining decidability results for such fields.

3.3.9 Mixed characteristic tame fields

While we do not know whether $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p(\zeta_{p^\infty})$ are (existentially) decidable, we will show that they admit decidable maximal immediate extensions. These are tame fields by Example 3.3.5(a).

Lemma 3.3.10 (Remark 2.23 [FF14]). Let (K, v) be a perfectoid field. Then (K, v) is maximal if and only if (K^\flat, v^\flat) is maximal.

Proof. Immediate from the tilting equivalence Theorem 3.1.31 and the fact that tilting preserves value groups and residue fields (see Lemma 3.1.7(b)). \square

Corollary 3.3.11. The valued field $(\mathbb{Q}_p(p^{1/p^\infty}), v_p)$ (resp. $(\mathbb{Q}_p(\zeta_{p^\infty}), v_p)$) admits a maximal immediate extensions which is decidable in L_{val} .

Proof. Recall from Corollaries 3.2.17(a) and 5.1.4(a) that $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}^b \cong \widehat{\mathbb{F}_p((t))}^{1/p^\infty}$ and that $\xi = [t] - p$. Let $\mathbb{F}_p((t^\Gamma))$ be the Hahn field with value group $\Gamma = \frac{1}{p^\infty}\mathbb{Z}$ and residue field \mathbb{F}_p . It is a (non-unique) maximal immediate extension of $\mathbb{F}_p((t))^{1/p^\infty}$. By Theorem 3.1.31, we may form $K = \mathbb{F}_p((t^\Gamma))^\sharp$ as in the diagram below

$$\begin{array}{ccc} K & & \mathbb{F}_p((t^\Gamma)) \\ | & & | \\ \widehat{\mathbb{Q}_p(p^{1/p^\infty})} & & \widehat{\mathbb{F}_p((t))}^{1/p^\infty} \end{array}$$

such that $\xi_K = [t] - p$ (see §3.1.22). By Lemma 3.3.10, we see that (K, v) is a maximal immediate extension of $(\mathbb{Q}_p(p^{1/p^\infty}), v_p)$. By Theorem 3.3.8 and Theorem 3.2.11, we see that (K, v) is decidable in L_{val} . The proof for $\mathbb{Q}_p(\zeta_{p^\infty})$ is similar. \square

Remark 3.3.12. Unraveling the description of $\mathbb{F}_p((t^\Gamma))^\sharp$ from §3.1.22, one sees that $\mathbb{F}_p((t^\Gamma))^\sharp = \text{Frac}(W(\mathbb{F}_p[[t^\Gamma]])/([t] - p))$ is the decidable maximal immediate extension of $\mathbb{Q}_p(p^{1/p^\infty})$ that we get from Corollary 3.3.11.

Remark 3.3.13. It is also true that \mathbb{Q}_p^{ab} admits a decidable maximal immediate extension (in this case unique) but this already follows from well-known results in the model theory of algebraically maximal Kaplansky fields (see pg.4 Part (f) [Kuh16a]).

In spite of Corollary 3.3.11, the following is worth noting:

Remark 3.3.14. A tree-like construction similar to Proposition 3.1.36(b) shows that there exist uncountably many pairwise elementary inequivalent maximal immediate extensions of $\mathbb{Q}_p(p^{1/p^\infty})$ (resp. $\mathbb{Q}_p(\zeta_{p^\infty})$). In particular, the valued field $\mathbb{Q}_p(p^{1/p^\infty})$ (resp. $\mathbb{Q}_p(\zeta_{p^\infty})$) has uncountably many *undecidable* maximal immediate extensions. Note that the tilts of all such fields will be maximal immediate extensions of $\mathbb{F}_p((t))^{1/p^\infty}$ and thus will be tame fields. As a consequence of Kuhlmann's Theorem 3.3.6, all of them will be decidable in L_{val} . Nevertheless, they will all be undecidable in L_t .

3.4 Applications: Congruences modulo p

3.4.1 Introduction

Goal

We shall now prove Theorem 3.4.6, which shows the existence of an algorithm that decides whether a system of polynomial equations and inequations, defined over \mathbb{Z} , has a solution modulo p over the valuation rings of our fields of interest.

Strategy

The proof is via a local field approximation argument, using the computations of Lemma 3.2.16. One eventually encodes the above problem in the existential theory of the tilt in L_{val} , where the Anscombe-Fehm Theorem 2.2.3 applies. This should be contrasted with Corollary 3.2.20, which requires decidability in the language L_t on the characteristic p side.

3.4.2 From residue rings to valuation rings

The crux of the argument lies in the following:

Proposition 3.4.3. Let $f_i(x), g_j(x) \in \mathbb{F}_p[x]$ be multi-variable polynomials in $x = (x_1, \dots, x_m)$ for $i, j = 1, \dots, n$. Then

$$\mathbb{F}_p[t^{1/p^\infty}]/(t) \models \exists x \bigwedge_{1 \leq i, j \leq n} (f_i(x) = 0 \wedge g_j(x) \neq 0) \iff$$

$$\mathbb{F}_p[[t]]^{1/p^\infty} \models \exists x \bigwedge_{1 \leq i, j \leq n} (v(f_i(x)) > v(g_j(x)))$$

Proof. First observe that

$$\mathbb{F}_p[[t]]^{1/p^\infty}/(t) \cong \varinjlim \mathbb{F}_p[[t^{1/p^n}]]/(t) \cong \varinjlim \mathbb{F}_p[t^{1/p^n}]/(t) \cong \mathbb{F}_p[t^{1/p^\infty}]/(t) \quad (\dagger)$$

" \Rightarrow ": Let $a \in (\mathbb{F}_p[t^{1/p^\infty}]/(t))^m$ be such that $f_i(a) = 0 \wedge g_j(a) \neq 0$, for $1 \leq i, j \leq n$ and let \tilde{a} be any lift of a in $\mathbb{F}_p[[t]]^{1/p^\infty}$ via the isomorphism (\dagger) . We see that $v(f_i(\tilde{a})) \geq 1 > v(g_j(\tilde{a}))$, for all $1 \leq i, j \leq n$.

" \Leftarrow ": Let $b \in (\mathbb{F}_p[[t]]^{1/p^\infty})^m$ be such that $v(g_j(b)) < v(f_i(b))$ for all $1 \leq i, j \leq n$. Set $\gamma_1 = \max\{v(g_j(b)) : j = 1, \dots, n\}$ and $\gamma_2 = \min\{v(f_i(b)) : i = 1, \dots, n\}$ and consider the open interval $I = (\gamma_1, \gamma_2) \subseteq \frac{1}{p^\infty}\mathbb{Z}^{\geq 0}$. Since $\frac{1}{p^\infty}\mathbb{Z}$ is dense in \mathbb{R} , we can find $q \in \frac{1}{p^\infty}\mathbb{Z}$ such that $1 \in qI$.

We now make use of the fact that for each $q \in \frac{1}{p^\infty}\mathbb{Z}^{>0}$, there is an embedding

$$\rho : \mathbb{F}_p((t))^{1/p^\infty} \rightarrow \mathbb{F}_p((t))^{1/p^\infty}$$

which maps $t \mapsto t^q$. Indeed, if $q \in \frac{1}{p^N}\mathbb{Z}^{>0}$ for some $N \in \mathbb{N}$, then there exists an embedding $\rho : \mathbb{F}_p((t))^{1/p^N} \rightarrow \mathbb{F}_p((t))^{1/p^N}$ mapping $t \mapsto t^q$, exactly as in Remark 7.9 [AF16]. Such a map can also be extended uniquely to the perfect hull $\mathbb{F}_p((t))^{1/p^\infty}$. Now let $\rho : \mathbb{F}_p((t))^{1/p^\infty} \rightarrow \mathbb{F}_p((t))^{1/p^\infty}$ be as above. Then, since $f_i(x), g_j(x) \in \mathbb{F}_p[x]$, we get

$$v(g_j(\rho(b))) = v(\rho(g_j(b))) = qv(g_j(b)) < qv(f_i(b)) = v(\rho(f_i(b))) = v(f_i(\rho(b)))$$

for all $1 \leq i, j \leq n$. We may thus replace our witness b with $a = \rho(b)$.

Since $1 \in qI$, we get $f_i(a) = 0 \pmod{(t)} \wedge g_j(a) \neq 0 \pmod{(t)}$, for all $i, j = 1, \dots, n$. The reduction of a modulo (t) , seen as a tuple in $\mathbb{F}_p[t^{1/p^\infty}]/(t)$ via (\dagger) , is the desired witness. \square

Remark 3.4.4. The same argument used in Lemma 3.4.3 shows that

$$k[t^{1/p^\infty}]/(t^{p-1}) \models \exists x \bigwedge_{1 \leq i, j \leq n} (f_i(x) = 0 \wedge g_j(x) \neq 0) \iff$$

$$k[[t]]^{1/p^\infty} \models \exists x \bigwedge_{1 \leq i, j \leq n} (v(f_i(x)) > v(g_j(x))) \iff$$

where $k = \mathbb{F}_p$ or $\overline{\mathbb{F}}_p$. The argument of Proposition 3.4.3 needs only a slight modification for the converse direction; one needs to take $q \in \frac{1}{p^\infty}\mathbb{Z}$ such that $(p-1) \in qI$ instead of $1 \in qI$ and the same proof goes through.

3.4.5 Proof of Theorem 3.4.6

We may now prove the following:

Theorem 3.4.6. *Let K be any of the fields $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{Q}_p(\zeta_{p^\infty})$ or \mathbb{Q}_p^{ab} . Then the existential theory of $\mathcal{O}_K/(p)$ is decidable in L_{rings} .*

Proof. By the Anscombe-Fehm Theorem 2.2.3, the valued fields $(\mathbb{F}_p((t))^{1/p^\infty}, v_t)$ and $(\overline{\mathbb{F}_p}((t))^{1/p^\infty}, v_t)$ are \exists -decidable in L_{val} . For $\mathbb{Q}_p(p^{1/p^\infty})$ the conclusion now follows from Lemma 3.2.16(a) and Proposition 3.4.3. For the other two fields, one has to use Lemma 3.2.16(b), (c) and Remark 3.4.4. \square

Remark 3.4.7. Note that Corollary 3.0.1 requires decidability in the language L_t on the characteristic p side. However, for the purposes of Theorem 3.4.6, the Anscombe-Fehm results in L_{val} turned out to be sufficient. Note that, if we were to proceed naively, we would have

$$\mathbb{F}_p[t^{1/p^\infty}]/(t) \models \exists x f(x) = 0 \wedge g(x) \neq 0 \iff$$

$$\mathbb{F}_p[[t]]^{1/p^\infty} \models \exists x, z(f(x)) = tz \wedge \forall w g(x) \neq tw$$

Note that the latter sentence uses a parameter for t . Nevertheless, the trick of Proposition 3.4.3 provides us with a different (equivalent) sentence which does not involve t .

3.5 Final remarks

3.5.1 An almost decidable field

In §2.3, we emphasized that *uniform* decidability of $(\mathcal{O}_K/(p^n))_{n \in \omega}$ is key for the decidability of (K, v) . Indeed, at least by assuming the decidability of $\widehat{\mathbb{F}_p((t))^{1/p^\infty}}$ in L_t , Proposition 3.1.36(b) allows us to produce undecidable valued fields (K, v) whose each individual residue ring $\mathcal{O}_K/(p^n)$ is decidable.

Proposition 3.5.2. Assume $\widehat{\mathbb{F}_p((t))^{1/p^\infty}}$ is decidable in L_t . Then there exists a mixed characteristic henselian valued field (K, v) such that:

1. The valued field (K, v) is undecidable in L_{val} .
2. For each $n \in \mathbb{N}$, the ring $\mathcal{O}_K/(p^n)$ is decidable in L_{rings} .
3. The value group (Γ_v, vp) decidable in L_{oag} .

Proof. For the convenience of the reader, we first review the construction from proof of Proposition 3.1.36(b). Set $\pi_{\alpha(0)} = p$ and $K_{\alpha(0)} = \mathbb{Q}_p$. For each $\alpha \in 2^\omega$, define inductively $K_{\alpha(n)} = K_{\alpha(n-1)}(((1+p)^{\alpha(n)} \cdot \pi_{\alpha(n-1)})^{1/p})$ and $\pi_{\alpha(n)} = ((1+p)^{\alpha(n)} \cdot \pi_{\alpha(n-1)})^{1/p}$ for $n \in \mathbb{N}$. Let $K_\alpha = \bigcup_{n \in \mathbb{N}} K_{\alpha(n)}$. Claim 2 of Proposition 3.1.36 shows that $\widehat{K_\alpha}^b = \widehat{\mathbb{F}_p((t))}^{1/p^\infty}$. We shall argue below that any K_α satisfies conditions (2) and (3). On the other hand, since $K_\alpha \not\cong K_\beta$ for $\alpha \neq \beta$ (see the proof of Proposition 3.1.36(b)), we will have that some K_α is undecidable in L_{val} .

Fix $\alpha \in 2^\omega$, set $\bar{a} = \sum_{k \geq 0} \alpha(k) \cdot p^k \in \mathbb{Z}_p$ and $\bar{\alpha}_n = \sum_{k=0}^{n-1} \alpha(k) \cdot p^k \in \mathbb{Z}$ for all $n \in \mathbb{N}$ ($\bar{\alpha}_0 = 0$ by convention). One sees that $(1+p)^{p^n} \equiv 1 \pmod{p^{n+1}\mathbb{Z}_p}$, for all $n \in \mathbb{N}$. It follows that the limit $\lim_{n \rightarrow \infty} (1+p)^{\bar{\alpha}_n}$ exists in \mathbb{Z}_p and we denote it by $(1+p)^{\bar{a}}$. Fix $n \in \mathbb{N}^{\geq 1}$ for the rest of the proof.

Claim: We have $t^\sharp = (1+p)^{\bar{a}} \cdot p$ and $t^\sharp \equiv (1+p)^{\bar{\alpha}_{n-1}} \cdot p \pmod{p^n \mathcal{O}_{K_\alpha}}$.

Proof. We have that $t = (\pi_{\alpha(0)} + (p), \pi_{\alpha(1)} + (p), \dots)$, via the identification $\widehat{\mathbb{F}_p[[t]]}^{1/p^\infty} \cong \varprojlim_{\Phi} \mathcal{O}_{K_\alpha}/(p)$ (see Remark 3.1.39). Since $\pi_{\alpha(n)}^{p^n} = (1+p)^{\bar{\alpha}_n} \cdot p$, we compute $t^\sharp = \lim_{n \rightarrow \infty} \pi_{\alpha(n)}^{p^n} = \lim_{n \rightarrow \infty} (1+p)^{\bar{\alpha}_n} \cdot p = (1+p)^{\bar{a}} \cdot p$. Since $(1+p)^{p^{n-1}} \equiv 1 \pmod{p^n \mathcal{O}_{L_\alpha}}$, we also get $t^\sharp \equiv (1+p)^{\bar{\alpha}_{n-1}} \cdot p \pmod{p^n \mathcal{O}_{K_\alpha}}$. \square *Claim*

As a consequence, we may take $\xi_\alpha = [t] - (1+p)^{\bar{a}} \cdot p$ as a generator of the ideal of \mathbf{A}_{inf} associated to K_α . Consider also the distinguished element $\xi' := [t] - (1+p)^{\bar{\alpha}_{n-1}} \cdot p \in \mathbf{A}_{\text{inf}}$ and the associated untilt (K', v') . We have an equality of ideals $(p^n, \xi_\alpha) = (p^n, \xi')$ in \mathbf{A}_{inf} and thus an isomorphism of rings $\mathcal{O}_{K_\alpha}/(p^n) \cong \mathcal{O}_{K'}/(p^n)$.

Having assumed that the valued field $(\widehat{\mathbb{F}_p((t))}^{1/p^\infty}, v_t)$ is decidable in L_t and since ξ' is $\mathbb{F}_p[t]$ -computable, we get that the valued field (K', v') is decidable in L_{val} by Theorem 3.0.2. In particular, the ring $\mathcal{O}_{K_\alpha}/(p^n) \cong \mathcal{O}_{K'}/(p^n)$ is decidable in L_{rings} . The value group $(\Gamma_\alpha, v_\alpha p) \cong (\frac{1}{p^\infty}\mathbb{Z}, 1)$ is also decidable in L_{oag} with a constant symbol for 1. Since n was arbitrary, this concludes the proof. \square

Remark 3.5.3. It seems plausible that a similar construction can be carried out with $\widehat{\mathbb{F}_p((t^\Gamma))}$ instead of $\widehat{\mathbb{F}_p((t))}^{1/p^\infty}$, where $\Gamma = \frac{1}{p^\infty}\mathbb{Z}$, thereby recovering the above result unconditionally. Nevertheless, we do not have a working example at present.

3.5.4 Reversing the direction

Given that our understanding of decidability problems in characteristic p is limited, our philosophy of reducing decidability questions from mixed characteristic to positive characteristic may seem impractical. Nevertheless, we have already seen two applications in §3.3 and §3.4. We also mention another application

in [Kar21a], which proves an *undecidability* result for the asymptotic theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ in the language L_{val} with a cross-section, again by transposing a result in positive characteristic.

We shall now demonstrate that the characteristic p difficulties in the language L_t are already encoded in the mixed characteristic setting, by showing a relative decidability result in the opposite direction:

Proposition 3.5.5. If $\mathbb{Q}_p(p^{1/p^\infty})$ is $\forall^1\exists$ -decidable in L_{val} , then $\mathbb{F}_p[[t]]^{1/p^\infty}$ is \exists^+ -decidable in L_t .

Proof. Let $f_i(X_1, \dots, X_m, T) \in \mathbb{F}_p[X_1, \dots, X_m, T]$ for $i = 1, \dots, n$. We claim that

$$\mathbb{F}_p[[t]]^{1/p^\infty} \models \exists x_1, \dots, x_m \left(\bigwedge_{1 \leq i \leq n} f_i(x_1, \dots, x_m, t) = 0 \right) \iff$$

$$\mathbb{F}_p[t^{1/p^\infty}]/(t) \models \forall y \in \mathfrak{m} \exists x_1, \dots, x_m \left(\bigwedge_{1 \leq i \leq n} f_i(x_1, \dots, x_m, y) = 0 \right)$$

It is enough to prove the claim since $\mathbb{Z}_p[p^{1/p^\infty}]/(p) = \mathbb{F}_p[t^{1/p^\infty}]/(t)$ by Lemma 3.2.16(a).

" \Rightarrow " : Let $(\alpha_1, \dots, \alpha_m) \in (\mathbb{F}_p[[t]]^{1/p^\infty})^m$ be such that

$$f_1(\alpha_1, \dots, \alpha_m, t) = f_2(\alpha_1, \dots, \alpha_m, t) = \dots = f_n(\alpha_1, \dots, \alpha_m, t) = 0$$

Let $y \in \mathfrak{m} \subset \mathbb{F}_p[[t]]^{1/p^\infty}$ and $\kappa \in \mathbb{N}$ be such that $y^{p^\kappa} \in \mathbb{F}_p[[t]]$. Then there exists a ring endomorphism ρ on $\mathbb{F}_p[[t]]$ mapping $t \mapsto y^{p^\kappa}$, which extends uniquely to $\mathbb{F}_p[[t]]^{1/p^\infty}$. Let $\beta_i := \rho(\alpha_i)$ for $i = 1, \dots, m$. Since $f_i(X_1, \dots, X_m, T) \in \mathbb{F}_p[X, T]$, we get

$$f_1(\beta_1, \dots, \beta_m, y^{p^\kappa}) = f_2(\beta_1, \dots, \beta_m, y^{p^\kappa}) = \dots = f_n(\beta_1, \dots, \beta_m, y^{p^\kappa}) = 0$$

and thus

$$f_1(\beta_1^{1/p^\kappa}, \dots, \beta_m^{1/p^\kappa}, y) = f_2(\beta_1^{1/p^\kappa}, \dots, \beta_m^{1/p^\kappa}, y) = \dots = f_n(\beta_1^{1/p^\kappa}, \dots, \beta_m^{1/p^\kappa}, y) = 0$$

In particular, the tuple $(x_1, \dots, x_m) := (\beta_1^{1/p^\kappa}, \dots, \beta_m^{1/p^\kappa})$ is a solution modulo (t) to the above system of equations.

" \Leftarrow " A generalized version of Greenberg's Theorem due to Moret-Bailly (see Corollary 1.2.2 [MB11]) shows that

$$f_1(x_1, \dots, x_m, t) = f_2(x_1, \dots, x_m, t) = \dots = f_n(x_1, \dots, x_m, t) = 0$$

has a solution in $\mathbb{F}_p[[t]]^{1/p^\infty}$ if and only if it has a solution modulo (t^N) , for all $N \in \mathbb{N}$. Equivalently, if and only if it has a solution modulo (t^{p^N}) for all $N \in \mathbb{N}$. Using the N -th iterated Frobenius, we see that for any $N \in \mathbb{N}$ we have

$$\mathbb{F}_p[[t]]^{1/p^\infty} \models \exists x_1, \dots, x_m \left(\bigwedge_{1 \leq i \leq n} f_i(x_1, \dots, x_m, t) = 0 \pmod{t^{p^N}} \right) \iff$$

$$\mathbb{F}_p[[t]]^{1/p^\infty} \models \exists x \left(\bigwedge_{1 \leq i \leq n} f_i(x_1, \dots, x_m, t^{1/p^N}) = 0 \pmod{t} \right)$$

and the latter is true by assumption, for any $N \in \mathbb{N}$. \square

Remark 3.5.6. By an identical argument, one can prove a similar result for $\mathbb{Q}_p(\zeta_{p^\infty})$ or \mathbb{Q}_p^{ab} . In the case of the latter, one gets that $\overline{\mathbb{F}}_p[[t]]^{1/p^\infty}$ is \exists^+ -decidable in L_t , provided that \mathbb{Q}_p^{ab} is $\forall^1\exists$ -decidable in L_{val} .

It would be nice to have a version of Proposition 3.5.5 for the full theories. Together with Corollary 3.0.1, this would yield a Turing equivalence between the theories of $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{F}_p((t))^{1/p^\infty}$. Nevertheless, Proposition 3.5.5 still suggests that if we eventually want to understand the theories of $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{Q}_p(\zeta_{p^\infty})$ and \mathbb{Q}_p^{ab} (even modest parts of their theories), we would have to face certain characteristic p difficulties, posed in Question 3.5.7 below:

- Question 3.5.7.** (a) Is $\text{Th}_{\exists^+}(\mathbb{F}_p[[t]]^{1/p^\infty})$ decidable in L_t ?
 (b) Is $\text{Th}_{\exists^+}(\overline{\mathbb{F}}_p[[t]]^{1/p^\infty})$ decidable in L_t ?

4

Interlude: Fontaine-Wintenberger

Introduction

The material presented in this chapter represents part of upcoming joint work with F. Jahnke. In chapter 3, we explained how ideas from perfectoid geometry can be useful in the model theory of henselian fields. We are now interested in the reverse direction. We will see that model-theoretic techniques can be used to understand certain arithmetic phenomena in perfectoid geometry. We refer to §3.1 for background on perfectoid fields and the tilting functor. The goal is to give a model-theoretic proof of the (generalized) Fontaine-Wintenberger isomorphism:

Theorem 4.0.1. *The absolute Galois groups of K and K^\flat are canonically isomorphic.*

In the form stated above, the result was stated and proved by Scholze [Sch12a] but it also follows from parallel work by Kedlaya-Liu [KL15]. The two approaches are quite different (cf. Remark 5.19 [Sch12a]). One should mention that there was earlier formative work by Fontaine-Wintenberger [FW79], which contained many of the key ideas and also instances of the above result. For this reason, the above result is commonly referred to as the (generalized) Fontaine-Wintenberger theorem (cf. §3 [Sch14], §2.1 [Sch12b]).

We briefly discuss the main ideas in Scholze's approach to the Fontaine-Wintenberger theorem, which is the more relevant for us. See §2.1 [Sch12b] for a more detailed exposition. We then explain the main ideas of our non-standard approach and draw comparisons with the classical approach.

Standard proof

Given a ring R , we write $R - \text{fét}$ for the category of finite étale extensions of R . When R is a field, this is simply the category of finite separable extensions of R . The goal is to provide an equivalence between the categories $K - \text{fét}$ and $K^\flat - \text{fét}$; the statement on Galois groups is a formal consequence.

This equivalence between $K - \text{fét}$ and $K^\flat - \text{fét}$ is achieved by a series of intermediate equivalences presented in the diagram below

$$\begin{array}{ccccc} K - \text{fét} & \xrightarrow{\cong} & \mathcal{O}_K^a - \text{fét} & \xrightarrow{\cong} & (\mathcal{O}_K/p)^a - \text{fét} \\ & & & & \downarrow \cong \\ K^\flat - \text{fét} & \xrightarrow{\cong} & \mathcal{O}_{K^\flat}^a - \text{fét} & \xrightarrow{\cong} & (\mathcal{O}_{K^\flat}/t)^a - \text{fét} \end{array}$$

Here $\mathcal{O}_K^a - \text{fét}$ is the category of *almost* finite étale extensions of \mathcal{O}_K and similarly for the other categories which have a superscript a . The notion of an almost étale extension is quite technical to define here. See §4 [Sch12a] for an exposition of Faltings' almost mathematics, which was developed systematically by Gabber-Ramero [GR03].

The equivalence between $K - \text{fét}$ and $\mathcal{O}_K^a - \text{fét}$ (resp. $K^\flat - \text{fét}$ and $\mathcal{O}_{K^\flat}^a - \text{fét}$) follows:

Theorem 4.0.2 (Tate/Gabber-Ramero). *If L/K is finite, then $\mathcal{O}_L/\mathcal{O}_K$ is almost finite étale. Similarly, if M/K^\flat is finite, then $\mathcal{O}_M/\mathcal{O}_{K^\flat}$ is almost finite étale.*

The positive characteristic version admits an easy proof due to the presence of Frobenius (see pg. 4 [Sch12b]). However, the proof in mixed characteristic is non-trivial. Finally, the equivalence between $\mathcal{O}_K^a - \text{fét}$ and $(\mathcal{O}_K/p)^a - \text{fét}$ (resp. $\mathcal{O}_{K^\flat}^a - \text{fét}$ and $(\mathcal{O}_{K^\flat}/t)^a - \text{fét}$) uses the fact that finite étale algebras lift uniquely over nilpotents (Theorem 4.17 [Sch12a]). We also note that Scholze [Sch12a] obtains an equivalence between the larger categories of *perfectoid spaces* over K and K^\flat , which vastly generalizes the above equivalence and in fact specializes to it.

Non-standard approach

In our non-standard approach, we aim to replace the above classical picture with the one below

$$\begin{array}{ccc} K_U - \text{fét} & \xrightarrow{\cong} & k_w - \text{fét} \\ & & \downarrow \cong \\ K_U^\flat - \text{fét} & \xrightarrow{\cong} & k_w^\flat - \text{fét} \end{array}$$

Here U is any non-principal ultrafilter on \mathbb{N} . Note that we have dispensed with the terminology of almost mathematics and that we have replaced the highly nilpotent rings \mathcal{O}_K/p and \mathcal{O}_{K^\flat}/t with the ordinary fields k_w and k_{w^\flat} . In particular, all categories in the above picture are plainly understood as categories of finite separable field extensions. This comes at the cost of replacing K and K^\flat with non-principal ultrapowers K_U and K_U^\flat respectively. The fields K_U and K_U^\flat can be equipped with suitable henselian valuations w and w^\flat respectively. We emphasize that these are *not* the non-standard counterparts of v and v^\flat but rather suitable coarsenings thereof. We then have the following facts:

1. Every finite extension of (K_U, w) (resp. (K_U^\flat, w^\flat)) is unramified. This stems from our work in §4.1 and can be seen as a non-standard version of the Tate/Gabber-Ramero theorem. However, our proof is easy, both in positive and mixed characteristic.
2. In particular, the horizontal arrows in the above diagram simply represent the equivalence between the category of unramified extensions of (K_U, w) (resp. (K_U^\flat, w^\flat)) and the category of separable extensions of k_w (resp. k_{w^\flat}).
3. We have a field isomorphism $k_w \cong k_{w^\flat}$. This is closely related to the vertical isomorphism between \mathcal{O}_K/p and \mathcal{O}_{K^\flat}/t in the classical picture.

It formally follows from the above that $G_{K_U} \cong G_{K_U^\flat}$, which already gives a quick proof of the weaker statement that $G_K \cong G_{K^\flat}$ in the language of profinite groups, as developed by Cherlin-Dries-Macintyre [CDM81]. A more refined analysis is required in order to prove that $G_K \cong G_{K^\flat}$. One first shows that K^\flat sits naturally inside k_w , as a relatively algebraically closed subfield. It thus makes sense to lift a finite extension of K^\flat to an (unramified) finite extension of K_U . We will prove that any such extension actually comes from a finite extension of K . It will then follow that $G_{K_U} \cong G_{k_w}$ restricts to $G_K \cong G_{K^\flat}$.

4.1 Taming perfectoid fields

All definitions and facts related to tame fields can be found in §3.3.1. While perfectoid fields are typically not tame (they can admit defect extensions), we will show that any \aleph_1 -saturated extension admits a tame coarsening. It is precisely in that sense that we can "tame" perfectoid fields. Moreover, the associated value group is going to be divisible.

4.1.1 Taming perfectoid fields of positive characteristic

We first recall the notion of the discriminant. Given a field K and $f(x) = a_0 + a_1x + \dots + a_nx^n \in K[x]$, with roots r_1, \dots, r_n , the discriminant $\text{disc}(f)$ of the polynomial $f(x)$ is equal to the quantity

$$\text{disc}(f) = a_n^{2n-2} \prod_{i < j} (r_i - r_j)^2$$

One readily checks that this is a symmetric polynomial in the r_i 's and can be therefore expressed as a (homogeneous) polynomial in the a_i 's by Galois theory. The next proof is essentially identical with the (easy) proof of the Tate/Gabber-Ramero theorem in positive characteristic (cf. pg. 4[Sch12b]):

Proposition 4.1.2. Let (K_0, v_0) be a perfect henselian valued field of positive characteristic and suppose $t \in \mathcal{O}_{v_0}$ is such that every finite extension of K_0 can be generated by a root of a polynomial $a_0 + a_1x + \dots + a_nx^n$, with $-v_0t \leq v_0a_i \leq v_0t$ whenever $a_i \neq 0$. Let (K, v) be an \aleph_1 -saturated elementary extension of (K_0, v_0) and w be the coarsest coarsening of v such that $wt > 0$. Then (K, w) is a tame field with divisible value group Γ_w .

Proof. Since henselianity is an elementary property, the valued field (K, v) is also henselian. By Lemma 7.13 [Dri12], we also get that (K, w) is henselian. Let Δ be the maximal convex subgroup of Γ_v such that $vt \notin \Delta$. It suffices to show that each finite extension of (K, w) is unramified.

Since K_0 is perfect, each finite extension of K_0 generated by a root of a separable polynomial $a_0 + a_1x + \dots + a_nx^n$ can also be generated by a root of $a_0^{1/p^m} + a_1^{1/p^m}x + \dots + a_n^{1/p^m}x^n$, for any $m \in \mathbb{N}$. Since the discriminant of a polynomial is itself a (homogeneous) polynomial in the coefficients, we see that for any $k \in \mathbb{N}$, any finite extension of K_0 can be generated by a root of some $f(x) \in K[x]$, with $-\frac{v_0t}{k} \leq v_0(\text{disc}(f)) \leq \frac{v_0t}{k}$. By Łoś' s Theorem, the same holds true for (K, v) . Moreover, by \aleph_1 -saturation of K , any finite extension L/K can be obtained by adjoining a root of an irreducible polynomial $f(x) \in K[x]$ with $v(\text{disc}(f)) \in \Delta$. Observe that $\pi_w(\text{disc}(f)) = \text{disc}(\bar{f}) \neq 0$ and therefore $\bar{f}(x) \in k_w[x]$ is separable. By Hensel's Lemma, the polynomial $\bar{f}(x) \in k_w[x]$ is also irreducible and therefore L/K is unramified with respect to w . \square

Remark 4.1.3. When Γ_{v_0} has a cofinal element, then any t with vt cofinal in Γ_{v_0} will satisfy the assumption of Proposition 4.1.2. This again relies on the observation that each finite extension of K_0 generated by a root of $a_0 + a_1x + \dots + a_nx^n$ can also be generated by a root of $a_0^{1/p^m} + a_1^{1/p^m}x + \dots + a_n^{1/p^m}x^n$, for any $m \in \mathbb{N}$. In particular, when Γ_{v_0} is of rank 1, then any $t \in \mathfrak{m}_{v_0}$ will do.

4.1.4 Taming perfectoid fields of mixed characteristic

We first record a coarsening lemma. Given a ring R with nilradical $N = \text{Nil}(R)$, we write R_{red} for the reduced ring R/N .

Lemma 4.1.5. Let (K, v) be a valued field and $\varpi \in \mathcal{O}_K$. Let w be the coarsest coarsening of v such that $w\varpi > 0$ and \bar{v} be the induced valuation on k_w . We will then have that $\mathcal{O}_{\bar{v}} = (\mathcal{O}_v/(\varpi))_{\text{red}}$.

Proof. By construction, we have that $\mathcal{O}_{\bar{v}} = \mathcal{O}_v/\mathfrak{m}_w$. It suffices to show that $\mathfrak{m}_w = \text{proj}^{-1}(\text{Nil}(\mathcal{O}_v/(\varpi)))$, where $\text{proj} : \mathcal{O}_v \rightarrow \mathcal{O}_v/(\varpi) : x \mapsto x + (\varpi)$ is the natural projection map. If $x \in \mathfrak{m}_w$, then $vx > \frac{1}{n} \cdot v\varpi$ for some $n \in \mathbb{N}$. It follows that $x^n \equiv 0 \pmod{(\varpi)}$, i.e., $\text{proj}(x) \in \text{Nil}(\mathcal{O}_v/(\varpi))$. Conversely, if $\text{proj}(x) \in \text{Nil}(\mathcal{O}_v/(\varpi))$, then $x^n \equiv 0 \pmod{(\varpi)}$ for some $n \in \mathbb{N}$. It follows that $vx > \frac{1}{n}v\varpi$ and therefore $x \in \mathfrak{m}_w$. \square

Lemma 4.1.6. Let (K, v) be an \aleph_1 -saturated mixed characteristic henselian valued field. Let w be the coarsest coarsening of v such that $wv > 0$. Then (K, w) is a henselian defectless field.

Proof. Let w_0 be the finest coarsening of v of residue characteristic 0, Δ_0 be the minimal convex subgroup such that $v\varpi \in \Delta_0$ and Δ be the maximal convex subgroup with $v\varpi \notin \Delta$. Consider the standard decomposition below

$$K \xrightarrow{\Gamma_v/\Delta_0} k_{w_0} \xrightarrow{\Delta_0/\Delta} k_w \xrightarrow{\Delta} k_v$$

The following claim is well-known. We include a proof for completeness:

Claim: The valued field (k_{w_0}, \bar{w}) is spherically complete.

Proof. Since $\Gamma_{\bar{w}} = \Delta_0/\Delta$ has cofinality \aleph_0 (being of rank 1), it suffices to show that for any decreasing sequence of closed balls $B_1 \supseteq B_2 \supseteq \dots$ in (k_{w_0}, \bar{w}) , we have that $\bigcap_{n \in \mathbb{N}} B_n \neq \emptyset$. Let $b_i \in B_i$ and $r_i \in \Gamma_{\bar{w}}$ be the radius of B_i . Let $a_i \in \mathcal{O}_{w_0}$ be such that $\pi_{w_0}(a_i) = b_i$ and $\gamma_i \in \Delta_0$ be such that $r_i = \gamma_i + \Delta$. By \aleph_1 -saturation of (K, v) , we may realize the partial type

$$p(x) = \{v(x - a_i) \geq \gamma_i : i \in \mathbb{N}\}$$

and thereby find $a \in \mathcal{O}_{w_0}$ such that $v(a - a_i) \geq \gamma_i$. In particular, for all $i \in \mathbb{N}$, we have $w(a - a_i) \geq \gamma_i + \Delta$ and therefore $\bar{w}(b - b_i) \geq r_i$, where $b = \pi_{w_0}(a)$. Equivalently, we get that $b \in \bigcap_{n \in \mathbb{N}} B_n$. \square *Claim*

Spherically complete fields are defectless by Theorem 11.27 [Kuh], so (k_{w_0}, \bar{w}) is defectless. The valued field (K, w_0) is also defectless, being of equal characteristic 0. We conclude that (K, w) is also defectless, since the associated place P_w is a composition of defectless places. \square

Proposition 4.1.7. Let (K_0, v_0) be a henselian valued field of mixed characteristic with $\mathcal{O}_v/(p)$ being semi-perfect and Γ_v being p -divisible. Let (K, v) be an \aleph_1 -saturated elementary extension of (K_0, v_0) and w be the coarsest coarsening of v such that $wp > 0$. Then (K, w) is a tame field. If v_0p is cofinal in Γ_{v_0} , then Γ_w is divisible.

Proof. By Lemma 4.1.6, the valued field (K, w) is henselian defectless. By Lemma 4.1.5, we have $\mathcal{O}_{\bar{v}} = (\mathcal{O}_v/(p))_{\text{red}}$. Since $\mathcal{O}_v/(p)$ is semi-perfect, so is $\mathcal{O}_{\bar{v}}$. It follows that the residue field $k_w = \text{Frac}(\mathcal{O}_{\bar{v}})$ is perfect. Finally, the value group $\Gamma_w = \Gamma_v/\Delta$ is p -divisible as a quotient of the p -divisible group Γ_v , where Δ is the maximal convex subgroup of Γ_v with $vt \notin \Delta$. We conclude that (K, w) is tame.

Suppose now that v_0p is cofinal in Γ_{v_0} . Fix $\gamma \in \Gamma_{v_0}$ and $n \in \mathbb{N}$. Since Γ_{v_0} is p -divisible and $\mathbb{Z}[\frac{1}{p}]$ is dense in \mathbb{Q} , we deduce the following: For any $\varepsilon \in \mathbb{Q}^{>0}$, we can find $\gamma_\varepsilon, \delta_\varepsilon \in \Gamma_{v_0}$ with $0 \leq \delta_\varepsilon \leq \varepsilon \cdot vp$ such that $\gamma = n \cdot \gamma_\varepsilon + \delta_\varepsilon$. By \aleph_1 -saturation of Γ_v , we see that for all $\gamma \in \Gamma_v$ we can find $\gamma' \in \Gamma_v$ and $\delta \in \Delta$ such that $\gamma = n \cdot \gamma' + \delta$. It follows that $\Gamma_w = \Gamma_v/\Delta$ is divisible. \square

4.2 Elementary Fontaine-Wintenberger

Keeping with our notation in §1.1.21, we write K_U and K_U^b for the ultrapowers of K and K^b with respect to an ultrafilter U on \mathbb{N} . The fields K_U and K_U^b naturally inherit henselian valuations which we denote by v_U and v_U^b respectively.

Proposition 4.2.1. Let K be a perfectoid field with tilt K^b and U be a non-principal ultrafilter on \mathbb{N} . There are valuations w and w^b on K_U and K_U^b respectively, such that (K_U, w) and (K_U^b, w^b) are tame, $k_w \cong k_{w^b}$ and with $\Gamma_w \cong \Gamma_{w^b}$ divisible. In particular, we have a canonical isomorphism $G_{K_U} \cong G_{K_U^b}$.

Proof. By Lemma 3.1.7(b), we may choose a pseudouniformizer t in K^b such that $\mathcal{O}_v/(p) \cong \mathcal{O}_{v^b}/(t)$. It then also follows that

$$\mathcal{O}_{v_U}/(p) \cong (\mathcal{O}_v/(p))_U \cong (\mathcal{O}_{v^b}/(t))_U \cong \mathcal{O}_{v_U^b}/(t)$$

Let w (resp. w^b) be the coarsest coarsening of v_U (resp. v_U^b) with $wp > 0$ (resp. $w^b t > 0$). By Lemma 4.1.5, we get that $k_w = \text{Frac}((\mathcal{O}_{v_U}/(p))_{\text{red}})$ and $k_{w^b} =$

$\text{Frac}((\mathcal{O}_{v_U}/(t))_{\text{red}})$. Since $\mathcal{O}_{v_U}/(p) \cong \mathcal{O}_{v_U^b}/(t)$, this yields $k_w \cong k_{w^b}$. The ultrapowers (K_U, v_U) and (K_U^b, v_U^b) are \aleph_1 -saturated by Corollary 2.2 [Kei70]. By Propositions 4.1.2 and 4.1.7, we will have that (K_U, w) and (K_U^b, w^b) are tame with divisible value groups. It follows that $G_{K_U} \cong G_{k_w} \cong G_{k_{w^b}} \cong G_{K_U^b}$. \square

Profinite groups are classically construed as structures in the Cherlin-Dries-Macintyre formalism [CDM81]. The basic fact needed here is that for a given a profinite group sentence σ , there is a sentence σ' in L_{rings} such that $G_K \models \sigma \iff K \models \sigma'$. In particular, if $L \equiv K$, then $G_L \equiv G_K$.

Corollary 4.2.2. *Let K be a perfectoid field with tilt K^b . Then $G_K \equiv G_{K^b}$.*

Proof. Let U be a non-principal ultrafilter on \mathbb{N} . We then have that $G_K \equiv G_{K_U} \cong G_{K_U^b} \equiv G_{K^b}$, where the middle isomorphism follows from Proposition 4.2.1. \square

4.3 Classical Fontaine-Wintenberger

4.3.1 K^b as a subfield of k_w

We first remind the reader our usual setting. Let K be a perfectoid field with tilt K^b . Let U be a non-principal ultrafilter on \mathbb{N} and (K_U, v_U) be the corresponding ultrapower. Let also w be the coarsest coarsenings such that $w^p > 0$ and (k_w, \bar{v}) be the associated residue field.

Recall that $\mathcal{O}_{\bar{v}} = ((\mathcal{O}_K/(p))_U)_{\text{red}}$ and $\mathcal{O}_{K^b} = \varprojlim_{\Phi} \mathcal{O}_K/(p)$. We may thus represent elements of \mathcal{O}_{K^b} as (x_0, x_1, \dots) where the $x_i \in \mathcal{O}_K/(p)$ are such that $x_{i+1}^p = x_i$. We exhibit K^b as a relatively algebraically closed subfield of k_w . In our upcoming work with F. Jahnke we even prove that $K^b \preceq k_w$. Here, for self-containment purposes, we will only show the former (weaker) statement:

Proposition 4.3.2. *There exists a valued field embedding $(K^b, v^b) \hookrightarrow (k_w, \bar{v})$ such that K^b is relatively algebraically closed in k_w .*

Proof. Define the map

$$\iota : \mathcal{O}_{K^b} \rightarrow \mathcal{O}_{\bar{v}} : (x_0, x_1, \dots) \mapsto \lim_{n \rightarrow U} x_n + N$$

Claim 1: The map ι is an injective local ring homomorphism.

Proof. It is clear that ι is a ring homomorphism. To see that $\iota(\mathfrak{m}_{K^\flat}) \subseteq \mathfrak{m}_{\bar{v}}$, note that $(x_0, x_1, \dots) \in \mathfrak{m}_{K^\flat}$ if and only if each x_i is in the maximal ideal of $\mathcal{O}_K/(p)$. By Łoś' Theorem, the latter condition implies that $\lim_{n \rightarrow U} x_n$ is in the maximal ideal of $(\mathcal{O}_K/(p))_U$ and hence $\lim_{n \rightarrow U} x_n + N$ is in the maximal ideal of $\mathcal{O}_{\bar{v}}$.

We now argue that ι is injective. Assume $\lim_{n \rightarrow U} x_n \in N$, so that $(\lim_{n \rightarrow U} x_n)^m = 0$ for some $m \in \mathbb{N}$. Let $S = \{n \in \mathbb{N} : x_n^m = 0\}$ and note that $S \in U$ by Łoś. Now suppose that $x_{n_0} \neq 0$ for some n_0 . It follows that $x_n^{p^{n-n_0}} \neq 0$ for $n \geq n_0$. Note that $[n_0 + \log_p m, \infty) \cap S \in U$. For any $n \in [n_0 + \log_p m, \infty) \cap S$, one obtains a contradiction since $x_n^m = 0 \Rightarrow x_n^{p^{n-n_0}} = 0$. \square *Claim 1*

The map induces an embedding of valued fields $(K^\flat, v^\flat) \hookrightarrow (k_w, \bar{v})$. We let $t \in \mathcal{O}_{K^\flat}$ be such that $\mathcal{O}_K/(p) \cong \mathcal{O}_{K^\flat}/(t)$. We may represent t as $(0, \pi_1, \pi_2, \dots)$ for some $\pi_n \in \mathcal{O}_K/(p)$ and set $\pi = \lim_{n \rightarrow U} \pi_n + N$. We also write $\tilde{\pi}_n$ for an arbitrary lift of π_n in \mathcal{O}_K . Note that $\iota((t)) \subseteq (\pi)$ and therefore ι induces an embedding $\mathcal{O}_{K^\flat}/(t) \hookrightarrow \mathcal{O}_{\bar{v}}/(\pi)$.

Claim 2: $\mathcal{O}_{K^\flat}/(t) \xrightarrow{\cong} \mathcal{O}_{\bar{v}}/(\pi)$.

Proof. Indeed, the embedding $\mathcal{O}_{K^\flat}/(t) \hookrightarrow \mathcal{O}_{\bar{v}}/(\pi)$ is elementary, being a composition of the elementary maps below

$$\mathcal{O}_{K^\flat}/(t) \xrightarrow{\cong} \mathcal{O}_K/(p) \xrightarrow{\delta} (\mathcal{O}_K/(p))_U \xrightarrow{\phi} \prod_{n \in \mathbb{N}} \mathcal{O}_K/(\tilde{\pi}_n)/U \xrightarrow{\cong} \mathcal{O}_{\bar{v}}/(\pi)$$

Here δ is the usual diagonal embedding, which is elementary by Łoś. The isomorphism ϕ is constructed by patching the isomorphisms

$$\mathcal{O}_K/(\tilde{\pi}_n) \xrightarrow{\cong} \mathcal{O}_K/(p) : x + (\tilde{\pi}_n) \mapsto x^{p^n} + (p)$$

Finally, since $N \subseteq (\pi)$, we have $\mathcal{O}_{\bar{v}}/(\pi) = (\mathcal{O}_K/(p))_U/(\pi) \cong \prod_{n \in \mathbb{N}} \mathcal{O}_K/(\tilde{\pi}_n)/U$.

\square *Claim 2*

Now we argue that K^\flat is algebraically closed in k_w . Consider a polynomial $f(X) \in \mathcal{O}_{K^\flat}[X]$ and suppose that $f(a) = 0$ for some $a \in \mathcal{O}_{\bar{v}}$. Since K^\flat is perfect, one may apply the generalized Greenberg's approximation theorem due to Moret-Bailly (see Corollary 1.2.2 [MB11]). It thus suffices to show that $f(X) \equiv 0 \pmod{t^N}$ has a solution in \mathcal{O}_{K^\flat} for every $N \in \mathbb{N}$. Equivalently, that $f(X) \equiv 0 \pmod{t^{p^N}}$ has a solution for every $N \in \mathbb{N}$. Equivalently, that $\Phi^{-N}f(X) \equiv 0 \pmod{t}$ has a solution in \mathcal{O}_{K^\flat} for every $N \in \mathbb{N}$. Here $\Phi^{-N}f(X)$ is the polynomial obtained from f_i by applying the N -th iterated inverse Frobenius map to its coefficients. Now $\Phi^{-N}f(X) \equiv 0 \pmod{(\pi)}$ has a solution in $\mathcal{O}_{\bar{v}}$, namely $\Phi^{-n}(a)$. By Claim 2, we have that $\mathcal{O}_{K^\flat}/(t) \preceq \mathcal{O}_{\bar{v}}/(\pi)$ and the conclusion follows. \square

We henceforth use the above embedding to identify K^b as a subfield of k_w . To deduce the classical Fontaine-Wintenberger theorem, it suffices to show that the isomorphism $G_{K_U} \cong G_{k_w}$ restricts to $G_K \cong G_{K^b}$. In other words, we need to check that the extensions of K_U coming from finite extensions of K are precisely those which are obtained by lifting a finite extension of K^b .

4.3.3 Small discriminants

Given a valued field (K, v) and a finite extension L/K , we will have that $L \cong K[x]/(f(x))$, where $f(x) \in \mathcal{O}_v[x]$ is monic. Indeed, we may assume that $f(x) \in \mathcal{O}_v[x]$ by making a change of variables of the form $x \mapsto xc$. In order to make $f(x)$ monic, write $f(x) = a_n x^n + \dots + a_1 x + a_0$. We now have $a_n^{n-1} f(x) = a_n^n x^n + \dots + a_n^{n-1} a_1 x + a_n^{n-1} a_0 = X^n + \dots + a_n^{n-2} a_1 X + a_n^{n-1} a_0$, where $X = a_n x$.

Lemma 4.3.4. Let (K, v) be a perfectoid field with pseudouniformizer ϖ and L/K be a finite extension. Then $L \cong K[x]/(f(x))$, where $f(x) \in \mathcal{O}_v[x]$ is monic such that $0 \leq v(\text{disc}(f)) < v\varpi$.

Proof. Let U be a non-principal ultrafilter on \mathbb{N} and (K_U, v_U) be the corresponding ultrapower. Let w be the finest coarsening of v_U such that $w\varpi > 0$. By Propositions 4.1.2 and 4.1.7, LK_U/K_U is unramified with respect to w and is therefore obtained by lifting a finite separable extension l_w/k_w . Write $l_w = k_w[x]/(\bar{f}(x))$, where $\bar{f}(x) \in k_w[x]$ is a separable irreducible polynomial. We may further assume that $\bar{f}(x) \in \mathcal{O}_{\bar{v}}[x]$ and that it is monic, by making a suitable change of variables, as we explained above.

Let $f(x) \in \mathcal{O}_w[x]$ be a monic lift of $\bar{f}(x)$ and note that $f(x) \in \mathcal{O}_{v_U}[x]$. We will have that $LK_U \cong K_U[x]/(f(x))$. We may write $f(x) = \lim_{i \rightarrow U} f_i(x)$, where $f_i(x) \in \mathcal{O}_v[x]$. By Łoś's Theorem, we see that $f_i(x)$ is monic, irreducible over K and $L \cong K[x]/(f_i(x))$, for almost all i . Since the discriminant of a polynomial is itself a (homogeneous) polynomial in the coefficients, we see that $\pi_w(\text{disc}(f)) = \text{disc}(\bar{f}) \neq 0$. It follows that $\text{disc}(f) \in \mathcal{O}_w^\times$. Since $f(x) \in \mathcal{O}_{v_U}[x]$, we get that $v(\text{disc}(f)) \in \Delta^{\geq 0}$. In particular, by Łoś's Theorem, we see that $0 \leq v(\text{disc}(f_i)) < v\varpi$, for almost all i . The conclusion follows. \square

Given a monic $f(x) \in K[x]$ and a root r , observe that $f'(r)^2 = \text{disc}(f)$.

Lemma 4.3.5. Let (K, v) be a henselian valued field with pseudouniformizer ϖ . Let $f(x) = a_0 + a_1 x + \dots + x^n \in \mathcal{O}_v[x]$ be an irreducible polynomial with $0 \leq v(\text{disc}(f)) < v\varpi$ and $g(x) = a'_0 + a'_1 x + \dots + x^n \in \mathcal{O}_v[x]$ be such that $a'_i \equiv a_i \pmod{\varpi}$. Then $K[x]/(f(x)) \cong K[x]/(g(x))$.

Proof. Set $L = K[x]/(g(x))$. It suffices to find $\alpha \in L$ such that $f(\alpha) = 0$. Recall that the discriminant is a (homogeneous) polynomial in the coefficients. Since $0 < v\varpi \leq vp$ and $a_i^p \equiv a_i \pmod{\varpi}$, we deduce that $\text{disc}(g)^p \equiv \text{disc}(f) \pmod{\varpi}$. Since $0 \leq v(\text{disc}(f)) < v\varpi$, it follows that $v(\text{disc}(g)) = \frac{1}{p}v(\text{disc}(f))$. Let $\beta \in L$ be a root of $g(x)$ and $L' = K(\beta)$. Since $a_i^p \equiv a_i \pmod{\varpi}$ and $0 < v\varpi \leq vp$, we compute that

$$0 = g(\beta)^p \equiv f(\beta^p) \pmod{\varpi} \Rightarrow f(\beta^p) \equiv 0 \pmod{\varpi}$$

Similarly, since $g'(\beta)^2 = \text{disc}(g)$ and thus $v(g'(\beta)) = \frac{1}{2}v(\text{disc}(g)) = \frac{1}{2p}v(\text{disc}(f)) \leq \frac{1}{2p}v\varpi$, we get that

$$g'(\beta)^p \equiv f'(\beta^p) \pmod{\varpi} \Rightarrow v(f'(\beta^p)) = p \cdot v(g'(\beta))$$

We get that $v(f(\beta^p)) \geq v\varpi > 2v(f'(\beta^p))$. By Hensel's Lemma, we deduce the existence of $\alpha \in L' \subseteq L$ with $f(\alpha) = 0$, which finishes the proof. \square

Let R be a ring and U be an ultrafilter on \mathbb{N} . Let $f(x) = a_0 + a_1x + \dots + a_nx^n \in R_U[x]$ and $f_i = a_{0,i} + a_{1,i}x + \dots + a_{n,i}x^n \in R[x]$, for $i \in \mathbb{N}$. We shall write $f(x) = \lim_{i \rightarrow U} f_i(x)$ if $a_k = \lim_{i \rightarrow U} a_{k,i}$, for each $k = 0, \dots, n$.

Lemma 4.3.6. A finite extension L' of K_U is of the form LK_U , for some finite extension L of K , if and only if $L' \cong K_U[x]/(f_*(x))$, where $f_*(x) = \lim_{i \rightarrow U} f_i(x)$ with $f_i \in \mathcal{O}_v[x]$ such that $0 \leq v(\text{disc}(f_0)) < v\varpi$ and $f_{i+1}(x)^p \equiv f_i(x) \pmod{\varpi\mathcal{O}_v[x]}$.

Proof. Suppose $f_*(x) = \lim_{i \rightarrow U} f_i(x)$ with $0 \leq v(\text{disc}(f_0)) < v\varpi$ and $f_{i+1}^p(x) \equiv f_i(x) \pmod{\varpi\mathcal{O}_v[x]}$. By Lemma 4.3.5, all fields $L_n = K[x]/(f_n(x))$ are isomorphic over K . By Łoś's Theorem, it follows that $K_U[x]/(f_*(x)) \cong K_U[x]/(f_0(x))$. Conversely, suppose L/K is finite and $L' = LK_U$. By Lemma 4.3.4, we will have that $L \cong K[x]/(f_0(x))$, for some monic $f_0(x) \in \mathcal{O}_v[x]$ with $0 \leq v(\text{disc}(f)) < v\varpi$. By Lemma 4.3.5 and induction, we also have $L \cong K[x]/(f_n(x))$, where $f_n(x) \in \mathcal{O}_v[x]$ are monic polynomials such that $f_{n+1}^p(x) \equiv f_n(x) \pmod{\varpi\mathcal{O}_v[x]}$. Note that said polynomials exist because $\mathcal{O}_v/(\varpi)$ is semi-perfect. Set $f_*(x) = \lim_{n \rightarrow U} f_n(x)$, for some choice of f_n 's as above, and note that $L' \cong K_U[x]/(f_*(x))$ by Łoś. \square

4.3.7 Finishing the proof

Throughout this section, we let (K, v) be a perfectoid field with pseudouniformizer ϖ . We also fix a non-principal ultrafilter U on \mathbb{N} and write K_U for the corresponding ultrapower. Recall that K_U is equipped with a valuation w , which is the finest coarsening such that $w\varpi > 0$, and that we have an induced valuation \bar{v} on the residue field k_w . Moreover, the tilt K^\flat sits inside k_w as an elementary substructure.

Proposition 4.3.8. Let (K, v) be a perfectoid field with pseudouniformizer ϖ . Let U be a non-principal ultrafilter on \mathbb{N} and (K_U, v_U) be the corresponding ultrapower. Let w be the finest coarsening such that $w\varpi > 0$. An extension L' of K_U is of the form LK_U , where L/K is finite, if and only if it is a lift of some finite extension lk_w , where l/K^\flat is finite.

Proof. By Lemma 4.3.6, we have $L' \cong K_U[x]/(f_*(x))$, where $f_*(x) = \lim_{i \rightarrow U} f_i(x)$ with $f_i \in \mathcal{O}_v[x]$ satisfying $0 \leq v(\text{disc}(f_0)) < v\varpi$ and $f_{i+1}^p(x) \equiv f_i(x) \pmod{\varpi\mathcal{O}_v[x]}$. One sees that $\bar{f}_*(x) \in K^\flat[x]$, where \bar{f}_* is the image of f_* via π_w . Note that $\bar{f}_*(x)$ is separable (and irreducible) because $v_v(\text{disc}(f_*)) \in \Delta$, where Δ is the maximal convex subgroup of Γ_{v_U} not containing $v_U\varpi$. We conclude that $\text{disc}(\bar{f}_*) = \pi_w(\text{disc}(f_*)) \neq 0$ and that L' is a lift of $k_w[x]/(\bar{f}_*(x))$.

Conversely, suppose we have a finite extension of the form $K^\flat[x]/(\bar{f}(x))$, where $\bar{f}(x) \in K^\flat[x]$. By making a change of variables, we may even assume that $\bar{f}(x) \in \mathcal{O}_{K^\flat}[x]$. Note that $\bar{f}(x)$ stays irreducible over k_w because by Lemma 4.3.2 we have that K^\flat is relatively algebraically closed in k_w . We now see that any lift $f(x) \in K_U[x]$ of $\bar{f}(x)$ is of the shape described in Lemma 4.3.6 and therefore the corresponding extension of K_U comes from a finite extension of K . \square

It is not difficult now to put all the pieces together. By Proposition 4.2.1, every finite extension of K_U is unramified with respect to w and therefore we have an equivalence of categories between $K_U - \text{fét}$ and $k_w - \text{fét}$. Note that $K - \text{fét}$ sits inside $K_U - \text{fét}$ as a full subcategory. Explicitly, the finite extension L/K gives rise to the composite extension LK_U/K_U . Similarly, $K^\flat - \text{fét}$ sits inside $k_w - \text{fét}$ as a full subcategory, using Lemma 4.3.2. By Proposition 4.3.8, the equivalence of categories between $K_U - \text{fét}$ and $k_w - \text{fét}$ restricts to an equivalence between the full subcategories $K - \text{fét}$ and $K^\flat - \text{fét}$. The above discussion is summarized in the diagram below:

$$\begin{array}{ccc} K_U - \text{fét} & \xrightarrow{\cong} & k_w - \text{fét} \\ \uparrow & & \uparrow \\ K - \text{fét} & \xrightarrow{\cong} & K^\flat - \text{fét} \end{array}$$

We have thus obtained:

Corollary 4.3.9 (Fontaine-Wintenberger). *Let K be a perfectoid field with tilt K^\flat . Then we have a canonical isomorphism $G_K \cong G_{K^\flat}$.*

It would be of great interest to also recover the equivalence between the categories of perfectoid spaces over K and K^\flat (Theorem 1.9 [Sch12a]) in a model-theoretic fashion. We mention that there are already some interesting (unpublished) efforts in this direction by Rideau-Scanlon.

5

Tamely ramified fields

Introduction

The material presented in this chapter is based on [Kar21b]. In a way, tame ramification is orthogonal to deep ramification, which was discussed in Chapter 3. We want to clarify early on that the fields of interest in this chapter are *not* tame fields. Indeed, far from it. Tame fields are fields which admit only tame extensions. On the other hand, we will be interested in fields which are tame over a base (an excellent DVR). The difference can be viewed as a bottom-up (rather than a top-down) approach to tameness.

More concretely, we aim to understand the theories of *tamely* ramified extensions of \mathbb{Q}_p and $\mathbb{F}_p((t))$, including the *infinitely* ramified ones. The scope of our results is restricted to existential theories. We will obtain an abundance of explicit examples of *infinitely* ramified extensions of \mathbb{Q}_p and $\mathbb{F}_p((t))$ whose theory is existentially decidable. The caveat is that a certain form of resolution of singularities is required. Before we state our results, let us first state the precise form which we will use:

Conjecture R (Log-Resolution). *Let X be a reduced, flat scheme of finite type over an excellent discrete valuation ring R . Then there exists a blow-up morphism $f : \tilde{X} \rightarrow X$ in a nowhere dense center $Z \subsetneq X$ such that*

1. \tilde{X} is a regular scheme.
2. $\tilde{X}_s = \tilde{X} \times_{\text{Spec} R} \text{Spec}(R/\mathfrak{m}_R)$ is a strict normal crossings divisor.

Some background material related to resolution of singularities is provided in §1.2.10. We obtain the following existential Ax-Kochen/Ershov principle, from which all applications will be deduced:

Theorem 5.0.1. *Assume Conjecture R. Suppose (K, v) and (L, w) are henselian with perfect residue fields and that they are tamely ramified over a valued field (F, v_0) with \mathcal{O}_F an excellent DVR. If $RV(K) \equiv_{\exists, RV(F)} RV(L)$, then $K \equiv_{\exists, F} L$ in L_{rings} .*

It should be noted that our notion of a "tamely ramified extension" used in Theorem 5.3.2 is not restricted to algebraic extensions but does indeed specialize to the ordinary notion in the case of an algebraic extension (see §5.1.5). This level of generality is essential for applications, even if one is merely interested in algebraic tamely ramified algebraic extensions of \mathbb{Q}_p or $\mathbb{F}_p((t))$ (see Remark 5.4.7). The excellence condition on the discrete valuation ring \mathcal{O}_F simply amounts to \widehat{F}/F being separable (see §1.2.10). For example, this is the case for the valued fields $(\mathbb{Q}(t), v_t)$, (\mathbb{Q}, v_p) or $(\mathbb{F}_p(t), v_t)$. In fact, all decidability consequences in §5.4 will be obtained by specializing (F, v_0) to one of the three examples mentioned above. Some background material on the RV-structure associated to a valued field is provided in §1.3.1.

The key ingredient in the proof of Theorem 5.3.2 is a certain form of Hensel's Lemma, which we call RV-Hensel's Lemma. This is completely parallel to the classical geometric version of Hensel's Lemma for smooth morphisms over henselian local rings, saying that the existence of an integral point is equivalent to the existence of a solution in the residue field. In general, the best one can hope for is not smoothness but rather strict normal crossings. This necessitates a variant of Hensel's Lemma for morphisms having strict normal crossings. However, to ensure the existence of an integral point, one must now require not just a solution in the residue field but rather an RV-solution, i.e., more residual information is needed. Most crucially, this only works when one of the multiplicities of the irreducible components of X_s is not divisible by p , the residue characteristic of the base ring. This condition plays well with tame ramification (as one might guess) and is the reason why the scope of our results is restricted to that setting.

Theorem 5.3.2 specializes to well-known Ax-Kochen/Ershov results in residue characteristic 0 and in the mixed characteristic *unramified* setting. Moreover, these Ax-Kochen/Ershov principles hold not only for the existential theories but also for the full-first order theories. The case of *finite* tame ramification in mixed characteristic and with perfect residue fields was proved recently in Corollary 5.9 [Lee20] (see also Remark 5.3.7).

At the same time, Theorem 5.3.2 implies conditional existential decidability results for $\mathbb{F}_p((t))$ and its finite extensions, which were already known by the work of Denef-Schoutens [DS03]. Our proof does not use Greenberg's approximation theorem, which is an important ingredient in [DS03]. In fact, in the case of $\mathbb{F}_p((t))$, the proof can be significantly simplified, as we explain in §5.5. Although Conjecture R is a more refined version of resolution than Conjecture 1 [DS03], the simpler proof explained in §5.5 uses only Conjecture 1 [DS03].

On the other hand, Theorem 5.3.2 applies also to the setting of *infinite* ramification. We highlight the following application regarding the maximal tamely ramified extension \mathbb{Q}_p^{tr} (resp. $\mathbb{F}_p((t))^{tr}$) of \mathbb{Q}_p (resp. $\mathbb{F}_p((t))$), which is of arithmetic significance:

Corollary 5.0.2. *Assume Conjecture R. Then the field \mathbb{Q}_p^{tr} (resp. $\mathbb{F}_p((t))^{tr}$) is existentially decidable in the language of rings (resp. L_t).*

Ramification theory provides us with nice explicit descriptions for the fields of Corollary 5.0.2 (see Fact 5.1.4). More generally, we deduce Corollary 5.4.6, which can be used to generate plenty of other existentially decidable infinitely ramified fields, both in mixed and positive characteristic (assuming Conjecture R). For instance, if $\ell \neq p$ is a prime, we get that $\mathbb{Q}_p(p^{1/\ell^\infty})$ (resp. $\mathbb{F}_p((t))(t^{1/\ell^\infty})$) is existentially decidable in the language of rings (resp. L_t). We note that the case of $\mathbb{Q}_p(p^{1/p^\infty})$ (resp. $\mathbb{F}_p((t))^{1/p^\infty}$), which was mentioned before, does not appear to be amenable to the methods presented here and probably requires some new ideas.

Apart from decidability consequences, certain new existential Ax-Kochen/Ershov phenomena become accessible (see §5.4.11). For example, if ℓ is a prime different from p , one can deduce (modulo Conjecture R) that $\mathbb{F}_p((t))(t^{1/\ell^\infty}) \preceq_1 \mathbb{F}_p((t^{\Gamma_\ell}))$, the latter being the Hahn series field with value group $\Gamma_\ell = \frac{1}{\ell^\infty}\mathbb{Z}$ and residue field \mathbb{F}_p . This should be contrasted with the classical example by Abhyankar, namely the Artin-Schreier equation $x^p - x - 1/t = 0$. This has a solution in the Hahn field $\mathbb{F}_p((t^{\Gamma_p}))$ with value group $\Gamma_p = \frac{1}{p^\infty}\mathbb{Z}$, namely $x = \sum_{n=1}^{\infty} t^{-1/p^n}$, but has no solution in $\mathbb{F}_p((t))^{1/p^\infty}$. Similarly, we have that $\mathbb{Q}_p(p^{1/\ell^\infty})$ is existentially closed in any of its maximal immediate extensions.

5.1 Tamely ramified extensions

For valued fields (K, v) and (L, w) , we shall abbreviate the residue fields by k and l respectively, and the value groups by Γ and Δ respectively. Although the notion of a tamely ramified extension is usually defined within the context of algebraic extensions, it will be essential for us to have an analogue for general (possibly transcendental) valued field extensions. Such a definition is given by Endler on pg. 180 [End72] and we follow his treatment.

5.1.1 Algebraic tamely ramified extensions

We first review the usual notion of an *algebraic* tamely ramified extension:

Definition 5.1.2. (a) A finite valued field extension $(L, w)/(K, v)$ is said to be tamely ramified if l/k is separable and $p \nmid [\Delta : \Gamma]$, where $p = \text{char}(k)$.

(b) An algebraic valued field extension $(L, w)/(K, v)$ is said to be tamely ramified if every finite subextension is tamely ramified.

Example 5.1.3. (a) The extension $(\mathbb{Q}_p(p^{1/n}), v_p)/(\mathbb{Q}_p, v_p)$ is tamely ramified if and only if $p \nmid n$.

(b) Any unramified extension of \mathbb{Q}_p or $\mathbb{F}_p((t))$ is automatically tamely ramified.

(c) If ℓ is a prime different than p , then $\mathbb{F}_p((t))(t^{1/\ell^\infty})$ is tamely ramified over $\mathbb{F}_p((t))$.

One can give an explicit description of the *maximal* tamely ramified algebraic extension of \mathbb{Q}_p and $\mathbb{F}_p((t))$, denoted by \mathbb{Q}_p^{tr} and $\mathbb{F}_p((t))^{tr}$ respectively:

Fact 5.1.4 (Corollary 1, pg. 32 [CF67]). We have that:

$$(a) \mathbb{Q}_p^{tr} = \mathbb{Q}_p^{ur}(\{p^{1/n} : (p, n) = 1\}) = \mathbb{Q}_p(\{\zeta_n, p^{1/n} : (p, n) = 1\}).$$

$$(b) \mathbb{F}_p((t))^{tr} = \mathbb{F}_p((t))^{ur}(\{t^{1/n} : (p, n) = 1\}) = \mathbb{F}_p((t))(\{\zeta_n, t^{1/n} : (p, n) = 1\}).$$

We stress that the notion of a tamely ramified extension defined above is subtly different from the notion of a tame extension in [Kuh16b]. The difference is that tamely ramified extensions are not required to be defectless. In many cases of interest, the base field is defectless (e.g., a local field). For instance, this applies in the treatment of tame ramification given in [Lan94], [CF67] and [Ser80]. This is also the case for us here (see Fact 5.1.11). Over such a defectless base, the notions "tamely ramified" and "tame" coincide. We favor the use of tamely ramified rather than tame extensions because the former readily extends in the transcendental context and will be sufficient for our purposes.

5.1.5 Transcendental tamely ramified extensions

We now extend the notion of a tamely ramified field extension to the context of transcendental valued field extensions. The need for this level of generality is explained in Remark 5.4.7. First, recall that an extension of fields l/k (not necessarily algebraic) is said to be *separable* if every finitely generated subextension l_1/k has a *separating transcendence basis*, i.e., a transcendence basis $T \subseteq l_1$ such that the extension $l_1/k(T)$ is separable algebraic.

Definition 5.1.6. A valued field extension $(L, w)/(K, v)$ is said to be tamely ramified if l/k is separable, the quotient group Δ/Γ is p -torsion-free, where $p = \text{char}(k)$.

Note that Definition 5.1.6 specializes to Definition 5.1.2 in the case of an algebraic extension.

Example 5.1.7. (a) Every valued field extension is tamely ramified when the residue characteristic is zero.

(b) The valued field extension $(\mathbb{Q}_p(p^{1/n}), v_p)/(\mathbb{Q}, v_p)$ is tamely ramified if and only if $p \nmid n$.

(c) Let $\mathbb{F}_p((t^\Gamma))$ be the Hahn series field with residue field \mathbb{F}_p and value group Γ . Suppose $1 \in \Gamma$ is a distinguished positive element and write t for t^1 . The valued field extension $(\mathbb{F}_p((t^\Gamma)), v_t)/(\mathbb{F}_p(t), v_t)$ is tamely ramified if and only if 1 is not p -divisible in Γ .

For the rest of the chapter, unless otherwise stated, a valued field extension is said to be tamely ramified if it is tamely ramified in the generalized sense of Definition 5.1.6.

5.1.8 Some comments on the defect

Definition

First let us recall some basic definitions. See §1 [Kuh10] for a more detailed overview. Given a valued field (K, v) and a finite field extension L/K of degree n , let w_1, \dots, w_r be the extensions of v to L . We then have the fundamental inequality

$$n \geq \sum_{i=1}^r e_i \cdot f_i$$

where $e_i = e(w_i/v)$ and $f_i = f(w_i/v)$ are the ramification and inertia degrees associated to w_i/v . We say that the extension L/K is *defectless* if equality holds. We also say that (K, v) is *defectless* if any finite extension as above is defectless.

Excellent vs Defectless

Let (F, v) be a valued field with valuation ring \mathcal{O}_F . We say that (F, v) is *separably defectless* if any finite separable extension F_1/F is defectless. Likewise (F, v) is *inseparably defectless* if any finite purely inseparable extension F_1/F is defectless.

Fact 5.1.9 (Theorem 3.3.5 [engprest]). Any discrete valued field is separably defectless.

However, when $\text{char}(F) > 0$, the valued field need not be inseparably defectless:

Example 5.1.10. Let (F, v) be as in Example 1.2.11. We see that $F(y^{1/p})/F$ is an inseparable immediate extension. In particular, it is not defectless.

Nevertheless, in the context of Theorem 5.3.2, our base ring \mathcal{O}_F will in fact be an *excellent* DVR. In this case, we record the following fact:

Fact 5.1.11. Let (F, v) be a discrete valued field. The following are equivalent:

1. (F, v) is defectless.
2. \mathcal{O}_F is excellent.

Proof. By Corollaire 7.6.6 [Gro65], \mathcal{O}_F is excellent if and only if it is a Japanese ring, which is equivalent to (F, v) being defectless by Théorème 2, pg. 143 [Bou06]. \square

5.2 RV-Hensel's Lemma

5.2.1 Hensel's Lemma

Motivation

Let $X \rightarrow \text{Spec}R$ be a *smooth* morphism, where R is a henselian local ring R with residue field κ . The *classical* geometric version of Hensel's Lemma allows us to lift κ -rational points of X_s to R -integral points of X (see e.g. Corollary 6.2.13 [Liu06]). Proposition 5.2.6 is an analogue of this, for the case where $X \rightarrow \text{Spec}R$ is not necessarily smooth but has *strict normal crossings*. In that case, one may lift RV-points of X to integral points of X , at least when one of the multiplicities of the irreducible components of X_s is not p -divisible, where $p = \text{char}(\kappa)$. For lack of a suitable reference, we shall spell out the details.

Hensel's Lemma for smooth morphisms

Recall the geometric version of Hensel's Lemma for (locally) smooth morphisms:

Lemma 5.2.2 (Hensel's Lemma for smooth morphisms). Let $\pi : X \rightarrow Y$ be a morphism of schemes over a henselian local ring R with residue field κ . Let $P \in Y(R)$ and suppose that the induced κ -rational point $y \in Y_s(\kappa)$ lifts to $x \in X_s(\kappa)$ and suppose that π is smooth at x . Then P lifts to an R -integral point of X , which specializes to x .

Proof. See Corollary 6.2.13 [Liu06]. \square

Remark 5.2.3 (Remark 6.2.14 [Liu06]). Corollary 6.2.13 [Liu06] is stated with a completeness assumption but henselianity is enough (cf. Proposition 2.3.5 [BLR90]).

5.2.4 RV-Hensel's Lemma

Throughout this section, the ring R will be a DVR with uniformizer π and residue field $\kappa = R/\pi R$. We set $p = \text{char}(\kappa)$. We also have a *henselian* valuation ring A extending R (typically non-discrete). We let \mathfrak{m}_A be the maximal ideal and $k = A/\mathfrak{m}_A$ be the residue field of A . Suppose X is a scheme over R . Given a scheme-theoretic point $x \in X_s$ (not necessarily closed), we write $x \in X_s(k)$ if there is a k -rational point of X_s above x . This allows us to identify the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ with a subfield of k . Given $a \in \mathcal{O}_{X,x}$, we write \bar{a} for the image of a inside k via this identification.

Proposition 5.2.6—which we call RV-Hensel's Lemma—is the key ingredient in the proof of Theorem 5.3.2. The proof of RV-Hensel's lemma goes via an étale local analysis of strict normal crossings around a point in the special fiber, eventually reducing to Lemma 5.2.5. The case $l = 0$ in Lemma 5.2.5 simply means that there are no T_i 's and gives the étale local picture around a closed point of the special fiber. The presence of the T_i 's is precisely to account for the non-closed case. Recall the map $\iota : k^\times \rightarrow \text{RV}(K^\times)$ from §1.3.1.

Lemma 5.2.5. Let $0 \leq n < m$ and $l \geq 0$ be integers. Let $\bar{F}(T) \in \kappa[T_1, \dots, T_l][T]$ be separable and irreducible over $\kappa(T_1, \dots, T_l)$ and $F(T) \in R[T_1, \dots, T_l][T]$ be any lift. Consider the affine scheme

$$Y = \text{Spec}(R[X_1, \dots, X_m, T_1, \dots, T_l, Z]/(Z \cdot X_1^{e_1} \cdot \dots \cdot X_n^{e_n} - \pi, F(Z)))$$

where $e_i \in \mathbb{Z}^{>0}$ for $i = 1, \dots, n$ and $p \nmid e_1$. Let $y \in Y_s$ be the point corresponding to the prime ideal (π, X_1, \dots, X_m) . Suppose $y \in Y_s(k)$ and that there exist $a_1, \dots, a_n \in \mathfrak{m}_A$ such that $\iota(\bar{Z}) \cdot \text{rv}(a_1^{e_1} \cdot \dots \cdot a_n^{e_n}) = \text{rv}(\pi)$. Then Y has an A -integral point lifting y .

Proof. The point $y \in Y_s(k)$ corresponds to a morphism of R -algebras

$$R[X_1, \dots, X_m, T_1, \dots, T_l, Z]/(Z \cdot X_1^{e_1} \cdot \dots \cdot X_n^{e_n} - \pi, F(Z)) \rightarrow k$$

with kernel (π, X_1, \dots, X_m) . Write $t_i = \bar{T}_i$ for the image of T_i in k and note that the t_i 's are algebraically independent. Let $\tilde{t}_i \in A$ be an arbitrary lift of t_i and $\tilde{f}(T)$ the polynomial $F(T)$ where we replace T_i with \tilde{t}_i . Likewise, the reduction $f(T)$ of $\tilde{f}(T)$ modulo \mathfrak{m}_A is the same as $\bar{F}(T)$ with T_i replaced by t_i . It is also separable since $\kappa(t_1, \dots, t_l) \cong \kappa(T_1, \dots, T_l)$ and $f(\bar{Z}) = 0$. Hensel's lemma allows us to choose $u \in A^\times$ so that $\tilde{f}(u) = 0$ and $\bar{u} = \bar{Z}$. By assumption, there exist $a_1, \dots, a_n \in A$ such that $u \cdot a_1^{e_1} \cdot \dots \cdot a_n^{e_n} = \pi \cdot \varepsilon$, with $\varepsilon \in 1 + \mathfrak{m}$. Since $p \nmid e_1$, Hensel's lemma provides

us with $\alpha \in A^\times$ such that $\alpha^{e_1} = \varepsilon$. Replacing a_1 with $a_1 \cdot \alpha$ gives us an A -integral point $P \in Y(A)$ corresponding to a morphism of R -algebras

$$R[X_1, \dots, X_m, T_1, \dots, T_l, Z]/(Z \cdot X_1^{e_1} \cdot \dots \cdot X_n^{e_n} - \pi, F(Z)) \rightarrow A$$

mapping $X_i \mapsto a_i$, $T_i \mapsto \tilde{t}_i$ and $Z \mapsto u$. One readily checks that P specializes to y . \square

Recall the definition of strict normal crossings in Definition 1.2.5(b).

Proposition 5.2.6 (RV-Hensel's Lemma). Let $X \rightarrow \text{Spec}R$ be a finite type morphism having strict normal crossings at $x \in X_s$. Write $\pi = h \cdot x_1^{e_1} \cdot \dots \cdot x_n^{e_n}$ in $\mathcal{O}_{X,x}$, where $e_i \in \mathbb{Z}^{>0}$, $h \in \mathcal{O}_{X,x}^\times$ and $\{x_1, \dots, x_n\}$ is part of a regular system of parameters of X at x . We further assume that $\kappa(x)/\kappa$ is separable and $p \nmid e_1$. Then X has an A -integral point lifting x if and only if $x \in X_s(k)$ and there exist $a_1, \dots, a_n \in \mathfrak{m}_A$ such that $\iota(\bar{h}) \cdot \text{rv}(a_1^{e_1} \cdot \dots \cdot a_n^{e_n}) = \text{rv}(\pi)$ in $\text{RV}(A)$.

Proof. " \Rightarrow " : Clear.

" \Leftarrow " : Shrinking X around x , if necessary, we may assume that $X = \text{Spec}(B)$. We may also assume that the x_i 's are regular functions on all of X . Let $\mathfrak{p} \subsetneq B$ be the prime ideal corresponding to x . Since $\kappa(x)/\kappa$ is finitely generated and separable, we may choose a separating transcendence basis, say $\{t_1, \dots, t_l\}$ (or \emptyset in case $x \in X_s$ is closed). Let $f(T) \in \kappa[t_1, \dots, t_l][T]$ be irreducible and separable over $\kappa(t_1, \dots, t_l)$ with $f(\bar{h}) = 0$. Let $\{\tilde{t}_1, \dots, \tilde{t}_l\}$ be a set of lifts in $B_{\mathfrak{p}}$. By shrinking X further, one may even assume that $\tilde{t}_1, \dots, \tilde{t}_l \in B$. Let $\tilde{f}(T) \in R[\tilde{t}_1, \dots, \tilde{t}_l][T]$ be a lift of $f(T)$.

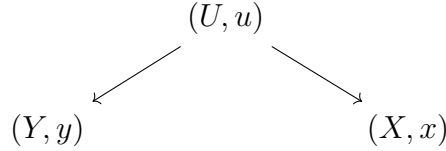
Claim: There is a common étale neighborhood (U, u) of (X, x) and (Y, y) , where (Y, y) is as in Lemma 5.2.5.

Proof. We pass to an étale neighborhood (U, u) of (X, x) , where $U = \text{Spec}(C)$ with $C = B[z]/(\tilde{f}(z))$ and u is the point of U_s corresponding to $\mathfrak{q} = (\mathfrak{p}, z - h)$. To verify the étaleness of $(U, u) \rightarrow (X, x)$ at u , observe that $\tilde{f}'(z) \notin \mathfrak{q}$. Indeed, otherwise we would have $f'(\bar{h}) = 0$, which is contrary to the assumption that $f(T)$ is separable. Finally, note that $(U, u) \rightarrow (X, x)$ is residually trivial, i.e., $\kappa(u) = \kappa(x)$. By Tag 02GU(8) [Stacks], we see that $(U, u) \rightarrow (X, x)$ is étale at u . In particular, we get that U is regular at u by Tag 025N [Stacks].

We introduce $Y = \text{Spec}(R[X_1, \dots, X_m, T_1, \dots, T_l, Z]/(Z \cdot X_1^{e_1} \cdot \dots \cdot X_n^{e_n} - \pi, F(Z)))$, where $F(T) \in R[T_1, \dots, T_l][T]$ is the same as $\tilde{f}(T)$ but with \tilde{t}_i replaced with T_i . Consider the morphism $U \rightarrow Y$, which corresponds contravariantly to the ring homomorphism sending $X_i \mapsto x_i$, $T_i \mapsto \tilde{t}_i$ and $Z \mapsto z$. Note that Y is regular at $y \in Y_s$ associated to the prime ideal (π, X_1, \dots, X_m) and that $U \rightarrow Y$ maps u to y .

We claim that $(U, u) \rightarrow (Y, y)$ is étale at u . First, note that $\{X_1, \dots, X_m\}$ is a regular system of parameters of Y at y . This maps via $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{U,u}$ to $\{x_1, x_2, \dots, x_m\}$, which is a regular system of parameters for U at u . We also need to argue that $\kappa(u)/\kappa(y)$ is finite separable. Indeed, we have a tower of finite extensions $\kappa(u)/\kappa(y)/\kappa(t_1, \dots, t_l)$ with $\kappa(u)/\kappa(t_1, \dots, t_l)$ being finite separable. It follows that $\kappa(u)/\kappa(y)$ is also finite separable. Étaleness of $(U, u) \rightarrow (Y, y)$ may now be verified using Lemma 2.1.4 [Nic13]. □*Claim*

We summarize the above in the following diagram



Note that $y \in Y_s(k)$, corresponding to the morphism

$$\kappa[X_1, \dots, X_m, T_1, \dots, T_l, Z]/(Z \cdot X_1^{e_1} \cdot \dots \cdot X_n^{e_n}, F(Z)) \rightarrow k$$

with $X_i \mapsto 0, T_i \mapsto t_i$ and $Z \mapsto \bar{h}$ (i.e., $\bar{Z} = \bar{h}$). Using our assumption, we may find $a_1, \dots, a_n \in \mathfrak{m}_A$ such that $\iota(\bar{Z}) \cdot \text{rv}(a_1^{e_1} \cdot \dots \cdot a_n^{e_n}) = \iota(\bar{h}) \cdot \text{rv}(a_1^{e_1} \cdot \dots \cdot a_n^{e_n}) = \text{rv}(\pi)$ in $\text{RV}(A)$. Lemma 5.2.5 implies the existence of an A -integral point of Y lifting y . Now apply Lemma 5.2.2 to the morphism $U_A \rightarrow Y_A$, induced from $U \rightarrow Y$ by base change. The morphism $U_A \rightarrow Y_A$ is étale at u_A (by base change) and we get an A -integral point of U lifting u . Finally, the latter induces an A -integral point of X lifting x . □

The assumption that one of the multiplicities not be p -divisible is necessary:

Example 5.2.7. Let $R = \mathbb{F}_p[[t]]$ and $A = \mathbb{F}_p[[t^{1/p}]]$. Consider

$$X = \text{Spec}(R[x]/((1+x) \cdot x^p - t))$$

Let P be the point at the origin of the special fiber, corresponding to the maximal ideal (x, t) . Although, $P \in X_s(\mathbb{F}_p)$ and $a = t^{1/p}$ has the property that $\text{rv}((1+a) \cdot a^p) = \text{rv}(t)$, one sees that X has no A -integral point lifting P . Indeed, this would yield a solution of $(1+x) \cdot x^p = t^p$ in $\mathbb{F}_p[[t]]$ with $v_t x > 0$. This would then imply that $1+x \in \mathbb{F}_p[[t^p]]$ and therefore $x \in \mathbb{F}_p[[t^p]]$. Setting $x = y^p$ and taking p -th roots, we get $(1+y) \cdot y^p = t$, which is impossible since the left hand side has p -divisible valuation.

5.3 Existential transfer for tamely ramified fields

5.3.1 Restating the main theorem

Theorem 5.3.2. *Assume Conjecture R. Suppose (K, v) and (L, w) are henselian and tamely ramified over a valued field (F, v_0) with \mathcal{O}_F an excellent DVR. If $RV(K) \equiv_{\exists, RV(F)} RV(L)$, then $K \equiv_{\exists, F} L$ in L_{rings} .*

For the valued fields (K, v) and (L, w) , we shall abbreviate the residue fields by k and l respectively, and the value groups by Γ and Δ respectively. We write π for a uniformizer of \mathcal{O}_F , $\kappa = \mathcal{O}_F/\pi\mathcal{O}_F$ for the residue field and $p = \text{char}(\kappa)$.

5.3.3 Geometric reformulation

We now reformulate our task in geometric terms. Using the disjunctive normal form and replacing conjunctions $\bigwedge_{i=1}^n f_i(X_1, \dots, X_m) \neq 0$ with a single inequation $\prod_{i=1}^n f_i(X_1, \dots, X_m) \neq 0$, every existential $L_{\text{rings}}(\mathcal{O}_F)$ -sentence is equivalent to a disjunction of sentences of the form

$$\phi = \exists X_1, \dots, X_m (f_1(X_1, \dots, X_m) = \dots = f_n(X_1, \dots, X_m) = 0 \text{ and } g(X_1, \dots, X_m) \neq 0)$$

for some $f_i(X_1, \dots, X_m), g(X_1, \dots, X_m) \in \mathcal{O}_F[X_1, \dots, X_m]$ and $n, m \in \mathbb{N}$. It is enough to focus on one such disjunct. We now define the affine \mathcal{O}_F -algebra

$$B = \mathcal{O}_F[X_1, \dots, X_m]/(f_1(X_1, \dots, X_m), \dots, f_n(X_1, \dots, X_m))$$

and let $X = \text{Spec}(B)$ be the associated affine scheme over $\text{Spec}(\mathcal{O}_F)$. Let $W \subseteq X_F$ be the (basic) Zariski open subset of the generic fiber, defined by the extra condition $g(X_1, \dots, X_m) \neq 0$. A witness of ϕ in \mathcal{O}_K (resp. \mathcal{O}_L) corresponds to an integral point $P \in X(\mathcal{O}_K)$ whose underlying K -rational point satisfies $P_K \in W(K)$.

From the above discussion and by symmetry, it suffices to prove the following:

Theorem 5.3.4. *Assume Conjecture R. Let (K, v) and (L, w) be valued field extensions of (F, v_0) with \mathcal{O}_F an excellent DVR. Suppose that $(K, v)/(F, v_0)$ is tamely ramified, (L, w) is henselian and that $RV(L) \models \text{Th}_{\exists, RV(F)} RV(K)$. Let X be a scheme of finite type over \mathcal{O}_F and $W \subseteq X_F$ be a Zariski open subset. Suppose there exists $P \in X(\mathcal{O}_K)$ with $P_K \in W(K)$. Then there also exists $Q \in X(\mathcal{O}_L)$ with $Q_L \in W(L)$.*

Let also $x \in X_s$ be the scheme-theoretic point (not necessarily closed) where P meets the special fiber. The proof of Theorem ?? is divided into two steps:

1. We treat the case where $X \rightarrow \text{Spec}\mathcal{O}_F$ has strict normal crossings at x and $W = X_F$.
2. We treat the general case using Conjecture R and an inductive argument on $\dim(X)$.

Strict normal crossings

First assume that $W = X_F$ and $X \rightarrow \text{Spec}\mathcal{O}_F$ has strict normal crossings. Since $X \rightarrow \text{Spec}\mathcal{O}_F$ has strict normal crossings at x , there exists a regular system of parameters $\{x_1, \dots, x_m\}$ in $\mathcal{O}_{X,x}$ such that $h \cdot x_1^{e_1} \cdot \dots \cdot x_n^{e_n} = \pi$, where π is a uniformizer of \mathcal{O}_F and $h \in \mathcal{O}_{X,x}^\times$.

The point P corresponds to a local \mathcal{O}_F -algebra homomorphism $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_K$ mapping $f \mapsto f(P)$. Taking valuations in the equation

$$h(P) \cdot x_1^{e_1}(P) \cdot \dots \cdot x_n^{e_n}(P) = \pi$$

and noting that $h(P)$ is a unit in \mathcal{O}_K , yields

$$\sum_{i=1}^n e_i \cdot vx_i(P) = v\pi$$

Since $\Gamma/\mathbb{Z}v\pi$ has no p -torsion elements, we get that $p \nmid e_i$, for some $i \in \{1, \dots, n\}$. Suppose $p \nmid e_1$, without loss of generality. Moreover, we have that $\kappa(x)/\kappa$ is separable because k/κ is separable and $x \in X_s(k)$. Now we may assume that L is $|K|^+$ -saturated as an L_{RV} -structure. Since $\text{RV}(L) \models \text{Th}_{\exists, \text{RV}(F)} \text{RV}(K)$, this furnishes us with an embedding $\rho : \text{RV}(\mathcal{O}_K) \hookrightarrow \text{RV}(\mathcal{O}_L)$ over $\text{RV}(\mathcal{O}_F)$ and a compatible embedding of residue fields $\bar{\rho} : k \rightarrow l$. In particular, we get $x \in X_s(l)$ and that there exist $a_1, \dots, a_n \in \mathfrak{m}_L$ such that $\iota(\bar{\rho}(\bar{h})) \cdot \text{rv}(a_1^{e_1} \cdot \dots \cdot a_n^{e_n}) = \text{rv}(\pi)$. Proposition 5.2.6 applies to give us an \mathcal{O}_L -integral point of X specializing to x .

Reduction to strict normal crossings

Our analysis of the general case follows the line of reasoning of the proof of Theorem 4.3 [DS03]. Let X be a scheme of finite type over \mathcal{O}_F and $W \subseteq X_F$ be Zariski open. We shall argue by induction on $\dim(X)$ that if there exists $P \in X(\mathcal{O}_K)$ with $P_K \in W(K)$, then there also exists $Q \in X(\mathcal{O}_L)$ with $Q_L \in W(L)$. The base case $\dim(X) = 0$ holds vacuously because there cannot exist $P \in X(\mathcal{O}_K)$ when $\dim(X) = 0$. We only need to explain how the inductive step works.

We shall argue that it is enough to assume that X is integral and affine. We may first assume that X is reduced by passing to its reduced underlying scheme. To see this, note that \mathcal{O}_K is reduced and therefore the integral point $P : \text{Spec}(\mathcal{O}_K) \rightarrow X$ factors (uniquely) as $\text{Spec}(\mathcal{O}_K) \xrightarrow{P'} X_{\text{red}} \rightarrow X$. We have an equality $|X| = |X_{\text{red}}|$ of underlying topological spaces and $P'(\eta_K) = P(\eta_K)$, where η_K is the generic point of $\text{Spec}(\mathcal{O}_K)$. In particular, we have $P'(\eta_K) \in W$. It would then suffice to prove the inductive step for X_{red} in place of X , so that we may assume $X = X_{\text{red}}$

to begin with. We may then assume that X is irreducible, replacing X with one of its irreducible components which contains the scheme-theoretic image of $P : \text{Spec}(\mathcal{O}_K) \rightarrow X$. Finally, replacing X with an affine neighborhood allows us to assume that $X = \text{Spec}(B)$, where B is an integral domain, which is a finitely generated \mathcal{O}_F -algebra with $\mathcal{O}_F \rightarrow B$ injective (here $\mathcal{O}_F \rightarrow B$ corresponds to the structure morphism $X \rightarrow \text{Spec}(\mathcal{O}_F)$).

It follows that B is a torsion-free \mathcal{O}_F -module and therefore flat by Tag 0539 [Stacks]. By Conjecture R, we will have a blow-up morphism $f : \tilde{X} \rightarrow X$ with $\tilde{X} \rightarrow \text{Spec}\mathcal{O}_F$ having strict normal crossings and f an isomorphism outside a nowhere-dense closed subscheme $Z \subsetneq X$, the center of f . If $P_K \in Z_K(K)$, then P is an integral point of Z and since $\dim(Z) < \dim(X)$, the conclusion follows from our induction hypothesis. Otherwise, we will have that $P_K \in X_F(K) - Z_F(K)$ and P_K lifts to $\tilde{P}_K : \text{Spec}(K) \rightarrow \tilde{X}$, using that $f_F : \tilde{X}_F \rightarrow X_F$ is an isomorphism outside $Z_F \subsetneq X_F$. By the valuative criterion of properness, the integral point P lifts to an \mathcal{O}_K -integral point of \tilde{X} as in the diagram below

$$\begin{array}{ccc} \text{Spec}K & \xrightarrow{\tilde{P}_K} & \tilde{X} \\ \downarrow & \nearrow \exists & \downarrow f \\ \text{Spec}\mathcal{O}_K & \xrightarrow{P} & X \end{array}$$

By the analysis of the strict normal crossings case, one also gets an \mathcal{O}_L -integral point of \tilde{X} . Now $W - Z_F$ is a Zariski dense open subset of X_F . To see this, note that $W, X - Z_F \subseteq X_F$ are non-empty and X_F is irreducible. By Theorem 2.4 [DS03], the \mathcal{O}_L -integral point of \tilde{X} can be chosen so that its underlying L -rational point is in $f_F^{-1}(W) - f_F^{-1}(Z_F)$, which is a Zariski dense open subset of $\tilde{X}_F - f_F^{-1}(Z_F)$, as f_F is an isomorphism outside Z_F and $W - Z_F$ is a Zariski dense open subset of X_F . This point induces an \mathcal{O}_L -integral point of $Q \in X(\mathcal{O}_L)$ with $Q_L \in W(L)$ via composition with f , which is what we wanted to show.

5.3.5 Some remarks on Theorem 5.3.2

Remark 5.3.6. By Fact 1.2.12, Conjecture R holds when the residue characteristic is 0 and the above proof becomes unconditional. In that case, it becomes automatically true that \mathcal{O}_F excellent (see Corollary 8.2.40(c) [Liu06]) and that the valued field extensions are tamely ramified (see Example 5.1.7(1)). Therefore, one recovers the *existential* version of Ax-Kochen/Ershov in residue characteristic 0. In [Den16], Denef manages to recover the full first-order Ax-Kochen/Ershov theorem in residue characteristic 0 using weak toroidalization of morphisms.

In the case of *finite* tame ramification in mixed characteristic and when the residue fields are perfect, the full first-order version of Theorem 5.3.2 has been proved *unconditionally* by J. Lee:

Remark 5.3.7 (see Corollary 5.9 [Lee20]). When $(K, v), (L, w)$ are henselian valued fields, finitely and tamely ramified over (\mathbb{Q}, v_p) with perfect residue fields, then J. Lee proves unconditionally that $\text{RV}(K) \equiv \text{RV}(L)$ implies $K \equiv L$ in L_{rings} .

Our proof did not use the assumption that F admits no defect extensions inside K (resp. L), which is part of the definition of a tamely ramified extension (see 5.1.2). The reason is that (F, v_0) is automatically defectless (see §5.1.8). If \mathcal{O}_F is not assumed to be excellent and K, L are only tamely ramified in the weak sense of Endler [End72], then the conclusion of Theorem 5.3.2 does not necessarily hold:

Example 5.3.8. Let R be the DVR that was introduced in Example 1.2.11, for which there exists $\alpha \in R$ such that $\alpha^{1/p} \in \widehat{R} - R^h$. Set $K = \text{Frac}(R^h)$, $L = \text{Frac}(\widehat{R})$ and $F = \text{Frac}(R)$. We have that $K \not\equiv_{\exists, F} L$, although $\text{RV}(K) \cong_{\text{RV}(F)} \text{RV}(L)$ (see Lemma 1.3.3).

5.4 Applications

5.4.1 Decidability

Motivation

In Remark 7.6 [AF16], the authors write:

"At present, we do not know of an example of a mixed characteristic henselian valued field (K, v) for which k and (Γ, v_p) are \exists -decidable but (K, v) is \exists -undecidable." The existence of such an example is demonstrated in Remark 3.6.9 [KK]. However, if we restrict ourselves to the tamely ramified setting and require that (K, v) admits a cross-section extending a cross-section of (\mathbb{Q}, v_p) , we indeed get such an Ax-Kochen/Ershov style statement in Corollary 5.4.6 (modulo Conjecture R). In fact, Corollary 5.4.6 is stated in a uniform fashion in all characteristics.

Existential Ax-Kochen/Ershov

For the sake of simplicity and concreteness, we take our base field (F, v_0) to be any of the valued fields $(\mathbb{Q}(t), v_t)$, (\mathbb{Q}, v_p) and $(\mathbb{F}_p(t), v_t)$. The associated valuation rings are indeed excellent DVRs (see Corollary 8.2.40 [Liu06]). The equal characteristic 0 and *unramified* mixed characteristic versions of Corollaries 5.4.2 and 5.4.6 are well-known. However, in this level of generality, the mixed characteristic and positive characteristic versions are new.

Corollary 5.4.2. *Assume Conjecture R. Suppose (K, v) and (L, w) are henselian valued fields tamely ramified over (\mathbb{Q}, v_p) (resp. $(\mathbb{F}_p(t), v_t)$ or $(\mathbb{Q}(t), v_t)$). Suppose (K, v) and (L, w) admit cross-sections that restrict to the same cross-section of (\mathbb{Q}, v_p) (resp. $(\mathbb{F}_p(t), v_t)$ or $(\mathbb{Q}(t), v_t)$). If $k \equiv_{\exists} l$ in L_{rings} and $(\Gamma, vp) \equiv_{\exists} (\Delta, wp)$ in L_{oag} (resp. $(\Gamma, vt) \equiv_{\exists} (\Delta, wt)$), then $K \equiv_{\exists} L$ in L_{rings} (resp. L_t).*

Proof. We focus on the mixed characteristic case—the proof applies verbatim to the other two cases. Let (L^*, w^*, s_L^*) be a $|K|^+$ -saturated elementary extension of (L, w, s_L) in the language L_{AKE} (see notation). In particular, the residue field l^* and the value group Δ^* are themselves $|k|^+$ -saturated and $|\Gamma|^+$ -saturated respectively. Since $l \models \text{Th}_{\exists} k$ and $(\Delta, wp) \models \text{Th}_{\exists}(\Gamma, vp)$, we get a field embedding $\rho : k \hookrightarrow l^*$ and an embedding of pointed ordered abelian groups $\sigma : (\Gamma, vp) \hookrightarrow (\Delta^*, w^*p)$.

The structure $(\text{RV}(K), \oplus, \cdot, \leq, 1, \infty)$ (resp. $(\text{RV}(L^*), \oplus, \cdot, \leq, 1, \infty)$) described in 1.3.1 is isomorphic to the structure $((k^{\times} \times \Gamma) \cup \{\infty\}, \oplus, \cdot, \leq, 1, \infty)$ (resp. $((l^{*\times} \times \Delta^*) \cup \{\infty\}, \oplus, \cdot, \leq, 1, \infty)$) via the identification described in 1.3.1. Moreover, these identifications are compatible with the identification of $(\text{RV}(\mathbb{Q}), \oplus, \cdot, \leq, 1, \infty)$ with $((\mathbb{F}_p^{\times} \times \mathbb{Z}) \cup \{\infty\}, \oplus, \cdot, \leq, 1, \infty)$, as the cross-sections of K and L^* extend the one of \mathbb{Q} . The maps ρ and σ combine to give us an embedding of RV-structures $\text{RV}(K) \hookrightarrow \text{RV}(L^*)$ over $\text{RV}(\mathbb{Q})$. Reversing the roles of K and L , we deduce that $\text{RV}(K) \equiv_{\exists, \text{RV}(\mathbb{Q})} \text{RV}(L)$ and the conclusion follows from Theorem 5.3.2. \square

The cross-section condition of Corollary 5.4.2 cannot be omitted, as the following example shows:

Example 5.4.3. Take $p \neq 2$ such that $2 \notin (\mathbb{F}_p^{\times})^2$ (e.g. $p = 3$). Let $(K, v) = (\mathbb{Q}_p(p^{1/2}), v_p)$ and $(L, w) = (\mathbb{Q}_p((2p)^{1/2}), v_p)$. It is clear that $k = l = \mathbb{F}_p$ and $(\Gamma, vp) = (\Delta, wp) \cong (\mathbb{Z}, 2)$. On the other hand, we have that $K \not\equiv_{\exists} L$ in L_{rings} . Indeed, if $p^{1/2} \in L$ this would imply that $2 \in (\mathbb{F}_p^{\times})^2$.

This should be contrasted with the *unramified* case, where the cross-section condition may be dropped. The equal characteristic 0 and mixed characteristic parts of Remark 5.4.4 are well-known. The (conditional) part (b) of Remark 5.4.4 can also be deduced from the method of Denef-Schoutens [DS03], although it is not explicitly mentioned in their paper:

Remark 5.4.4. (a) When (K, v) is *unramified*, meaning that vp (resp. vt) is the smallest positive element of Γ , there is always an elementary extension admitting a cross-section which extends the standard cross-section $n \mapsto p^n$ (resp. $n \mapsto t^n$) of (\mathbb{Q}, v_p) (resp. $(\mathbb{F}_p(t), v_t)$ or $(\mathbb{Q}(t), v_t)$). This follows from Proposition 5.4 [Dri12].

In particular, one can drop the cross-section condition in Corollary 5.4.2 in the unramified setting.

(b) Assuming Conjecture R, we deduce that if (K, v) is a henselian valued field extending $(\mathbb{F}_p(t), v_t)$ with $k = \mathbb{F}_p$ and $(\Gamma, vt) \equiv_{\exists} (\mathbb{Z}, 1)$ in L_{oag} with a constant for the value of t , then $K \equiv_{\exists} \mathbb{F}_p((t))$ in L_t . It is worth noting that the condition $(\Gamma, vt) \equiv_{\exists} (\mathbb{Z}, 1)$ is simply equivalent to asking that vt be minimal positive in Γ (cf. Corollary 1.6 [Wei90]).

Remark 5.4.5. If we do not ask for (K, v) and (L, w) to be tameily ramified over (\mathbb{Q}, v_p) (resp. $(\mathbb{F}_p(t), v_t)$), the conclusion of Corollary 5.4.2 may fail (see Example 5.4.12).

Corollary 5.4.6. *Assume Conjecture R. Suppose (K, v) is henselian and tameily ramified over (\mathbb{Q}, v_p) (resp. $(\mathbb{F}_p(t), v_t)$ or $(\mathbb{Q}(t), v_t)$). Suppose (K, v) admits a cross-section that extends a cross-section of (\mathbb{Q}, v_p) (resp. $(\mathbb{F}_p(t), v_t)$ or $(\mathbb{Q}(t), v_t)$). Then K is existentially decidable in L_{rings} relative to k in L_{rings} and (Γ, vp) in L_{oag} with a constant for the value of p (resp. t).*

Proof. We again focus on the mixed characteristic version. Assume k (resp. Γ) is existentially decidable in L_{rings} (resp. L_{oag} with a constant for vp). Consider the L_{AKE} -theory

$$T = \text{Hen} \cup T_{\text{res}} \cup T_{\text{vg}} \cup \text{Diag}_K(\mathbb{Q})$$

where Hen is the usual axiom-schema capturing Hensel's Lemma and

$$T_{\text{res}} = \{\phi \in L_{\text{rings}} : \phi \text{ existential and } k \models \phi\} \cup \{\phi \in L_{\text{rings}} : \phi \text{ universal and } k \models \phi\}$$

and

$$T_{\text{vg}} = \{\phi \in L_{\text{oag}, vp} : \phi \text{ existential and } \Gamma \models \phi\} \cup \{\phi \in L_{\text{oag}, vp} : \phi \text{ universal and } \Gamma \models \phi\}$$

and $\text{Diag}_K(\mathbb{Q})$ is the atomic diagram of \mathbb{Q} in K in L_{AKE} . By our assumptions on k and Γ , we have that the above axiomatization is recursive.

Claim: For every existential or universal sentence $\phi \in L_{\text{rings}}$, we have $T \models \phi \iff K \models \phi$.

Proof. If $T \models \phi$, then clearly $K \models \phi$. For the converse, suppose that $K \models \phi$ and let $(L, w, s_L) \models T$ with residue field l , value group Δ and cross-section $s_L : \Delta \rightarrow L^\times$. Note that $\Delta/\mathbb{Z}wp$ has no p -torsion elements, using that $(\Delta, wp) \equiv_{\exists} (\Gamma, vp)$ and that $\Gamma/\mathbb{Z}vp$ has no p -torsion elements. Since (\mathbb{Q}, v_p) is defectless and l/\mathbb{F}_p is separable, we deduce that $(L, w)/(\mathbb{Q}, v_p)$ is tameily ramified. Since $L \models \text{Diag}_K(\mathbb{Q})$, we have that s_L and s_K restrict to the same cross-section of (\mathbb{Q}, v_p) . By Corollary 5.4.2, we see that $L \models \phi$. \square *Claim*

In particular, the theory T is *existentially complete* with respect to L_{rings} , meaning that for every existential sentence $\phi \in L_{\text{rings}}$ either $T \models \phi$ or $T \models \neg\phi$. A brute-force enumeration of all proofs from the axioms of T now yields an effective procedure for deciding whether $K \models \phi$, for any existential sentence $\phi \in L_{\text{rings}}$. \square

Remark 5.4.7. Even if (K, v) is algebraic over (\mathbb{Q}, v_p) , the model (L, w) constructed in the proof of Corollary 5.4.6 will generally be highly transcendental, the residue field l will also be highly transcendental and Δ will be of rank greater than 1. It is therefore essential—even if one is merely interested in algebraic extensions—that we have proved Theorem 5.3.2 in this level of generality.

Once again, the cross-section condition cannot be omitted from Corollary 5.4.6. We provide a counter-example in equal characteristic 0:

Example 5.4.8. For each $\alpha \in 2^\omega$, we define an equal characteristic 0 valued field K_α as follows. Set $(K_{\alpha(0)}, v_t) = (\mathbb{Q}_2((t)), v_t)$ and $\pi_{\alpha(0)} = t$. We now define inductively:

- (1) $K_{\alpha(n)} = K_{\alpha(n-1)}(\pi_{\alpha(n-1)}^{1/2})$ and $\pi_{\alpha(n)} = \pi_{\alpha(n-1)}^{1/2}$ if $\alpha(n) = 0$ or
- (2) $K_{\alpha(n)} = K_{\alpha(n-1)}((2 \cdot \pi_{\alpha(n-1)})^{1/2})$ and $\pi_{\alpha(n)} = (2 \cdot \pi_{\alpha(n-1)})^{1/2}$ if $\alpha(n) = 1$.

More succinctly, for $n \in \mathbb{N}$ we have that $K_{\alpha(n)} = \mathbb{Q}_2((t))((2^{\bar{\alpha}_n} \cdot t)^{1/2^n})$, where $\bar{\alpha}_n = \sum_{k=0}^{n-1} \alpha(k) \cdot 2^k$. We let $K_\alpha = \bigcup_{n \in \mathbb{N}} K_{\alpha(n)}$. For every $\alpha \in 2^\omega$, we have that (K_α, v_t) is henselian, being an algebraic extension of $(\mathbb{Q}_2((t)), v_t)$. It has value group $(\Gamma_\alpha, vp) = (\frac{1}{2^\infty}\mathbb{Z}, 1)$ and residue field $k_\alpha = \mathbb{Q}_2$, both of which are decidable. Indeed, the former is decidable using results by Robinson-Zakon [RZ60] and the latter is decidable by Ax-Kochen/Ershov.¹

If $\alpha \neq \beta$, then $K_\alpha \not\equiv_{\exists} K_\beta$ in L_t . Indeed, suppose that $K_\alpha \equiv_{\exists} K_\beta$ and let $n \in \omega$ be least such that $\alpha(n) \neq \beta(n)$ and say $\alpha(n) = 1$. We would then have $a, b \in K_\alpha$ such that $a^{2^n} = 2^{\bar{\alpha}_n} \cdot t$ and $b^{2^n} = 2^{\bar{\beta}_n} \cdot t$. It follows that $c^{2^n} = 2^{2^{n-1}}$, where $c = \frac{a}{b}$. Reducing this equation over the residue field $k_\alpha = \mathbb{Q}_2$, we get $\bar{c}^{2^n} = 2^{2^{n-1}} \Rightarrow 2 \cdot v_2 \bar{c} = 1$, which has no solution in the value group \mathbb{Z} of \mathbb{Q}_2 . We conclude that $K_\alpha \not\equiv_{\exists} K_\beta$ in L_t . Since 2^ω is uncountable and there are countably many Turing machines, there must exist an $\alpha \in 2^\omega$ such that K_α is \exists -undecidable in L_t .

¹The former needs some explanation: Let T be the theory in L_{oag} that requires of Γ that it is regularly dense, that $[\Gamma : 2\Gamma] = 1$ and $[\Gamma : p\Gamma] = p$ for any prime $p > 2$. By Theorem 4.4 [RZ60] (see also the proof), T is model-complete in $L_{\text{oag}} \cup \{P_n : n \in \mathbb{N}\}$ (where $P_n(x) \leftrightarrow \exists y(ny = x)$) and also complete. Consider the expansion $L_{\text{oag}} \cup \{P_n : n \in \mathbb{N}\} \cup \{1\}$ and the theory T' , which in addition asks that $1 > 0$ and that $p \nmid 1$ if $p > 2$. It is clear that T' is still model-complete and that $(\frac{1}{2^\infty}\mathbb{Z}, 1)$ is a prime model of T' . The decidability of $(\frac{1}{2^\infty}\mathbb{Z}, 1)$ in L_{oag} follows.

Remark 5.4.9. The algorithm provided by the proof of Corollary 5.4.6 is effective only in theory, meaning that it is very difficult to implement in practice. A similar remark applies to the proof of Denef-Schoutens [DS03] (see Remark 5.5.7(b)). In §5.5, we will present an alternative (conditional) proof of the existential decidability of $\mathbb{F}_p((t))$, which becomes effective once an effective desingularization algorithm in positive characteristic becomes available (see Remark 5.5.7(b)).

Proof of Corollary 5.0.2

Among the fields that are existentially decidable, the maximal tamely ramified extensions of \mathbb{Q}_p and $\mathbb{F}_p((t))$ occur naturally in ramification theory and are of arithmetic significance.

Corollary 5.4.10 (Ramification fields). *Assume Conjecture R. Then the field \mathbb{Q}_p^{tr} (resp. $\mathbb{F}_p((t))^{tr}$) is existentially decidable in L_{rings} (resp. L_t).*

Proof. From Fact 5.1.4, one sees that both of these fields have residue field $\overline{\mathbb{F}}_p$ and value group $\mathbb{Z}_{(p)}$. The field $\overline{\mathbb{F}}_p$ is decidable in L_{rings} and $(\mathbb{Z}_{(p)}, 1)$ is decidable in L_{oag} with a constant symbol for 1. The latter is an application of Robinson-Zakon [RZ60] (cf. footnote 1). Moreover, the field \mathbb{Q}_p^{tr} (resp. $\mathbb{F}_p((t))^{tr}$) admits a cross-section mapping $\gamma \mapsto p^\gamma$ (resp. $\gamma \mapsto t^\gamma$). The conclusion follows from Corollary 5.4.6. \square

Other examples include $\mathbb{Q}_p(p^{1/\ell^\infty})$ and $\mathbb{F}_p((t))(t^{1/\ell^\infty})$, where ℓ is a prime different from p . These are again existentially decidable in L_{rings} and L_t respectively.

5.4.11 Taming Abhyankar's example

For an application of a different kind, we present a tame variant of the following famous example due to Abhyankar [Abh56]. It is also presented by Kuhlmann in Example 3.13 [Kuh11] in relation to the defect:

Example 5.4.12. Let $(K, v) = (\mathbb{F}_p((t))^{1/p^\infty}, v_t)$ and $(L, w) = (\mathbb{F}_p((t^{\Gamma_p})), v_t)$ be the Hahn series field with value group $\Gamma_p = \frac{1}{p^\infty}\mathbb{Z}$ and residue field \mathbb{F}_p . We observe that $\text{RV}(K) \cong_{\text{RV}(\mathbb{F}_p((t)))} \text{RV}(L)$ but $(K, v) \not\cong_{\exists, \mathbb{F}_p((t))} (L, w)$ since the Artin-Schreier equation $x^p - x - 1/t = 0$ has a solution in L but not in K . Note that both K, L admit a cross-section which sends $\gamma \mapsto t^\gamma$. This example therefore demonstrates why Corollary 5.4.2 is not true without the tameness assumption.

Our version of Abhyankar's example is obtained by replacing p -power roots of t with ℓ -power roots and exhibits a totally different behaviour (as expected):

Example 5.4.13. Fix any prime $\ell \neq p$. Consider the valued fields $(K, v) = (\mathbb{F}_p((t))(t^{1/\ell^\infty}), v_t)$ and $(L, w) = (\mathbb{F}_p((t^{\Gamma_\ell})), v_t)$, with the latter being the Hahn series field with value group $\Gamma_\ell = \frac{1}{\ell^\infty}\mathbb{Z}$ and residue field \mathbb{F}_p . We observe that $\text{RV}(K) \cong \text{RV}(L)$ and by Theorem 5.3.2 we get that $(K, v) \equiv_{\exists, \mathbb{F}_p((t^{1/\ell^n}))} (L, w)$, for all $n \in \mathbb{N}$. It follows that $K \preceq_1 L$ in L_{rings} .

Remark 5.4.14. Similarly, we have that $\mathbb{Q}_p(p^{1/\ell^\infty})$ is existentially closed in every maximal immediate extension.

5.5 Revisiting Denef-Schoutens

5.5.1 Overview of Denef-Schoutens

In Theorem 4.3 [DS03], Denef-Schoutens proved the existential decidability of $\mathbb{F}_p((t))$ in L_t , assuming resolution of singularities for schemes over fields (Conjecture 1 [DS03]). We state Conjecture 1 [DS03] below for the convenience of the reader:

Conjecture 1. *Let X be a reduced scheme of finite type over a field k . Then there exists a blow-up morphism $f : \tilde{X} \rightarrow X$ in a nowhere dense center $Z \subsetneq X$ such that \tilde{X} is regular.*

Using Conjecture 1, Denef-Schoutens reduce the problem of existential decidability of $\mathbb{F}_p((t))$ in L_t to the problem of deciding whether a given scheme Y of finite type over $\mathbb{F}_p[[t]]$ has an $\mathbb{F}_p[[t]]$ -integral point (see the proof of Theorem 4.3 [DS03]). They solve the latter using an *effective* Greenberg approximation theorem (Theorems 3.2, 6.1 [BDLD79]). In particular, they prove *unconditionally* that the *positive* existential theory of $\mathbb{F}_p[[t]]$ is decidable in L_t (Proposition 3.5 [DS03]).

5.5.2 Comparison with Denef-Schoutens

Assuming Conjecture R, our Theorem 5.3.2 also implies the existential decidability of $\mathbb{F}_p((t))$ in L_t (see Corollary 5.4.6). Moreover, by Remark 5.4.4(b) we have a simple system of axioms which captures the existential theory of $\mathbb{F}_p((t))$ in L_t . Note that our proof does not make use of Greenberg's approximation theorem. On the other hand, Conjecture R is more refined than Conjecture 1. We shall now provide a simplified proof in the case of $\mathbb{F}_p((t))$, which only relies on Conjecture 1.

5.5.3 Simplified proof for $\mathbb{F}_p((t))$

We refer to Appendix ?? for background material related to computational algebraic geometry. Let X_0 be a *given* affine scheme of finite type over $\mathbb{F}_p[t]$ and W a given Zariski open subset of the generic fiber $(X_0)_{\mathbb{F}_p(t)}$. Our task is to decide whether there exists $P \in X_0(\mathbb{F}_p[[t]])$ such that $P_\eta \in W(\mathbb{F}_p((t)))$, where P_η is the underlying $\mathbb{F}_p((t))$ -rational point. We note that X and W may be viewed as algorithmic inputs in the sense of 1.2.15.

We shall also write $X = X_0 \times_{\text{Spec}(\mathbb{F}_p[t])} \text{Spec}(\mathbb{F}_p[[t]])$ for the base change via $\text{Spec}(\mathbb{F}_p[[t]]) \rightarrow \text{Spec}(\mathbb{F}_p[t])$ and X_s for the special fiber.

Non-singular case

Our proof relies on the observation that one can avoid using the effective Greenberg approximation theorem to check if X has an $\mathbb{F}_p[[t]]$ -integral point, in case X is *regular* at all points $x \in X_s$. Instead, one can use a more elementary fact. Since we were not able to find a reference for Proposition 5.5.4 in published literature, we shall spell out the details. We do mention however that Proposition 5.5.4 follows easily from Proposition 2, pg.61 [BLR90].

Proposition 5.5.4. Let R be a DVR with residue field κ and $f : X \rightarrow \text{Spec}R$ be a morphism of finite type. Let P be an R -integral point of X , meeting the special fiber at $x \in X_s(\kappa)$ and suppose that X is regular at x . Then f is smooth at x .

Proof. By the definition of a smooth morphism (Definition 4.3.35 [Liu06]), it suffices to show that $f : X \rightarrow \text{Spec}R$ is flat at x and that X_s is smooth at x , as an algebraic variety over κ . To this end, we may assume that X is the local scheme $\text{Spec}(A)$, where A is a regular local ring (by replacing X with $\text{Spec}(\mathcal{O}_{X,x})$).

Claim 1: The morphism f is flat at x .

Proof. Since A is a regular local ring, it is also an integral domain (Proposition 4.2.11 [Liu06]). Since X admits an R -integral point, we get that $f : R \rightarrow A$ is injective. We conclude that A is a torsion-free R -module and therefore flat (see Tag 0539 [Stacks]). □*Claim 1*

An integral point $P : \text{Spec}R \rightarrow X$ corresponds to a section of $f : X \rightarrow \text{Spec}R$. The maps f and P induce ring homomorphisms $f^* : R/\mathfrak{m}_R \rightarrow A/\mathfrak{m}_A$ and $P^* : A/\mathfrak{m}_A \rightarrow R/\mathfrak{m}_R$ such that $P^* \circ f^* = id$. The latter condition means that P^* is surjective. On the other hand, a ring homomorphism between fields is always injective, whence $P^* : A/\mathfrak{m}_A \xrightarrow{\cong} R/\mathfrak{m}_R = \kappa$ is an isomorphism. Since R is a DVR,

we may consider $\mathfrak{m}_R/\mathfrak{m}_R^2$ as a 1-dimensional κ -vector subspace of $\mathfrak{m}_A/\mathfrak{m}_A^2$, spanned by $\pi + \mathfrak{m}_A^2$, where π is a uniformizer of R . One may extend $\{\pi + \mathfrak{m}_A^2\}$ to a basis of $\mathfrak{m}_A/\mathfrak{m}_A^2$, say $\{\pi + \mathfrak{m}_A^2, x_1 + \mathfrak{m}_A^2, \dots, x_n + \mathfrak{m}_A^2\}$ with $x_i \in \mathfrak{m}_A$.

Claim 2: The scheme X_s is smooth at x .

Proof. Let $B = A \otimes_R R/\mathfrak{m}_R \cong A/\pi A$ be the local ring of the special fiber at x . We will then have that $\{x_1 + \mathfrak{m}_B^2, \dots, x_n + \mathfrak{m}_B^2\}$ is a κ -basis for $\mathfrak{m}_B/\mathfrak{m}_B^2$. By flatness, we get that $\dim(B) = \dim(A) - 1$ (Theorem 4.3.12 [Liu06]) and therefore that the local ring B is regular, i.e. X_s is regular at x . Since $\kappa(x) = \kappa$, we get that X_s is smooth at x (Proposition 4.3.30 [Liu06]). $\square_{\text{Claim 2}}$

\square

We refer to Definition 1.2.16, which allows us to view affine schemes of finite type (over a computable base ring) as algorithmic inputs.

Corollary 5.5.5. *There exists an algorithm to decide whether a given regular affine scheme X_0 of finite type over $\mathbb{F}_p[t]$ has an $\mathbb{F}_p[[t]]$ -integral point.*

Proof. We keep our notation from §5.5.3; $X = X_0 \times_{\text{Spec}(\mathbb{F}_p[t])} \text{Spec}(\mathbb{F}_p[[t]])$ and X_s is the special fiber of X_0 , which is also identified with the special fiber of X . Say $X_0 = \text{Spec}(A)$ where $A = \mathbb{F}_p[t, x_1, \dots, x_m]/(f_1, \dots, f_s)$ with $f_i \in \mathbb{F}_p[t, x_1, \dots, x_m]$. Any $\mathbb{F}_p[[t]]$ -integral point of X_0 corresponds to an $\mathbb{F}_p[[t]]$ -integral point of one of its irreducible components. Since we may effectively compute the irreducible components of X_0 (see Fact 1.2.17(a)), this allows us to further assume that X_0 is irreducible. Since X_0 is already reduced (being regular), we have that X_0 is integral.

For any $x \in X_s$, we have that X_0 is regular at x if and only if X is regular at x (see Tag 0BG6(2) [Stacks]). Together with our assumption, this gives that X is regular at all points $x \in X_s$. Now X_s has finitely many \mathbb{F}_p -rational points, the set of which is computable by brute-force. It is therefore enough to check whether a given $x \in X_s(\mathbb{F}_p)$ lifts to an $\mathbb{F}_p[[t]]$ -integral point. By Proposition 5.5.4 and Hensel's Lemma (Corollary 6.2.13 [Liu06]), we equivalently need to check if $X \rightarrow \text{Spec}(\mathbb{F}_p[[t]])$ is smooth at x .

Claim 1: We can effectively check if $X \rightarrow \text{Spec}(\mathbb{F}_p[[t]])$ is flat at x .

Proof. Equivalently, we need to see if t is a zero-divisor in A . Since A is an integral domain, we need to check if $t = 0$ in A (i.e., if $X_0 = X_s$). Equivalently, we need to check if the generic fiber $(X_0)_{\mathbb{F}_p(t)} = \text{Spec}(\mathbb{F}_p(t)[x_1, \dots, x_m]/(f_1, \dots, f_s))$ is non-empty. Equivalently, we need to see if $(f_1, \dots, f_s) = (1)$ in $\mathbb{F}_p(t)[x_1, \dots, x_m]/(f_1, \dots, f_s)$, which can be checked effectively by Fact 1.2.20. $\square_{\text{Claim 1}}$

Note that $X_s = \text{Spec}(\mathbb{F}_p[x_1, \dots, x_m]/(\bar{f}_1, \dots, \bar{f}_s))$, where $\bar{f}_i(x_1, \dots, x_m) = f_i(0, x_1, \dots, x_m)$. Finally, we have the following:

Claim 2: We can effectively check whether x is a smooth \mathbb{F}_p -rational point of X_s .

Proof. This can be done by using the Jacobian criterion for smoothness (Theorem 4.2.19 [Liu06]). More precisely, one needs to check if $\text{rank}(J_x) = m - \dim_x X_s$, where $J_x := (\frac{\partial \bar{f}_i}{\partial x_j}(x))_{1 \leq i \leq s, 1 \leq j \leq m}$. To this end, we note that:

1. The number $\text{rank}(J_x)$ is computable by Gauss elimination.
2. The number $\dim_x X_s$ is also computable. One needs to compute the irreducible decomposition $X_s = \bigcup_{i=1}^n X_i$ using Fact 1.2.17(a) and calculate $\max_{x \in X_i} \dim X_i$ using Fact 1.2.17(b).

□ *Claim 2*

Claim 2 finishes the proof. □

General case

We now sketch the case where X_0 is a general affine scheme over $\mathbb{F}_p[t]$. This follows again by induction on $\dim(X_0)$. If $\dim(X_0) = 0$, then X_0 cannot possibly have any $\mathbb{F}_p[[t]]$ -integral points. For the inductive step, we first view X_0 as a scheme of finite type over \mathbb{F}_p via $X_0 \rightarrow \text{Spec}(\mathbb{F}_p[t]) \rightarrow \text{Spec}(\mathbb{F}_p)$. As in the proof of Corollary 5.5.5, we may assume that X_0 is reduced, by computing *effectively* $(X_0)_{\text{red}}$ (see Fact 1.2.17(a) and Remark 1.2.19) and replacing X_0 with $(X_0)_{\text{red}}$. We may therefore take X_0 to be reduced and W a *non-empty* Zariski open subset of the generic fiber $(X_0)_{\mathbb{F}_p(t)}$. We note that the case $W = \emptyset$ can again be checked effectively: If $W^c = V(J)$ and $(X_0)_{\mathbb{F}_p(t)} = V(I)$, it suffices to calculate \sqrt{J} (Fact 1.2.17(a)) and check if $I = J$ (see Fact 1.2.20). In that case, the output of our algorithm is that there does not exist $P \in X_0(\mathbb{F}_p[[t]])$ with $P_\eta \in W(\mathbb{F}_p((t)))$.

Now Conjecture 1 provides us with a blow-up morphism $Y_0 \rightarrow X_0$ in a nowhere dense center $Z \subsetneq X_0$ with Y_0 regular. The center of such a blow-up can be calculated by brute-force, as explained in Remark 4.1 [DS03]. The affine charts of Y_0 can be computed explicitly from the blow-up ideal. Using Corollary 5.5.5 for each affine piece separately, one may then check if $Y = Y_0 \times_{\text{Spec}(\mathbb{F}_p[t])} \text{Spec}(\mathbb{F}_p[[t]])$ has an $\mathbb{F}_p[[t]]$ -integral point. Moreover, if Y has an $\mathbb{F}_p[[t]]$ -integral point, its underlying rational point is also a regular point of Y , using that the regular locus $\text{Reg}(Y)$ is open in Y . Arguing as in the proof of Theorem 4.3 [DS03], we see that there are two scenarios:

1. Either $Y(\mathbb{F}_p[[t]]) \neq \emptyset$, which implies the existence of $P \in X(\mathbb{F}_p[[t]])$ with $P_\eta \in W(\mathbb{F}_p((t)))$ (using Theorem 2.4 [DS03]) or
2. The problem is reduced to the lower-dimensional scheme Z , which can be solved by our induction hypothesis.

Remark 5.5.6. (a) In Theorem 4.3 [DS03], Denef-Schoutens desingularize the *generic* fiber of X . As a consequence, the scheme Y produced in the proof of Theorem 4.3 [DS03] has regular generic fiber but Y need not be regular at points $y \in Y_s$. On the other hand, by desingularizing X_0 , our scheme Y_0 (and hence Y) is regular at all $y \in Y_s$. This simplifies the task of deciding whether $Y(\mathbb{F}_p[[t]]) = \emptyset$, because of Corollary 5.5.5.

(b) The state of the art in desingularization theory allows the present method to work unconditionally for $\dim(X_0) = 3$, whereas the method of [DS03] works for $\dim(X_0) = 4$. This is an unfortunate consequence of desingularizing X_0 instead of the generic fiber of X .

Remark 5.5.7. (a) One defect of this method, besides the one mentioned in Remark 5.5.6(b), is that it does not provide us with an *unconditional* proof of the decidability of the *positive* existential theory of $\mathbb{F}_p[[t]]$ in L_t .

(b) One advantage of this method is that once an *effective* desingularization algorithm is known in characteristic p , the above proof can be converted into an actually effective (not merely theoretically terminating) algorithm. On the other hand, the effective Greenberg approximation theorem used by Denef-Schoutens (see Theorem 3.1, Remark 3.3 [DS03]) is only effective in theory, as it ultimately relies on the brute-force algorithm explained in the proof of Theorem 6.1 [BDLD79].

5.6 Local uniformization

The result of this section was stated and sketched in the first version of [Kar21b]. We explain how local uniformization—a local form of resolution of singularities—suffices in order to prove the existential decidability of $\mathbb{F}_p((t))$ in L_t . Using local uniformization, we will show the following *existential* Ax-Kochen/Ershov principle, from which the existential decidability of $\mathbb{F}_p((t))$ in L_t follows easily:

Proposition 5.6.1. Assume local uniformization in positive characteristic. Let (K, v) and (L, w) be henselian valued fields over $(\mathbb{F}_p(t), v_t)$ with residue field \mathbb{F}_p . Suppose further that vt (resp. wt) is minimal positive in Γ (resp. Δ). Then $K \equiv_{\exists, \mathbb{F}_p(t)} L$ in L_{rings} .

Similarly to the decidability results obtained in §5.4.1, the resulting algorithm is highly impractical. For this reason, §5.5 is still relevant.

5.6.2 Background

For a concise overview on the problem of local uniformization, see §1 [Tem08b]. More detailed accounts may be found in [Spi20] and [Kuh00]. Kuhlmann's survey [Kuh00] also draws connections with the model theory of henselian fields.

Center of a valuation

To give a precise form of local uniformization, we need the notion of *center* of a valuation on an integral scheme over a field:

Definition 5.6.3. Let X be an integral scheme over k . Consider the function field $k(X)$ and let $V \subseteq k(X)$ be a valuation ring. The set $\{x \in X : V \subseteq \mathcal{O}_{X,x} \text{ and } \mathfrak{m}_V \cap \mathcal{O}_{X,x} = \mathfrak{m}_{X,x}\}$ is called the center of the valuation at X .

The center of V is a closed irreducible subscheme of X , i.e., a scheme-theoretic point of X , but may well be empty. For instance, it is empty when $X = \text{Spec}(k[t])$ and $V \subseteq k(x)$ is the valuation corresponding to the point at infinity (i.e., $V = k[y]_{(y)}$ with $y = x^{-1}$). When the center of V at X is non-empty, we say that V is *centered* at X . Note that in the case $X = \text{Spec}(A)$, this happens precisely when $A \subseteq V$. In that case, the center is the closed irreducible subscheme of X corresponding to the prime ideal $\mathfrak{q} = A \cap \mathfrak{m}_V$ of A .

Formulation

We now define the notion of *local uniformization* along a valuation and subsequently state the precise form of local uniformization. Our exposition follows §3.5 [Spi20], where more historical details can be found.

Definition 5.6.4. Let X be an integral scheme over a field k and $V \subseteq k(X)$ be a valuation ring centered at X . A local uniformization of X along V is a proper birational morphism $f : Y \rightarrow X$ such that V is centered at a non-singular point of Y .

Note that a resolution of singularities $f : Y \rightarrow X$, i.e., a proper birational morphism with Y regular, readily gives a local uniformization of X along *any* valuation ring V centered at X . We are now in a position to formulate the conjecture:

Conjecture 5.6.5 (Local Uniformization). Let X be an integral scheme of finite type over a field k and $V \subseteq k(X)$ be a valuation ring centered at X . Then there exists a local uniformization of X along V .

Zariski [Zar40] has established local uniformization over fields of characteristic 0. As with resolution of singularities, the problem of local uniformization in positive characteristic is wide open. Temkin's inseparable local uniformization [Tem08b] achieves local uniformization up to an *inseparable* alteration. This means that the map f is not necessarily birational but there is still some control over the induced function field extension, namely $k(Y)/k(X)$ is finite and purely inseparable. Interestingly enough, there is also a somewhat orthogonal result, due to Kuhlmann-Knaf [KK09], which achieves local uniformization up to a *separable* alteration.

5.6.6 $\mathbb{F}_p((t))$ and local uniformization

A valuative warm-up

It will be useful to analyze what local uniformization says in terms of V -integral points, where V is a valuation ring. Fix an integral scheme X over a field k , a valuation ring V with $\text{Frac}(V) = K$ and suppose that we are given a *dominant* morphism $P : \text{Spec}V \rightarrow X$. In particular, since P is dominant, one sees that $k(X) \subseteq K$ and V induces a valuation on $k(X)$ with valuation ring $V_1 = k(X) \cap V$. Assuming local uniformization of X along V_1 and using the valuative criterion of properness, we get a proper birational morphism $f : Y \rightarrow X$ and a lift $\tilde{P} : \text{Spec}V \rightarrow Y$ of P as in the diagram below

$$\begin{array}{ccc} \text{Spec}K & \xrightarrow{\tilde{P}_K} & \tilde{X} \\ \downarrow & \nearrow \exists & \downarrow f \\ \text{Spec}\mathcal{O}_K & \xrightarrow{P} & X \end{array}$$

such that $\tilde{P}(\mathfrak{m}_V)$ is non-singular, being the center of V_1 at Y .

Proof of Proposition 5.6.1

We now explain the proof of Proposition 5.6.1, from which the existential decidability of $\mathbb{F}_p((t))$ in L_t (modulo local uniformization) follows routinely. We first prove an easy generalization of Proposition 5.5.4:

Proposition 5.6.7. Let R be a DVR with uniformizer π , residue field κ and $f : X \rightarrow \text{Spec}R$ be a morphism of finite type. Let also $R \subseteq S$ be a local ring extending R such that $\mathfrak{m}_S = \pi S$ and P be an S -integral point of X , meeting the special fiber at a closed point $x \in X_s$. Suppose that X is regular at x with $\kappa(x)/\kappa$ separable. Then f is smooth at x .

Proof. By the definition of a smooth morphism (Definition 4.3.35 [Liu06]), it suffices to show that $f : X \rightarrow \text{Spec}R$ is flat at x and that X_s is smooth at x , as a scheme over κ . To this end, we may assume that X is the local scheme $\text{Spec}(A)$, where A is a regular local ring (by replacing X with $\text{Spec}(\mathcal{O}_{X,x})$).

Claim 1: The morphism f is flat at x .

Proof. Since A is a regular local ring, it is also an integral domain (Proposition 4.2.11 [Liu06]). Since X admits an S -integral point, we get that $f : R \rightarrow A$ is injective. We conclude that A is a torsion-free R -module and therefore flat (see Tag 0539 [Stacks]). □*Claim 1*

An integral point $P : \text{Spec}S \rightarrow X$ gives rise to a commutative diagram

$$\begin{array}{ccc} & A & \\ f \nearrow & & \searrow P \\ R & \xrightarrow{\subseteq} & S \end{array}$$

Note that $\pi \notin \mathfrak{m}_A^2$ because otherwise we would have that $\pi \in \mathfrak{m}_S^2$, which is contrary to our assumption that $\mathfrak{m}_S = \pi S$. We thus consider $\mathfrak{m}_R/\mathfrak{m}_R^2$ as a 1-dimensional $\kappa(x)$ -vector subspace of $\mathfrak{m}_A/\mathfrak{m}_A^2$, spanned by $\pi + \mathfrak{m}_A^2$, where π is a uniformizer of R . One may extend $\{\pi + \mathfrak{m}_A^2\}$ to a basis of $\mathfrak{m}_A/\mathfrak{m}_A^2$, say $\{\pi + \mathfrak{m}_A^2, x_1 + \mathfrak{m}_A^2, \dots, x_n + \mathfrak{m}_A^2\}$ with $x_i \in \mathfrak{m}_A$.

Claim 2: The scheme X_s is smooth at x .

Proof. Let $B = A \otimes_R R/\mathfrak{m}_R \cong A/\pi A$ be the local ring of the special fiber at x . We will then have that $\{x_1 + \mathfrak{m}_B^2, \dots, x_n + \mathfrak{m}_B^2\}$ is a κ -basis for $\mathfrak{m}_B/\mathfrak{m}_B^2$. By flatness, we get that $\dim(B) = \dim(A) - 1$ (Theorem 4.3.12 [Liu06]) and therefore that the local ring B is regular, i.e. X_s is regular at x . Since $\kappa(x)/\kappa$ is separable, we get that X_s is smooth at x (Proposition 4.3.30 [Liu06]). □*Claim 2*

□

As in §5.3.3, we may reformulate our task in geometric terms. Let X be a scheme of finite type over $\mathbb{F}_p[t]_{(t)}$ and W a given Zariski open subset of the generic fiber $X_{\mathbb{F}_p(t)}$. Suppose that we have $P \in X(\mathcal{O}_K)$ with $P_\eta \in W(K)$. We aim to show that there is $Q \in X(\mathcal{O}_L)$ with $Q_\eta \in W(L)$. Arguing as in §5.3.3, we may assume that X is integral and that W is non-empty. We proceed by induction on $\dim(X)$.

First assume that $P : \text{Spec}\mathcal{O}_K \rightarrow X$ is dominant. We may view X as a scheme over \mathbb{F}_p by $X \rightarrow \text{Spec}\mathbb{F}_p[t]_{(t)} \rightarrow \text{Spec}\mathbb{F}_p$. Now local uniformization provides us with a proper birational map $f : Y \rightarrow X$, so that $\text{Spec}\mathcal{O}_K \rightarrow X$ lifts to $\text{Spec}\mathcal{O}_K \rightarrow Y$ as in the diagram below

$$\begin{array}{ccc}
\mathrm{Spec}K & \longrightarrow & Y \\
\downarrow & \nearrow \exists \tilde{P} & \downarrow f \\
\mathrm{Spec}\mathcal{O}_K & \xrightarrow{P} & X
\end{array}$$

and with $\tilde{P}(\mathfrak{m}_K)$ being a regular point of Y (cf. §5.6.6). By assumption, we have that wt is minimal positive in Δ , which is equivalent to $\mathfrak{m}_L = t\mathcal{O}_L$. Proposition 5.6.7 now applies to show that the integral point $\tilde{P} \in Y(\mathcal{O}_K)$ meets the special fiber at a point $y \in Y_s$ such that $Y \rightarrow \mathrm{Spec}\mathbb{F}_p[t]_{(t)}$ is smooth at y . By Hensel's Lemma, y can also be lifted to an \mathcal{O}_L -integral point of Y . Moreover, Theorem 2.4 [DS03] implies the existence of $\tilde{Q} \in Y(\mathcal{O}_L)$ with $\tilde{Q}_\eta \in f^{-1}W(L)$. Finally, precomposing with f , we see that \tilde{Q} induces $Q \in X(\mathcal{O}_L)$ with $Q_\eta \in W(L)$.

Finally, if $P : \mathrm{Spec}\mathcal{O}_K \rightarrow X$ is not dominant, then P is an integral point of a closed subscheme $Z \subsetneq X$, where $Z = \overline{\{\xi\}}$ and $\xi = P(\eta_K)$. Since X is integral, we see that $\dim Z < \dim X$ and the conclusion follows from our induction hypothesis.

6

The theory of all p -adic fields

Introduction

The material presented in this chapter is based on [Kar21a]. Let $L_{\text{val},\times}$ be the language of valued fields with a cross-section (see §1.3.5). Ax-Kochen [AK65] and independently Ershov [Ers65] showed that \mathbb{Q}_p , equipped with the normalized cross section $s : n \mapsto p^n$, is decidable in $L_{\text{val},\times}$. More generally, we showed in §2.4 that every finite extension of \mathbb{Q}_p is also decidable in $L_{\text{val},\times}$, for a suitable choice of a cross-section \cdot . Combining the results of [AK65] with the theory of pseudofinite fields, Ax showed in Theorem 17 [Ax68] that the (asymptotic) theory of $\{\mathbb{Q}_p : p \in \mathbb{P}\}$ is decidable in $L_{\text{val},\times}$ and also that the (asymptotic) theory of $\{\mathbb{Q}_p(\zeta_n) : p \nmid n\}$, namely the collection of all finite *unramified* extensions of \mathbb{Q}_p , is decidable in $L_{\text{val},\times}$. Recall that the asymptotic theory of a class \mathcal{C} of L -structures is defined to be the set of sentences $\phi \in L$ which are true in all but finitely many $M \in \mathcal{C}$.

In sharp contrast, we have that $\mathbb{F}_p((t))$ is *undecidable* in $L_{\text{val},\times}$. This was already known to Ax (unpublished) and an elementary proof was given later by Becker-Denef-Lipshitz [BDL80], which was also reworked by Cherlin in §4 [Che82]. Pheidas [Phe87] generalized the result for $k((t))$, where k is an *arbitrary* field of characteristic p , and showed that already the *existential* $L_{\text{val},\times}$ -theory is undecidable. The decidability problem for $\mathbb{F}_p((t))$ in the language of rings L_{rings} is still an open problem.

An important related open question in mixed characteristic is whether $\text{Th}(\{K : [K : \mathbb{Q}_p] < \infty\})$, i.e. the theory of *all* finite p -adic extensions, is decidable. This question was first raised (in print) by Derakhshan-Macintyre in §9 [DM22] for the language of rings L_{rings} . In Theorem 9.1 [DM22], they showed that the

theory of adèle rings of *all* number fields in L_{rings} is decidable if and only if $\text{Th}(\{K : [K : \mathbb{Q}_p] < \infty\})$ is decidable in L_{rings} , for each prime p . It is also natural to consider $\text{Th}(\{K : [K : \mathbb{Q}_p] < \infty\})$ in any of the standard expansions L of L_{rings} (see e.g., pg. 20-22 [Der20]), and ask whether it is decidable (Problem 6.2 [Der20]).

In the present chapter, we resolve negatively the asymptotic version of this problem in the presence of a cross-section. While each *individual* finite p -adic extension is decidable in $L_{\text{val}, \times}$, the asymptotic theory of all of them is less well-behaved:

Theorem 6.0.1. *The asymptotic $L_{\text{val}, \times}$ -theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ is undecidable.*

We note that for each K with $[K : \mathbb{Q}_p] < \infty$ there are many choices of a cross-section and *any* such choice will lead to an undecidability result (see Convention 6.1.6). By carefully keeping track of quantifiers, we will prove in §6.3.6 that already the asymptotic $\exists\forall$ -theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ is undecidable in $L_{\text{val}, \times}$.

The key idea of the proof will be to use highly ramified p -adic fields to approximate $\mathbb{F}_p((t))$ à la Krasner-Kazhdan-Deligne (see Section 6.1) and then adapt Pheidas' proof of the undecidability of $\mathbb{F}_p((t))$ in $L_{\text{val}, \times}$ (see [Phe87]). In more detail, the proof of Theorem 6.0.1 consists of the following steps:

1. Encode the asymptotic theory of totally ramified p -adic fields, i.e., the asymptotic theory of $\{K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty\}$ (see the proof of Theorem 6.3.7).
2. Observe that $\mathcal{O}_K/(p) \cong \mathbb{F}_p[t]/(t^e)$, for K/\mathbb{Q}_p totally ramified of degree e , and thereby encode the asymptotic theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ in L_t with a predicate for powers of t (see Corollary 6.1.7).
3. Encode the asymptotic theory of *truncated fragments* of $(\mathbb{N}; 0, 1, +, |_p)$ (see §6.2.2), where $m |_p n$ if and only if $n = p^s \cdot m$ for some $s \in \mathbb{N}$.
4. Show that the latter is undecidable by encoding the Diophantine problem of $(\mathbb{N}; 0, 1, +, |_p)$ (see §6.2.8).

We note that Derakhshan-Macintyre had already suggested that the difficulty in understanding the (asymptotic) theory of p -adic fields must lie in unbounded ramification (see the last paragraph of [DM22]). This is precisely the ingredient that makes the transition to positive characteristic work and is the reason why this asymptotic class exhibits a different behavior from the two asymptotic theories mentioned in the first paragraph of the introduction.

6.1 Local field approximation

6.1.1 Motivation

A powerful philosophy, often referred to as the Krasner-Kazhdan-Deligne principle (due to [Kra56], [Kaz86] and [Del84]), says that a highly ramified p -adic field K (e.g., $K = \mathbb{Q}_p(p^{1/n})$ with n large) is in many respects "close" to a positive characteristic valued field. Although this is reflected in many aspects of K , perhaps the most elementary one is that the residue ring $\mathcal{O}_K/(p)$ "approximates" a positive characteristic valuation ring (see Lemma 6.1.3, Remark 6.1.4). Our goal in this section will be to prove Corollary 6.1.7, which will serve as a bridge between Pheidas' work in positive characteristic and our problem in mixed characteristic (see §6.2.1).

6.1.2 Computations modulo p

Let K/\mathbb{Q}_p be a finite *totally ramified* extension of degree n , with value group Γ and residue field k . For completeness, we record here the following computation:

Lemma 6.1.3. For each uniformizer π of \mathcal{O}_K we have an isomorphism $\mathcal{O}_K/(p) \cong \mathbb{F}_p[t]/(t^n)$, which maps the image of π in $\mathcal{O}_K/(p)$ to the image of t in $\mathbb{F}_p[t]/(t^n)$.

Proof. Write $\mathcal{O}_K = \mathbb{Z}_p[\pi]$, where π is a root of an Eisenstein polynomial $E(t) = t^n + a_{n-1}t^{n-1} + \dots + a_0 \in \mathbb{Z}_p[t]$ (see Proposition 11, pg.52 [Lan94]). In particular, we have that $a_i \equiv 0 \pmod{p\mathbb{Z}_p}$ and the reduction of $E(t)$ modulo $p\mathbb{Z}_p$ is equal to $\overline{E}(t) = t^n \in \mathbb{F}_p[t]$. We now compute

$$\mathcal{O}_K/(p) = \mathbb{Z}_p[t]/(p, E(t)) \cong \mathbb{F}_p[t]/(\overline{E}(t)) = \mathbb{F}_p[t]/(t^n)$$

and observe that the above isomorphism sends $\pi + (p)$ to $t + (t^n)$. □

Remark 6.1.4. If K, π are as above, then $\mathcal{O}_K/(\pi^n) = \mathcal{O}_K/(p) \cong \mathbb{F}_p[t]/(t^n) \cong \mathbb{F}_p[[t]]/(t^n)$ and Kazhdan says that K is n -close to $\mathbb{F}_p[[t]]$ (see §0 [Kaz86]). In what follows, it will indeed be useful to think of $\mathcal{O}_K/(p)$ as being very close to $\mathbb{F}_p[[t]]$ for large n (see 6.2.1).

Residue rings

For each $n \in \mathbb{N}$, we view $\mathbb{F}_p[t]/(t^n)$ as an $L_t \cup P$ -structure, where L_t is the language of rings L_{rings} together with a constant symbol for t and P is a unary predicate, whose interpretation is the set $\{0, 1, t, \dots, t^{n-1}\}$. Note that we have tacitly replaced the equivalence class $t^k + (t^n)$ with t^k , which is harmless and common when dealing with truncated/modular arithmetic.

Interpreting $\mathbb{F}_p[t]/(t^n)$

As a consequence of Lemma 6.1.3, we obtain:

Proposition 6.1.5. Let K/\mathbb{Q}_p be totally ramified of degree $n \in \mathbb{N}$ and $s : \Gamma \rightarrow K^\times$ be a cross-section. The structure $\mathbb{F}_p[t]/(t^n)$ in $L_t \cup P$ is $\exists\forall$ -interpretable in K in the language $L_{\text{val}, \times}$. Moreover, the reduction map $\phi \mapsto \phi_{\Delta_K}$ does not depend on K or the choice of s .

Proof. Let γ be the minimal positive element in Γ . It is definable by the \forall -formula in the free variable $\gamma \in \Gamma$ written below

$$\gamma > 0 \wedge \forall \delta \in \Gamma^{>0} (\gamma \leq \delta)$$

henceforth abbreviated by (γ minimal positive).

We now define a 1-dimensional interpretation Δ_K of $\mathbb{F}_p[t]/(t^n)$ in K . Take $\partial_{\Delta_K}(x)$ to be the formula $x \in \mathcal{O}$. The reduction map on unnested atomic formulas is described as follows:

1. If $\phi(x)$ is the formula $x = 0$ (resp. $x = 1$ and $x = t$), we take $\phi_{\Delta_K}(x)$ to be the formula $\exists y \in \mathcal{O} (x = py)$ (resp. $\exists y \in \mathcal{O} (x = 1 + py)$ and $\exists \gamma \in \Gamma [(\gamma \text{ minimal positive}) \wedge (x = s(\gamma))]$).
2. If $\phi(x, y)$ is the formula $x = y$, we take $\phi_{\Delta_K}(x, y)$ to be the formula $\exists z \in \mathcal{O} (x = y + pz)$.
3. If $\phi(x, y, z)$ is $x \diamond y = z$, then we take ϕ_{Δ_K} to be the formula $\exists w \in \mathcal{O} (x \diamond y = z + pw)$, where \diamond is either \cdot or $+$.
4. If $\phi(x)$ is the formula $x \in P$, we take $\phi_{\Delta_K}(x)$ to be the formula $\exists \gamma \in \Gamma^{\geq 0} (x = s(\gamma))$.

The coordinate map $f_{\Delta_K} : \mathcal{O}_K \rightarrow \mathbb{F}_p[t]/(t^n)$ is the projection modulo p , given by Lemma 6.1.3. The isomorphism $\mathcal{O}_K/(p) \cong \mathbb{F}_p[t]/(t^n)$ identifies π with t and the image of $s(\Gamma^{\geq 0})$ in $\mathcal{O}_K/(p)$ is equal to $\{0, 1, t, \dots, t^{n-1}\}$, via the above identification. One readily checks that the above data defines an $\exists\forall$ -interpretation of the $L_t \cup P$ -structure $\mathbb{F}_p[t]/(t^n)$ in the $L_{\text{val}, \times}$ -structure K .

Finally, the reduction map $\phi \mapsto \phi_{\Delta_K}$ does not depend on K , because of the inductive construction of the reduction map (see Proposition 1.1.8) and the fact that $\phi \mapsto \phi_{\Delta_K}$ does not depend on K when ϕ is any of the unnested atomic formulas listed above. \square

Convention 6.1.6. For the rest of the chapter, fix once and for all a choice of a cross-section $s_K : \Gamma \rightarrow K^\times$, for each finite extension K of \mathbb{Q}_p . For each such K , a cross-section does indeed exist and corresponds to a choice of a uniformizer for \mathcal{O}_K .

Corollary 6.1.7. *The asymptotic existential $L_t \cup P$ -theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is Turing reducible to the asymptotic existential-universal $L_{\text{val}, \times}$ -theory of $\{K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty\}$.*

Proof. For K/\mathbb{Q}_p totally ramified of degree n , let Δ_K be the interpretation of the $L_t \cup P$ -structure $\mathbb{F}_p[t]/(t^n)$ in the $L_{\text{val}, \times}$ -structure K , provided by Proposition 6.1.5. The reduction map of Δ_K does not depend on K and will simply be denoted by $\phi \mapsto \phi_\Delta$.

Claim: For any existential $\phi \in \text{Sent}_{L_t \cup P}$, the sentence ϕ_Δ is equivalent to an $\exists\forall$ -sentence.

Proof. For sentences ϕ of the form $\exists x\psi(x)$, where $\psi(x)$ is a quantifier-free formula without negations, this follows from the fact that Δ_K is an $\exists\forall$ -interpretation. We may therefore focus on formulas $\psi(x)$ of the form $f(x_1, \dots, x_m, t) \neq 0$ (resp. $f(x_1, \dots, x_m, t) \notin P$). Such a formula is logically equivalent to $\neg f(x_1, \dots, x_m, y) = 0 \wedge y = t$ (resp. $\neg f(x_1, \dots, x_m, y) \in P \wedge y = t$). Now $\neg(f(x_1, \dots, x_m, y) = 0)_\Delta$ (resp. $\neg(f(x_1, \dots, x_m, y) \in P)_\Delta$) is universal and $(y = t)_\Delta$ is existential-universal. The conclusion follows. \square *Claim*

For any $\phi \in \text{Sent}_{L_t \cup P}$ and any totally ramified extension K/\mathbb{Q}_p of degree n , we have that $\mathbb{F}_p[t]/(t^n) \models \phi \iff K \models \phi_\Delta$. For any given $n \in \mathbb{N}$, there are finitely many totally ramified extensions K/\mathbb{Q}_p of degree n (Proposition 14 [Lan94]). It follows that

$$\begin{aligned} \mathbb{F}_p[t]/(t^n) \models \phi \text{ for almost all } n \in \mathbb{N} &\iff \\ K \models \phi_\Delta \text{ for almost all } K \text{ with } [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty & \end{aligned}$$

The conclusion follows from the Claim. \square

6.2 Truncations of $(\mathbb{N}; 0, 1, +, |_p)$

6.2.1 Motivation

Pheidas' work

In Theorem 1 [Phe87], Pheidas showed that the Diophantine problem for $(\mathbb{N}; 0, 1, +, |_p)$ is undecidable. The proof goes by defining multiplication via a positive existential formula and using Matiyasevich's negative solution to Hilbert's tenth problem

[Mat70]. In Lemma 1(c) [Phe87], it is shown that the Diophantine problem for $(\mathbb{N}; 0, 1, +, |_p)$ can be encoded in the existential theory of $\mathbb{F}_p((t))$ in L_t with a predicate P for $\{0, 1, t, t^2, \dots\}$. It follows that the latter is also undecidable.

Plan of action

Our intuition that $\mathbb{F}_p[t]/(t^n)$ approximates $\mathbb{F}_p[[t]]$ when n is large, suggests that we can adapt Pheidas' strategy and show that the asymptotic existential theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is also undecidable in $L_t \cup P$. This will be established in Proposition 6.3.2 and in combination with Corollary 6.1.7 will pave the way for proving Theorem 6.0.1 in Section 6.3.6.

6.2.2 p -divisibility

The notion of p -divisibility was introduced by Denef in [Den79a], in order to show that the Diophantine problem of a polynomial ring of positive characteristic is undecidable. Given $n, m \in \mathbb{N}$ and $p \in \mathbb{P}$, we write $n |_p m$ if $m = p^s n$, for some $s \in \mathbb{N}$. Let us also write $L_{p\text{-div}} = \{0, 1, +, |_p\}$ for the language of addition and p -divisibility.

Encoding $|_p$ in $\mathbb{F}_p[t]/(t^n)$

The following result is due to Pheidas:

Lemma 6.2.3 (Lemma 1(a) [Phe87]). Let $n, m \in \mathbb{N}$ with $0 < n \leq m$. Then $n |_p m$ if and only if there exists $a \in \mathbb{F}_p[[t]]$ such that $t^{-m} - t^{-n} = a^{-p} - a^{-1}$.

Remark 6.2.4. If $n |_p m$, then the proof of Lemma 1(a) [Phe87] provides $a = (t^{-n \cdot p^{s-1}} + t^{-n \cdot p^{s-2}} \dots + t^{-n})^{-1}$. Note that we have slightly rephrased the original formulation of Lemma 1(a) [Phe87], so that the witness a has positive valuation.

We shall use a truncated version of Lemma 6.2.3, whose proof is identical, modulo some additional bookkeeping:

Lemma 6.2.5. Let $n, m, N \in \mathbb{N}$ with $0 < n \leq m < N/3$. Then $n |_p m$ if and only if there exists $\alpha \in \mathbb{F}_p[t]/(t^N)$ such that $\alpha^p(t^n - t^m) = t^n t^m (1 - \alpha^{p-1})$ and $\alpha^{3p} \neq 0$ in $\mathbb{F}_p[t]/(t^N)$.

Proof. " \Rightarrow ": Let $a = (t^{-n \cdot p^{s-1}} + t^{-n \cdot p^{s-2}} \dots + t^{-n})^{-1} \in \mathbb{F}_p[[t]]$. After clearing denominators in Lemma 6.2.3, we get that $a^p(t^n - t^m) = t^n t^m (1 - a^{p-1})$. Reducing the equation modulo t^N , yields $\alpha^p(t^n - t^m) = t^n t^m (1 - \alpha^{p-1})$, where α is the image of a in $\mathbb{F}_p[[t]]/(t^N) \cong \mathbb{F}_p[t]/(t^N)$. Note that $v_t \alpha^{3p} = 3m < N$ and thus $\alpha^{3p} \neq 0$ in $\mathbb{F}_p[t]/(t^N)$.

" \Leftarrow ": Let $n = p^r k$ and $m = p^l i$ with $p \nmid k, i$. Let α be as in our assumption and $a \in \mathbb{F}_p[[t]]$ be a lift of α . Since $\alpha^{3p} \neq 0$, we get that $v_t a^p < N/3$. We will have by assumption that $t^{-m} - t^{-n} = a^{-p} - a^{-1} + t^{N-m-n} z a^{-p}$, for some $z \in \mathbb{F}_p[[t]]$. Set $\varepsilon = t^{N-m-n} z a^{-p}$ and note that $v_t \varepsilon > 0$ because $m, n, v_t a^p < N/3$ and $v_t z \geq 0$. By Lemma 1 [Phe87], we may find $a_1, a_2 \in \mathbb{F}_p[[t]]$ such that $t^{-n} - t^{-k} = a_1^{-p} - a_1^{-1}$ and $t^{-m} - t^{-i} = a_2^{-p} - a_2^{-1}$. We compute that

$$t^{-i} - t^{-k} = b^p - b + \varepsilon$$

where $b = a^{-1} + a_1^{-1} - a_2^{-1}$.

We claim that $i = k$. Otherwise, the left hand side must have negative valuation. Since $v_t \varepsilon > 0$, this forces the right hand side to have p -divisible valuation. This is contrary to the fact that $p \nmid i, k$. It follows that $i = k$ and thus $n \mid_p m$. \square

Interpreting I_n in $\mathbb{F}_p[t]/(t^n)$

Motivated by Lemma 6.2.5, we introduce for each $n \in \mathbb{N}$ the $L_{p\text{-div}} \cup \{\infty\}$ -structure $I_n = (\{0, 1, \dots, n-1, \infty\}; 0, 1, \infty, \oplus, \mid_p)$, where:

- For $x, y \in I_n$, we have $x \mid_p y$ if $y = p^s x$ for some $s \in \mathbb{N}$ and $1 \leq x, y < n/3$.
- The operation $\oplus : I_n \times I_n \rightarrow I_n$ stands for truncated addition, i.e. given $x, y \in \{0, 1, \dots, n-1\}$ we have that $x \oplus y = x + y$ if $x + y < n$ and ∞ otherwise. Moreover, $\infty \oplus x = x \oplus \infty = \infty$ for all $x \in \{0, 1, \dots, n-1, \infty\}$.

Proposition 6.2.6. For each $n \in \mathbb{N}$, there is an \exists -interpretation Γ_n of the $L_{p\text{-div}} \cup \{\infty\}$ -structure I_n in the $L_t \cup P$ -structure $\mathbb{F}_p[t]/(t^n)$. Moreover, the reduction map $\phi \mapsto \phi_{\Gamma_n}$ does not depend on \mathbb{N} .

Proof. Take $\partial_{\Gamma_n}(x)$ to be the formula $x \in P$. The reduction map of Γ_n on unnested atomic formulas is described as follows:

1. If ϕ is the formula $x = 0$ (resp. $x = 1$, $x = \infty$ and $x = y$), we take ϕ_{Γ_n} to be the formula $x = 1$ (resp. $x = t$, $x = 0$ and $x = y$).
2. If $\phi(x, y, z)$ is $x + y = z$, then we take $\phi_{\Gamma_n}(x, y, z)$ to be the formula $x \cdot y = z$.
3. If $\phi(x, y)$ is the formula $x \mid_p y$, we take $\phi_{\Gamma_n}(x, y)$ to be the formula $\exists z(z^p(x - y) = x \cdot y(1 - z^{p-1}) \wedge z^{3p} \neq 0)$.

The coordinate map $f_{\Gamma_n} : \{0, 1, t, \dots, t^{n-1}\} \rightarrow I_n$ is equal to $v|_P$, i.e., the valuation map v restricted on P . Condition (3) of Definition 1.1.7 is readily verified for unnested atomic formulas of type (1). For the formula described in (2), one has to use the isomorphism of monoids $(P, \cdot, 1) \cong (\{0, 1, \dots, n-1, \infty\}, \oplus, 0)$. Using Lemma 6.2.5, one also verifies it for the unnested atomic formula of type (3). We deduce that Γ_n is an interpretation. It is clear from the description of the reduction map on unnested formulas that Γ_n is existential.

Finally, the reduction map $\phi \mapsto \phi_{\Gamma_n}$ does not depend on $n \in \mathbb{N}$, because of the inductive construction of the reduction map (see Proposition 1.1.8) and the fact that $\phi \mapsto \phi_{\Gamma_n}$ does not depend on n when ϕ is any of the unnested atomic formulas listed above. □

Corollary 6.2.7. *The (resp. asymptotic) $L_{p\text{-div}} \cup \{\infty\}$ -theory of $\{I_n : n \in \mathbb{N}\}$ is Turing reducible to the (resp. asymptotic) $L_t \cup P$ -theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$.*

Proof. Immediate from Proposition 6.2.6. □

6.2.8 Undecidability of fragments

Using the undecidability of the Diophantine problem over $(\mathbb{N}; 0, 1, +, |_p)$ (Theorem 1 [Phe87]) as a black box, we shall prove that the asymptotic *existential* theory of $\{I_n : n \in \mathbb{N}\}$ is also undecidable.

The proof becomes more transparent by using matrix norms. Recall the *maximum absolute row sum* norm $\|\cdot\|_\infty$ on the set of all matrices over \mathbb{R} , which is defined as $\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$, for $A \in M_{m \times n}(\mathbb{R})$. One readily checks that $\|\cdot\|_\infty$ is both sub-additive and sub-multiplicative, meaning that $\|A+B\|_\infty \leq \|A\|_\infty + \|B\|_\infty$ and $\|A \cdot B\|_\infty \leq \|A\|_\infty \cdot \|B\|_\infty$, whenever the operations are defined. Note also that when $A = (a_1, \dots, a_n)^\top \in M_{n \times 1}(\mathbb{N})$, we have $\|A\|_\infty = \max\{a_i : i = 1, \dots, n\}$.

For $A = (a_{ij})$ and $B = (b_{ij})$ in $M_{m \times n}(\mathbb{N})$, it will be convenient to use the notation $A |_p B$, which means that $a_{ij} |_p b_{ij}$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$.

Proposition 6.2.9. *The asymptotic existential $L_{p\text{-div}} \cup \{\infty\}$ -theory of $\{I_n : n \in \mathbb{N}\}$ is undecidable. Moreover, let T be the subtheory consisting of sentences ϕ of the form $\exists x \psi(x)$, where $x = (x_1, \dots, x_n)$ and $\psi(x)$ is a conjunction of a quantifier-free formula without negations with a formula of the form $\bigwedge_{i=1}^n N x_i \neq \infty$, for some $N \in \mathbb{N}$. Then T is undecidable.*

Proof. We shall encode the Diophantine problem of $(\mathbb{N}; 0, 1, +, |_p)$ in T . Let Σ be an arbitrary system in variables $\mathbf{x} = (x_1, \dots, x_n)^\top \in M_{n \times 1}(\mathbb{N})$ of the form

$$\begin{cases} A_1\mathbf{x} + \mathbf{b}_1 = A_2\mathbf{x} + \mathbf{b}_2 \\ A_3\mathbf{x} + \mathbf{b}_3 \mid_p A_4\mathbf{x} + \mathbf{b}_4 \end{cases} \quad (\Sigma)$$

where $A_i \in M_{m \times n}(\mathbb{N})$ and $\mathbf{b}_i \in M_{m \times 1}(\mathbb{N})$. Consider also the formula $\Sigma(x) \in \text{Form}_{L_{p\text{-div}}}$ associated with the system Σ .

Claim: We have that

$$(\mathbb{N}; 0, 1+, |_p) \models \exists x \Sigma(x) \iff T \models \exists x (\Sigma(x) \bigwedge_{i=1}^n 3Mx_i \neq \infty)$$

where $M = \max_{1 \leq i \leq 4} \{\|A_i\|_\infty + \|\mathbf{b}_i\|_\infty\}$.

Proof. " \Rightarrow ": Consider a witnessing tuple $\mathbf{c} = (c_1, \dots, c_n)^\top \in M_{n \times 1}(\mathbb{N})$. If $\mathbf{c} = \mathbf{0}$, then the conclusion is clear. Otherwise, consider $m = \|\mathbf{c}\|_\infty = \max\{c_i : i = 1, \dots, n\} \geq 1$ and choose $N \in \mathbb{N}$ such that $N > 3 \cdot M \cdot m$. Using the sub-additive and sub-multiplicative properties of $\|\cdot\|_\infty$, we see that

$$\|A_i\mathbf{c} + \mathbf{b}_i\|_\infty \leq \|A_i\|_\infty \|\mathbf{c}\|_\infty + \|\mathbf{b}_i\|_\infty \leq m \cdot (\|A_i\|_\infty + \|\mathbf{b}_i\|_\infty) \leq M \cdot m < N/3$$

for $i = 1, \dots, 4$. In particular, we get that both \oplus and $|_p$ specialize to their ordinary counterparts in \mathbb{N} and that $\Sigma(\mathbf{c})$ holds true in I_N , viewing \mathbf{c} as a tuple in I_N^n . Each conjunct $3Mx_i \neq \infty$ also holds true for c_i because $3Mc_i \leq 3M \cdot m < N$.

" \Leftarrow ": Let $\mathbf{c} = (c_1, \dots, c_n)^\top \in I_N^n$ be a witness of the sentence $\exists x (\Sigma(x) \bigwedge_{i=1}^n 3Mx_i \neq \infty)$, for some $N > 3 \cdot M$. Since $3Mc_i \neq \infty$, we get that $c_i < N/3M$ for $i = 1, \dots, n$. We therefore get that

$$\|A_i\mathbf{c} + \mathbf{b}_i\|_\infty \leq \|A_i\|_\infty \|\mathbf{c}\|_\infty + \|\mathbf{b}_i\|_\infty < \frac{N}{3M} \cdot (\|A_i\|_\infty + \|\mathbf{b}_i\|_\infty) \leq \frac{N}{3M} \cdot M = N/3$$

for $i = 1, \dots, 4$. In particular, we get that both \oplus and $|_p$ specialize to their ordinary counterparts in \mathbb{N} . The corresponding tuple in \mathbb{N}^n is the desired witness. \square_{Claim}

The conclusion follows from the fact that the Diophantine problem over $(\mathbb{N}; 0, 1, +, |_p)$ is undecidable (Theorem 1 [Phe87]). \square

6.3 Proof of the main Theorem

6.3.1 Truncated polynomial rings

Proposition 6.3.2. The asymptotic existential $L_t \cup P$ -theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is undecidable.

Proof. Let T be as in Proposition 6.2.6. We shall argue that the asymptotic existential $L_t \cup P$ -theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is Turing reducible to T . The conclusion will then follow from Proposition 6.2.9. Let Γ_n be the \exists -interpretation of the $L_{p\text{-div}} \cup \{\infty\}$ -structure I_n in the $L_t \cup P$ -structure $\mathbb{F}_p[t]/(t^n)$, provided by Proposition 6.2.6. The reduction map of Γ_n does not depend on $n \in \mathbb{N}$ and will simply be denoted by $\phi \mapsto \phi_\Gamma$. Since Γ_n is existential, whenever $\psi(x)$ is a quantifier-free $L_{p\text{-div}} \cup \{\infty\}$ -formula, the formula $\psi_\Gamma(x)$ is an existential formula. Moreover, from the description of the reduction map on unnested atomic formulas (see the proof of Proposition 6.2.6), we have $(\bigwedge_{i=1}^n Nx_i \neq \infty)_\Gamma = \bigwedge_{i=1}^n (Nx_i \neq \infty)_\Gamma = \bigwedge_{i=1}^n x_i^N \neq 0$. It follows that T is Turing reducible to the asymptotic existential $L_t \cup P$ -theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$. \square

6.3.3 Totally ramified extensions

Proposition 6.3.4. The asymptotic $\exists\forall$ -theory of $\{K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty\}$ is undecidable in $L_{\text{val}, \times}$.

Proof. This follows from Corollary 6.1.7 and Proposition 6.3.2. \square

Remark 6.3.5. (a) The same proof applies verbatim to any *infinite* collection of totally ramified extensions of \mathbb{Q}_p , e.g., $\{\mathbb{Q}_p(\zeta_{p^n}) : n \in \mathbb{N}\}$ or $\{\mathbb{Q}_p(p^{1/n}) : n \in \mathbb{N}\}$.

(b) We do not know if the asymptotic existential theory of $\{K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty\}$ is decidable or not in $L_{\text{val}, \times}$.

6.3.6 Finite extensions

Theorem 6.3.7. *The asymptotic $\exists\forall$ -theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ is undecidable in $L_{\text{val}, \times}$.*

Proof. Let T be the theory in question and T_{tot} be the asymptotic theory of $\{K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty\}$ in L_{val} with a cross-section. We shall encode T_{tot} in T . Given an $\exists\forall$ -sentence ϕ , we see that

$$T_{\text{tot}} \models \phi \iff T \models (k = \mathbb{F}_p) \rightarrow \phi$$

The formal counterpart of $(k = \mathbb{F}_p) \rightarrow \phi$ is logically equivalent to an $\exists\forall$ -sentence. It follows that the asymptotic existential-universal $L_{\text{val},\times}$ -theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ is Turing reducible to the asymptotic existential-universal $L_{\text{val},\times}$ -theory of $\{K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty\}$. The conclusion follows from Proposition 6.3.4. \square

6.4 Final remarks

6.4.1 L_{val} vs L_{rings}

In view of [CDLM13] and by possibly increasing the complexity, one may replace $L_{\text{val},\times}$ with the 1-sorted language of rings L_{rings} together with a unary predicate P for the image of the cross-section in K . We need to use the following:

Fact 6.4.2 (Theorem 2 [CDLM13]). There is an $\exists\forall$ -formula $\phi(x)$ in L_r such that $\mathcal{O}_K = \phi(K)$ for any henselian valued field K with finite or pseudo-finite residue field.

Remark 6.4.3. In fact, Theorem 2 [CDLM13] is about the existence of an *existential* formula in $L_{\text{rings}} \cup P_2^{\text{AS}}$, where $P_2^{\text{AS}}(x) = \exists y(x = y^2 + y)$. However, any existential sentence in $L_{\text{rings}} \cup P_2^{\text{AS}}$ is equivalent to an $\exists\forall$ -sentence in L_{rings} .

By Fact 6.4.2, the asymptotic theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ in L_{val} with a cross-section can be encoded in the asymptotic theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ in $L_{\text{rings}} \cup P$. By Theorem 6.0.1, the latter is undecidable. Our use of the cross-section/predicate formalism is essential and we do not know whether the (asymptotic) theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ is decidable in L_{rings} .

6.4.4 Residue rings and $\mathbb{F}_p((t))$

If the (asymptotic) theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ is decidable in L_{rings} , then by Corollary 6.1.7 this would also yield a positive answer to the following question:

Question 6.4.5. Is the (asymptotic) theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ decidable in L_{rings} ?

Observation 6.4.6. The asymptotic *existential* theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is decidable in L_{rings} .

Proof. Let $\mathbb{F}_p[t^{1/\infty}]$ be the direct limit of the *injective* system $\varinjlim \mathbb{F}_p[t^{1/n}]$, where $\phi_{nm} : \mathbb{F}_p[t^{1/n}] \hookrightarrow \mathbb{F}_p[t^{1/m}]$ is the natural inclusion map for $n \mid m$. We write $\mathbb{F}_p[t^{1/n}]/(t)$ and $\mathbb{F}_p[t^{1/\infty}]/(t)$ for the quotients modulo t . Note that $\phi_{nm}(t \cdot \mathbb{F}_p[t^{1/n}]) \subseteq t \cdot \mathbb{F}_p[t^{1/m}]$ and therefore the induced maps $\overline{\phi_{nm}} : \mathbb{F}_p[t^{1/n}]/(t) \hookrightarrow \mathbb{F}_p[t^{1/m}]/(t)$ and $\overline{\phi_n} : \mathbb{F}_p[t^{1/n}]/(t) \hookrightarrow \mathbb{F}_p[t^{1/\infty}]/(t)$ are also injective.

Claim 1: The asymptotic existential L_{rings} -theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is equal to $\text{Th}_{\exists}(\mathbb{F}_p[t^{1/\infty}]/(t))$.

Proof. For each $n \in \mathbb{N}$, we have a ring isomorphism

$$\mathbb{F}_p[t^{1/n}]/(t) \cong \mathbb{F}_p[t, X]/(X^n - t, t) \cong \mathbb{F}_p[X]/(X^n) \cong \mathbb{F}_p[t]/(t^n)$$

Now if $\phi \in L_{\text{rings}}$ is existential, then $\mathbb{F}_p[t^{1/\infty}]/(t) \models \phi$ if and only if $\mathbb{F}_p[t^{1/n}]/(t) \models \phi$ for all sufficiently large n . The conclusion follows. $\square_{\text{Claim 1}}$

Finally, we prove:

Claim 2: $\text{Th}_{\exists}(\mathbb{F}_p[t^{1/\infty}]/(t))$ is decidable in L_{rings} .

Proof. A straightforward adaptation of Proposition 3.4.3 shows that

$$\mathbb{F}_p[t^{1/\infty}]/(t) \models \exists x \bigwedge_{1 \leq i, j \leq n} (f_i(x) = 0 \wedge g_j(x) \neq 0) \iff$$

$$\mathbb{F}_p[[t]](t^{1/\infty}) \models \exists x \bigwedge_{1 \leq i, j \leq n} (v(f_i(x)) > v(g_j(x)))$$

where $f_i(x), g_j(x) \in \mathbb{F}_p[x]$ are any multi-variable polynomials in $x = (x_1, \dots, x_m)$ for $i, j = 1, \dots, n$. Finally, the henselian valued field $\mathbb{F}_p((t))(t^{1/\infty})$ is existentially decidable in L_{val} by Corollary 7.5 [AF16]. $\square_{\text{Claim 2}}$

\square

Remark 6.4.7. (a) Observation 6.4.6 should be contrasted with Proposition 6.3.2, which shows that the asymptotic existential theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is undecidable in $L_t \cup P$.

(b) The proof of Observation 6.4.6 does not go through for the language L_t , as the ring isomorphisms in the proof of Claim 1 do not respect t . We do not know if the asymptotic existential theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is decidable in L_t .

(c) On the other hand, the asymptotic *positive* existential theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ in L_t is equal to $\text{Th}_{\exists^+} \mathbb{F}_p[[t]]$ and is decidable by an *effective* version of Greenberg's approximation theorem (see Theorem 3.1 and Theorem 6.1 [BDLD79]).

7

Valued fields with a total residue map

Introduction

The material presented in this chapter is based on [Kar22]. Let k be a field. Consider the power series field $k((t))$ and let $\text{res} : k((t)) \rightarrow k$ be the *total residue map*

$$\text{res} : \sum_{i=-N}^{\infty} c_i t^i \mapsto c_0$$

Viewing $k((t))$ as a k -vector space, we see that $\text{res} : k((t)) \rightarrow k$ is a linear form which extends the usual residue map $\pi : k[[t]] \rightarrow k$ to all of $k((t))$ (hence the name *total*). The motivation for studying the model-theory of this structure is twofold:

1. In complex analysis, one defines the *residue* of a complex meromorphic function f at an isolated singularity $a \in \mathbb{C}$, denoted by $\text{Res}(f, a)$ or $\text{Res}_a(f)$. The map res is essentially the same as Res_0 on $\mathbb{C}((t))$. More precisely, we have $\text{res}(t \cdot f) = \text{Res}_0(f)$. For all model-theoretic purposes, one can work interchangeably with either one of those functions.
2. On the other hand, Becker-Denef-Lipschitz [BDL80] showed that $\text{res} : \mathbb{F}_p((t)) \rightarrow \mathbb{F}_p$ is definable in $\mathbb{F}_p((t))$ in L_t , which is defined to be L_{rings} together with a constant symbol for t . The crucial observation was that $\text{res}(x) = 0$ holds for $x \in \mathbb{F}_p((t))$, precisely when there exist $x_0, x_1, \dots, x_{p-1} \in \mathbb{F}_p((t))$ such that

$$x = x_0^p - x_0 + tx_1^p + \dots + t^{p-1}x_{p-1}^p$$

Using this, they showed that $\mathbb{F}_p((t))$ is undecidable in the language of valued fields with a cross-section. This was in sharp contrast with the result by Ax-Kochen [AK66] and Ershov [Ers65]: For any decidable field k of characteristic

0, the power series field $k((t))$ is decidable in the language of valued fields with a cross-section.

Towards understanding the model theory of $\mathbb{F}_p((t))$, it may be instructive to isolate such definable functions/predicates and study them over $k((t))$, where k is not necessarily finite. Enriching $k((t))$ with such functions/predicates, even when k is infinite, allows us to gauge the complexity of said functions/predicates over $\mathbb{F}_p((t))$ itself and may give us new insights into its theory. For instance, if $k((t))$ turns out to be undecidable in the resulting expansion—which is a common theme as we will see—this may be a strong indication that large part of the complexity of the theory of $\mathbb{F}_p((t))$ lies in that particular function/predicate. Also, once we establish the undecidability of $k((t))$ in the resulting expansion, we can then ask if that function/predicate is in fact definable in the language of rings, at least for suitable infinite fields k of positive characteristic. This would then yield an undecidability result in the language of rings. The approach described above is largely influenced by Cherlin [Che82], especially Problems 3 and 4 in §5 [Che82].

In the present chapter focuses on the case of the total residue map $\text{res} : k((t)) \rightarrow k$. We will study the model theory of power series fields $k((t))$, equipped with $\text{res} : k((t)) \rightarrow k$. One can also take this a step further and study valued fields—not necessarily power series fields—enriched with a total residue map $\text{res} : K \rightarrow k$. In the axiomatic setting, the *total residue* map $\text{res} : K \rightarrow k$ is assumed to be a linear form extending $\pi : \mathcal{O}_K \rightarrow k$. We consider the theory of equal characteristic valued fields K with a total residue map $\text{res} : K \rightarrow k$ and also a lift $\iota : k \rightarrow K$ of the residue field. We call $\text{VF}_{\text{res}, \iota}$ the resulting theory. The standard model is $(k((t)), \text{res})$, which was discussed above.

At first glance, the theory $\text{VF}_{\text{res}, \iota}$ seems like an innocent variant of VF. However, we will show the following:

Theorem 7.0.1. *The theory $\text{VF}_{\text{res}, \iota}$ does not admit a model companion.*

This should be contrasted with the classical theorem by A. Robinson: the theory VF of valued fields admits a model companion, namely ACVF, the theory of algebraically closed valued fields. We also note that the theory VF_ι , of valued fields with a lift $\iota : k \rightarrow K$ of the residue field, still admits a model companion. The model companion of VF_ι is simply the theory ACVF_ι described by Hrushovski-Kazhdan §6 [HK09] (there it is called T_{loc}). Indeed, they prove a quantifier-elimination result for ACVF_ι , which in fact—according to §6.1 [HK09]—goes back to F. Delon.

Building on the ideas that go into the proof Theorem 7.0.1, we will then be able to show:

Theorem 7.0.2. *Let k be an infinite field. Then $(k((t)), \text{res})$ is undecidable in L_{res} .*

Here L_{res} is the three-sorted language L_{val} of valued fields, together with a function symbol for $\text{res} : k((t)) \rightarrow k$. In fact, we prove that already the $\exists\forall$ -theory is undecidable (at least when t is added in the language) and also that $(\mathbb{N}, +, \cdot)$ is interpretable (without parameters). A more refined analysis shows also that the subring $k[t] \subseteq k((t))$ is definable in L_{res} with a parameter for t (see Corollary 7.2.26). The proof also applies to show that the Puiseux series field $\bigcup_{n \in \mathbb{N}} k((t^{1/n}))$ is undecidable in L_{res} , whenever k is an *infinite* field. As a consequence of Theorem 7.2.11, we get that $(\mathbb{C}((t)), \text{Res}_0)$ is undecidable, where $\text{Res}_0 : \mathbb{C}((t)) \rightarrow \mathbb{C} : f \mapsto \text{Res}_0(f)$ maps f to its complex residue at 0 (see Corollary 7.2.13). As was mentioned above, when k is finite, then $\text{res} : k((t)) \rightarrow k$ is definable in L_{val} and in that case the decidability problem over $(k((t)), \text{res})$ in L_{res} is equivalent to the decidability problem over $k((t))$ in L_{val} . The proofs given here do not seem to shed any light on that case.

7.1 Valued Fields with a total residue map

Let L_{val} be the three-sorted language of valued fields with sorts for the field, the value group and the residue field, equipped with a function symbol for the valuation $v : K \rightarrow \Gamma \cup \{\infty\}$.

7.1.1 Axiomatization of $\text{VF}_{\text{res}, \iota}$

We now study valued fields of equal characteristic, together with a lifting $\iota : k \rightarrow K$ and a *total* residue map $\text{res} : K \rightarrow k$, which is a linear form extending the residue map $\pi : \mathcal{O}_K \rightarrow k$. To simplify notation, we shall henceforth identify k with its field of representatives $\iota(k)$. Call $L_{\text{res}, \iota}$ the three-sorted language L_{val} together with function symbols for ι and res . Consider the following set of axioms:

1. (K, v) is a valued field of equal characteristic and $\iota : k \rightarrow K$ is a field homomorphism such that $\pi \circ \iota = \text{id}_k$.
2. We have that $\text{res}|_{\mathcal{O}_K} = \pi$.
3. $\text{res} : K \rightarrow k$ is k -linear, i.e. $\text{res}(\lambda a + \mu b) = \lambda \text{res}(a) + \mu \text{res}(b)$, for all $\lambda, \mu \in k$.

Let $\text{VF}_{\text{res}, \iota}$ be the $L_{\text{res}, \iota}$ -theory generated by the above axioms. Some standard models of $\text{VF}_{\text{res}, \iota}$ are presented below:

Example 7.1.2. (a) We have $(k((t)); v_t, \text{res}, \iota) \models \text{VF}_{\text{res}, \iota}$, where $\text{res} : \sum_{i=-N}^{\infty} c_i t^i \mapsto c_0$ and $\iota : k \rightarrow k((t))$ is the obvious lifting.

(b) Similarly, we have $(\text{P}(k); v_t, \text{res}, \iota) \models \text{VF}_{\text{res}, \iota}$, where $\text{P}(k) = \bigcup_{n \in \mathbb{N}} k((t^{1/n}))$ is the Puiseux series field, $\text{res} : \text{P}(k) \rightarrow k : \sum_{i=-N}^{\infty} c_i t^{i/n} \mapsto c_0$ and $\iota : k \rightarrow \text{P}(k)$ is the obvious lifting.

7.1.3 Extensions

Recall the following from linear algebra:

Definition 7.1.4. Given a k -vector space V , a subspace $W \subseteq V$ and $v_1, \dots, v_n \in V$, we say that v_1, \dots, v_n are linearly independent over W if for $\alpha_1, \dots, \alpha_n \in k$, we have that

$$\sum_{i=1}^n \alpha_i v_i \in W \Rightarrow \alpha_i = 0 \text{ for all } i = 1, \dots, n$$

More succinctly, we require that the images of v_1, \dots, v_n in V/W be k -linearly independent.

Lemma 7.1.5. Let $(L, w, \iota_L)/(K, v, \iota_K)$ be an extension of valued fields with liftings of their residue fields. If $\beta_1, \dots, \beta_n \in l$ are k -linearly independent and $a_1, \dots, a_n \in K$, then

$$w(\beta_1 a_1 + \dots + \beta_n a_n) = \min_{1 \leq i \leq n} v a_i$$

Proof. We may assume that $v(a_1) = \dots = v(a_n)$. For each $1 \leq i \leq n$, there exist $\alpha_i \in k$ and ε_i with $v \varepsilon_i > 0$, such that $a_i = \alpha_i \cdot a_1 + \varepsilon_i \cdot a_1$. It follows that

$$\sum_{i=1}^n \beta_i a_i = a_1 \cdot \sum_{i=1}^n \alpha_i \beta_i + a_1 \cdot \sum_{i=1}^n \beta_i \varepsilon_i$$

Since the β_i 's are k -linearly independent, we get that $\sum_{i=1}^n \alpha_i \beta_i \neq 0$. In particular, we get that $w(a_1 \cdot \sum_{i=1}^n \alpha_i \beta_i) = v a_1$. Since $v(a_1 \cdot \sum_{i=1}^n \beta_i \varepsilon_i) = v a_1 + v(\sum_{i=1}^n \beta_i \varepsilon_i) > v a_1$, we deduce that $w(\beta_1 a_1 + \dots + \beta_n a_n) = v a_1$. □

Given a valued field extension $(L; w, \iota_L)/(K; v, \iota)$, we denote by $\langle K \rangle_l$ the l -linear span of K inside L . Note that $\langle K \rangle_l$ is isomorphic to $K \otimes_k l$ as an l -vector space.

Lemma 7.1.6. Let $(L, w, \iota_L)/(K, v, \iota_K)$ be an extension of valued field with liftings of their residue fields. Then $\mathcal{O}_L \cap \langle K \rangle_l = \langle \mathcal{O}_K \rangle_l$.

Proof. The conclusion will follow easily from the following:

Claim: Let $a_1, \dots, a_n \in K$ be k -linearly independent over \mathcal{O}_K . Then a_1, \dots, a_n are also l -linearly independent over \mathcal{O}_L .

Proof. Suppose $\beta'_1, \dots, \beta'_n \in l$ are such that

$$\sum_{i=1}^n \beta'_i a_i \in \mathcal{O}_L$$

Let $\{\beta_1, \dots, \beta_m\}$ be a k -linear basis of $\langle \beta'_1, \dots, \beta'_n \rangle_k$ and write $\beta'_i = \sum_{j=1}^m c_{ij} \cdot \beta_j$ with $c_{ij} \in k$. We will then have that

$$\sum_{i=1}^n \beta'_i a_i = \sum_{i=1}^n \left(\beta_i \sum_{j=1}^m c_{ij} a_j \right)$$

Since the β_i 's are k -linearly independent, Lemma 7.1.5 implies that

$$w\left(\sum_{i=1}^n \left(\beta_i \sum_{j=1}^m c_{ij} a_j \right)\right) = \min_{1 \leq i \leq n} \left\{ v\left(\sum_{j=1}^m c_{ij} a_j\right) \right\}$$

For $i = 1, \dots, n$, it follows that

$$\sum_{j=1}^m c_{ij} a_j \in \mathcal{O}_K \Rightarrow c_{ij} = 0$$

for all $j = 1, \dots, m$. We conclude that $\beta'_i = 0$ for all $i = 1, \dots, n$ and that the a_i 's are l -linearly independent over \mathcal{O}_L . \square *Claim*

Now let V be a complement of the k -vector subspace $\mathcal{O}_K \subseteq K$, i.e., we have $K = \mathcal{O}_K \oplus V$. We will then have that $\langle K \rangle_l = \langle \mathcal{O}_K \rangle_l + V_l$. By the claim, we get that $V_l \cap \mathcal{O}_L = \{0\}$ and since $\langle \mathcal{O}_K \rangle_l \subseteq \mathcal{O}_L$, we conclude that $\mathcal{O}_L \cap \langle K \rangle_l = \langle \mathcal{O}_K \rangle_l$. \square

Lemma 7.1.7. Let $(K; v, \text{res}_K, \iota_K) \models \text{VF}_{\text{res}, \iota}$ and $(L; w, \iota_L)/(K; v, \iota_K)$ be an extension of valued fields with liftings of their residue fields. Let $\{e_i : i \in I\} \subseteq L$ be l -linearly independent over $\mathcal{O}_L + \langle K \rangle_l$. One may then extend $\pi_L : \mathcal{O}_L \rightarrow l$ to an l -linear map $\text{res}_L : L \rightarrow l$ such that $(L; w, \text{res}_L, \iota_L)$ is a model of $\text{VF}_{\text{res}, \iota}$ extending $(K; v, \text{res}_K, \iota_K)$ with $\text{res}(e_i) = 0$.

Proof. First we extend $\text{res}_K : K \rightarrow k$ to $\text{res}_L : \langle K \rangle_l \rightarrow l$ by extension of scalars; more explicitly, we will have $\text{res}_L(\beta \cdot a) = \beta \cdot \text{res}_K(a)$, for $\beta \in l$ and $a \in K$. We also require that res_L restricts to π_L on \mathcal{O}_L . By Lemma 7.1.6, if $b \in \mathcal{O}_L \cap \langle K \rangle_l$, we may write $b = \sum_{i=1}^n \beta_i a_i$, with $\beta_i \in l$ and $a_i \in \mathcal{O}_K$. We now compute that

$$\text{res}_L(b) = \sum_{i=1}^n \beta_i \cdot \text{res}_L(a_i) = \sum_{i=1}^n \beta_i \cdot \text{res}_K(a_i) = \sum_{i=1}^n \beta_i \cdot \pi_K(a_i) = \pi_L(b)$$

We may therefore extend $\text{res}_K : K \rightarrow k$ to $\text{res}_L : \mathcal{O}_L + \langle K \rangle_l \rightarrow l$ so that it restricts to res_K on K and also to π_L on \mathcal{O}_L . Finally, by elementary linear algebra, we may extend the linear map res_L on L by requiring that $e_i \mapsto 0$ and get a model $(L; w, \text{res}_L, \iota_L)$ of $\text{VF}_{\text{res}, \iota}$ extending $(K; v, \text{res}_K, \iota_K)$, as required. \square

Remark 7.1.8. There is nothing special about 0 in Lemma 7.1.7. One can ask that $\text{res}(e_i) = \alpha_i$, where $\{\alpha_i : i \in I\} \subseteq l$ is an arbitrary subset.

7.1.9 Existentially closed models

Extension by an infinitesimal

Fact 7.1.10 (Corollary 2.2.3 [EP05]). Let (K, v) be a valued field with value group Γ and residue field k . Let Γ' be an ordered abelian group extending Γ and $\gamma \in \Gamma'$ be torsion-free over Γ , i.e., if $n \cdot \gamma \in \Gamma$, then $n = 0$. Then there is exactly one valuation w on $K(x)$ extending v with $w(x) = \gamma$. We have $\Gamma_{K(x)} = \Gamma \oplus \mathbb{Z}\gamma$, with the ordering induced by Γ' , and $k_{K(x)} = k$.

Remark 7.1.11. A lifting $\iota : k \rightarrow K$ of the valued field (K, v) naturally induces a lifting of $(K(x), w)$, namely $\iota_{K(x)} : k \rightarrow K(x) : \alpha \mapsto \iota(\alpha)$.

Remark 7.1.12. As a special case of Fact 7.1.10, there is a unique valuation w on $K(x)$ such that $wx > \Gamma$. We will have $k_{K(x)} = k$ and $\Gamma_{K(x)} = \mathbb{Z}\gamma \oplus_{lex} \Gamma$.

Lemma 7.1.13. Let $(K, v, \text{res}, \iota) \models \text{VF}_{\text{res}, \iota}$. Let w be the unique valuation on $K(x)$ such that $wx > \Gamma$ and also let $\iota_{K(x)} : k \rightarrow K(x)$ be the induced lifting. Then there is a map $\text{res}_{K(x)} : K(x) \rightarrow k$, such that $(K(x); w, \text{res}_{K(x)}, \iota_{K(x)})$ is a model of $\text{VF}_{\text{res}, \iota}$ extending $(K; v, \text{res}, \iota)$ and with $\text{res}_{K(x)}(x^{-m}) = 0$, for all $m \in \mathbb{N}$.

Proof. By Lemma 7.1.7, it suffices to show that the elements $x^{-1}, x^{-2}, \dots, x^{-n}$ are k -linearly independent over $K + \mathcal{O}_{K(x)}$. Indeed, let $c \in K$ and $\beta_1, \dots, \beta_n \in k$, with some $\beta_i \neq 0$. We will have that

$$w\left(\sum_{i=1}^n \beta_i x^{-i} + c\right) = \min_{1 \leq i \leq n} \{-iwx, vc\} < 0$$

since $wx > -vc$.

□

Lemma 7.1.14. Let $(K; v, \text{res}, \iota)$ be an e.c. model of $\text{VF}_{\text{res}, \iota}$ and $n \in \mathbb{N}$. Then there exists $a \in \mathfrak{m} - \{0\}$ such that $\text{res}(a^{-m}) = 0$ for $m = 1, \dots, n$.

Proof. Let $(K(x); w, \text{res}_{K(x)}, \iota_{K(x)})$ be as in Lemma 7.1.13. Since $(K; v, \text{res}, \iota)$ is an e.c. model of $\text{VF}_{\text{res}, \iota}$, we will have that

$$(K; v, \text{res}, \iota) \preceq_1 (K(x); w, \text{res}_{K(x)}, \iota_{K(x)})$$

Since $x \in \mathfrak{m}_{K(x)} - \{0\}$ and $\text{res}_{K(x)}(x^{-m}) = 0$, for $m = 1, \dots, n$, we may also find $a \in \mathfrak{m} - \{0\}$ such that $\text{res}(a^{-m}) = 0$, for $m = 1, \dots, n$. □

Gauss extension

Fact 7.1.15 (Corollary 2.2.2 [EP05]). Let (K, v) be a valued field with value group Γ and residue field k . Then there is exactly one extension w of v to $K(x)$ such that $w(x) = 0$ and \bar{x} is transcendental over k . For this w , we have $k_{K(x)} = k(\bar{x})$ and $\Gamma_{K(x)} = \Gamma$.

Remark 7.1.16. The proof of Corollary 2.2.2 [EP05] provides us with

$$w(a_0 + a_1x + \dots + a_nx^n) = \min_{1 \leq i \leq n} v(a_i)$$

and $w(\frac{f}{g}) = w(f) - w(g)$, for any $f, g \in K[x]$.

The valuation w described in Fact 7.1.15 and Remark 7.1.16 is known as the *Gauss extension* of v from K to $K(x)$.

Remark 7.1.17. A lifting $\iota : k \rightarrow K$ automatically induces a lifting $\iota_{K(x)} : k(\bar{x}) \rightarrow K(x)$ such that $\iota|_{K(x)} = \iota$ and $\iota_{K(x)}(\bar{x}) = x$. As usual, we shall identify \bar{x} with its lift x to simplify notation.

Lemma 7.1.18. Let $(K, v, \text{res}, \iota) \models \text{VF}_{\text{res}, \iota}$ and $a, c \in K$ with $a \in \mathfrak{m} - \{0\}$ and $vc < \mathbb{Z}va$. Let w be the Gauss extension of v to $K(x)$ and $\iota_{K(x)} : k(x) \rightarrow K(x)$ be the induced lifting. There is a map $\text{res}_{K(x)} : K(x) \rightarrow k(x)$ such that $(K(x); w, \text{res}_{K(x)}, \iota_{K(x)})$ is a model of $\text{VF}_{\text{res}, \iota}$ extending $(K; v, \text{res}, \iota)$ and with $\text{res}_{K(x)}(\frac{c}{1-ax}) = 0$.

Proof. By Lemma 7.1.7, it suffices to show that the element $\frac{c}{1-ax}$ is $k(x)$ -linearly independent over $\mathcal{O}_{K(x)} + \langle K \rangle_{k(x)}$. Note that $\langle K \rangle_{k(x)} = K[x]_{k[x] - \{0\}}$, the latter being the localization of $K[x]$ at $k[x] - \{0\} \subseteq K[x]$. Suppose that

$$q(x) \cdot \frac{c}{1-ax} + \frac{f(x)}{g(x)} \in \mathcal{O}_{K(x)}$$

for some $q(x) \in k(x)^\times$, $f(x) \in K[x]$, $g(x) \in k[x] - \{0\}$. Dividing by $q(x)$, we may assume that $q(x) \equiv 1$ to begin with. Therefore, we have

$$\frac{c}{1-ax} + \frac{f(x)}{g(x)} \in \mathcal{O}_{K(x)}$$

for some $f(x) \in K[x]$, $g(x) \in k[x] - \{0\}$. Choose $f(x)$ and $g(x)$ as above, such that $\deg(f)$ is minimum; note that $f(x)$ cannot be 0 and thus $\deg(f)$ is well-defined. Write $f(x) = f_0 + \dots + f_nx^n$ with $f_i \in K$ and $g(x) = g_0 + \dots + g_nx^n$

with $g_i \in k$. Note that $w((1-ax)g(x)) = w(1-ax) + w(g(x)) = 0$ and therefore $w(cg(x) + (1-ax)f(x)) \geq 0$. By the definition of w , this is equivalent to

$$v(f_0 + cg_0) \geq 0 \quad (1) \quad \text{and} \quad v(f_m + cg_m - af_{m-1}) \geq 0 \quad (2) \quad \text{and} \quad v(af_n) \geq 0 \quad (3)$$

for $m = 1, \dots, n$.

Claim 1: We have that $vf_0 < 0$.

Proof. Indeed, suppose that $vf_0 \geq 0$. We will then have that $\frac{c}{1-ax} + \frac{f(x)-f_0}{g(x)} \in \mathcal{O}_{K(x)}$. If $g_0 = 0$, then write $f(x) - f_0 = x \cdot \tilde{f}(x)$ and $g(x) = x\tilde{g}(x)$ and note that $\frac{c}{1-ax} + \frac{\tilde{f}(x)}{\tilde{g}(x)} \in \mathcal{O}_{K(x)}$. Since $\deg(\tilde{f}) < \deg(f)$, this contradicts our minimality assumption on $\deg(f)$. Therefore $g_0 \neq 0$. We now see that $v(cg_0 + f_0) = \min\{v(cg_0), vf_0\} = vc < 0$, which contradicts (1). □*Claim 1*

Next, we prove:

Claim 2: For each $m = 0, \dots, n$, there exists $k_m \in \mathbb{N}$ such that $vf_m = vc + k_m va$.

Proof. We proceed inductively. For $m = 0$, recall from (1) that $v(f_0 + cg_0) \geq 0$. By Claim 1, we have that $vf_0 < 0$. We deduce that $g_0 \neq 0$ and therefore $vf_0 = v(cg_0) = vc$. For $m > 0$, recall that $v(f_m + cg_m - af_{m-1}) \geq 0$. By our induction hypothesis, we have $vf_{m-1} = vc + k_{m-1}va$, for some $k_{m-1} \in \mathbb{N}$. If $g_m = 0$, this gives $vf_m = vc + (k_{m-1} + 1)va$ and we may take $k_m = k_{m-1} + 1$. Suppose that $g_m \neq 0$. Since $va > 0$, we see that $v(af_{m-1}) > v(c)$ and thus $v(cg_m - af_{m-1}) = v(cg_m) = vc$. By (2), we have that $v(f_m + cg_m - af_{m-1}) \geq 0$. Since $vc < 0$, we deduce $vf_m = vc$ and $k_m = 0$. This finishes the proof of the inductive step. □*Claim 2*

In particular, we will have that $vf_n = vc + k_n va$, for some $k_n \in \mathbb{N}$. Since $vc < \mathbb{Z}va$, this finally gives $v(af_n) = vc + (k_n + 1)va < 0$, because of our assumption that $vc < \mathbb{Z}va$. This contradicts (3). □

Lemma 7.1.19. Let $(K; v, \text{res}, \iota)$ be an e.c. model of $\text{VF}_{\text{res}, \iota}$. Let $a \in \mathfrak{m} - \{0\}$ be such that $\text{res}(a^{-m}) = 0$, for all $m \in \mathbb{N}$. Let also $c \in K$ be such that $vc < \mathbb{Z}va$. For every $\alpha \in k$, there exists $\beta \in k^\times$ such that $\text{res}(\frac{c}{1-a\beta}) = 0$.

Proof. By Lemma 7.1.18 and since $(K; v, \text{res}, \iota)$ is an e.c. model of $\text{VF}_{\text{res}, \iota}$, we will have that

$$(K; v, \text{res}, \iota) \preceq_1 (K(x); w, \text{res}_{K(x)}, \iota_{K(x)})$$

Since $\text{res}_{K(x)}(\frac{c}{1-ax}) = 0$, there is $\beta \in k^\times$ with $\text{res}(\frac{c}{1-a\beta}) = 0$. □

7.1.20 Non-existence of a model companion of $\text{VF}_{\text{res},\iota}$

Generalities on model companions

Definition 7.1.21 (Definition 3.2.8 [TZ12]). Let T be a theory. A theory T^* is a model companion of T if the following conditions are satisfied:

- (a) Every model of T embeds into a model of T^* .
- (b) Every model of T^* embeds into a model of T .
- (c) T^* is model-complete.

Fact 7.1.22 (Theorem 3.2.9 [TZ12]). A theory T has, up to equivalence, at most one model companion T^* .

Definition 7.1.23. (a) Let M, N be L -structures. Suppose that $N \models \phi \Rightarrow M \models \phi$ for any existential sentence $\phi \in L(M)$. Then we say that M is *existentially closed* (or e.c.) in N and write $M \preceq_1 N$.

(b) Let T be a theory. A model $M \models T$ is said to be an existentially closed (or e.c.) model of T if $M \preceq_1 N$, for all $N \models T$.

Fact 7.1.24 (Theorem 3.2.14 [TZ12]). For any theory T , the following are equivalent:

- (a) T has a model companion T^* .
- (b) The e.c. models of T form an elementary class.

Moreover, if T^* exists, we have that $T^* =$ "the theory of e.c. models of T ".

Proof of Theorem 7.0.1

Theorem 7.1.25. *The theory $\text{VF}_{\text{res},\iota}$ does not admit a model companion.*

Proof. Assume otherwise and let T be the model companion of $\text{VF}_{\text{res},\iota}$. By Fact 7.1.24, the theory T is precisely the theory of e.c. models of $\text{VF}_{\text{res},\iota}$. Let $(K_0; v_0, \text{res}_0, \iota_0) \models T$ be \aleph_1 -saturated. By Lemma 7.1.14 and \aleph_1 -saturation, we may find $a \in \mathfrak{m}_0 - \{0\}$ such that $\text{res}_0(a^{-n}) = 0$ for $n > 0$. Let $(K; v, \text{res}, \iota) := (K_0; v_0, \text{res}_0, \iota_0)^U$ be an ultrapower of K_0 , where U is a non-principal ultrafilter on $\mathcal{P}(\mathbb{N})$. By Łoś's Theorem, we will have that $(K_0; v_0, \text{res}_0, \iota_0) \preceq (K; v, \text{res}, \iota)$. In particular, we will have $(K; v, \text{res}, \iota) \models T$ and therefore $(K; v, \text{res}, \iota)$ is an e.c. model of $\text{VF}_{\text{res},\iota}$.

Set $a^* := \lim_{n \rightarrow U} a^{-n}$. Since U is non-principal, we will have that $va^* < \mathbb{Z}va$. By Lemma 7.1.19, we may find $\beta \in k - \{0\}$ such that $\text{res}(\frac{a^*}{1-a\beta}) = 0$. By Łoś's Theorem, we will have that

$$\{n \in \mathbb{N} : K_0 \models \exists y \in k^\times (\text{res}_0(\frac{a^{-n}}{1-ay}) = 0)\} \in U$$

In particular, there exists $n \in \mathbb{N}$ and $\beta_0 \in \iota(k_0) - \{0\}$ such that $\text{res}(\frac{a^{-n}}{1-a\beta_0}) = 0$.

Note that

$$\frac{a^{-n}}{1-a\beta_0} \equiv \beta_0^n + a^{-1}\beta_0^{n-1} + \dots + a^{-n} \pmod{\mathfrak{m}}$$

Since $\text{res} : K \rightarrow k$ is k -linear, $\text{res}(\mathfrak{m}) = \{0\}$ and $\text{res}(a^{-m}) = 0$ for all $m \in \mathbb{N}$, we compute that

$$\text{res}\left(\frac{a^{-n}}{1-a\beta_0}\right) = \text{res}(\beta_0^n + a^{-1}\beta_0^{n-1} + \dots + a^{-n}) = \sum_{m=0}^n \beta_0^{m-i} \text{res}(a^{-m}) = \beta_0^n$$

This forces $\beta_0 = 0$, which is a contradiction. It follows that $\text{VF}_{\text{res},\iota}$ does not admit a model companion. \square

7.2 Undecidability of $k((t))$ with a total residue map

7.2.1 Definability

We consider the $L_{\text{res},\iota}$ -structure $(k((t)); v_t, \text{res}, \iota)$, which is a model of $\text{VF}_{\text{res},\iota}$ by Lemma 7.1.2. In fact, the lifting can be recovered from res according to the following:

Lemma 7.2.2. Let k be any field. The subfield $k \subseteq k((t))$ is \emptyset -definable in L_{res} . In particular, the lifting $\iota : k \rightarrow k((t))$ is \emptyset -definable in L_{res} .

Proof. We claim that $k = \{x \in k((t)) : \forall y(\text{res}(x \cdot y) = \text{res}(x) \cdot \text{res}(y))\}$. The inclusion " \subseteq " is clear: for all $x \in k$ and $y \in k((t))$, we have $\text{res}(x \cdot y) = x \cdot \text{res}(y) = \text{res}(x) \cdot \text{res}(y)$, using that $\text{res} : k((t)) \rightarrow k$ is k -linear and that $\text{res}(x) = x$ for $x \in k$. For the converse, write $x = \alpha + x'$ with $\alpha \in k$, $\text{res}(x') = 0$ and $x' \neq 0$. For $n = vx'$, we have

$$\text{res}(x \cdot t^{-n}) = \alpha \cdot \text{res}(t^{-n}) + \text{res}(x' \cdot t^{-n}) = \text{res}(x' \cdot t^{-n}) = \text{ac}(x') \neq 0$$

It follows that $\text{res}(x \cdot t^{-n}) \neq \text{res}(x) \cdot \text{res}(t^{-n}) = \text{res}(x) \cdot 0 = 0$. We have thus shown that the subfield $k \subseteq k((t))$ is \emptyset -definable in L_{res} . Finally, note that $\iota(\alpha) = a$ if and only if $a \in \iota(k)$ and $\text{res}(a) = \alpha$. \square

Because of Lemma 7.2.2, we will henceforth restrict our focus on the L_{res} -structure $(k((t)), \text{res})$. In complete analogy with the integer part $[r]$ of a real number $r \in \mathbb{R}$, one may define the *principal part* $[a]$ of $a \in k((t))$.

Definition 7.2.3. For $a \in c_n t^{-n} + \dots + c_0 + tk[[t]]$, we define the principal part of a to be $[a] = c_n t^{-n} + \dots + c_0 \in k[t^{-1}]$.

Each polynomial $p \in k[t^{-1}]$ gives rise to a function $p : k \rightarrow k : \alpha \mapsto p(\alpha)$ by simply evaluating p at $t^{-1} = \alpha$.

Comment 7.2.3.1. *We have refrained from making the substitution $z = t^{-1}$ and working with $k((1/z))$. Although this is arguably natural in the present context, it is not so natural in other parts of the chapter and would probably cause confusion.*

Lemma 7.2.4. We have $\text{res}(\frac{a}{1-ty}) = [a](y)$, for all $a \in k((t))$ and $y \in k$.

Proof. Write $a = c_n t^{-n} + \dots + c_0 + \epsilon$, where $n \in \mathbb{N}$ and $\epsilon \in tk[[t]]$. For $y \in k$, we compute

$$\begin{aligned} \frac{a}{1-ty} &= (c_n t^{-n} + \dots + c_0 + \epsilon)(1 + ty + \dots + t^n y^n + \dots) \\ &\equiv (c_n t^{-n} + \dots + c_0)(1 + ty + \dots + t^n y^n) \pmod{tk[[t]]} \end{aligned}$$

It follows that

$$\text{res}\left(\frac{a}{1-ty}\right) = \sum_{0 \leq i, j \leq n} \text{res}(t^{j-i}) c_i y^j = \sum_{0 \leq i, j \leq n} \delta_{ij} c_i y^j = \sum_{0 \leq i \leq n} c_i y^i = [a](y)$$

for every $y \in k$ (here δ_{ij} is the Kronecker symbol). □

Remark 7.2.5. Lemma 7.2.4 does *not* show that the function

$$[\] : k((t)) \rightarrow k((t)) : a \mapsto [a]$$

is definable in $L_{\text{res}, t}$. Instead, Lemma 7.2.4 shows that

$$k((t)) \rightarrow \text{Hom}(k, k) : a \mapsto (y \mapsto [a](y))$$

is definable in $L_{\text{res}, t}$.

For $a \in k((t))$, we introduce the set $Z([a]) = \{y \in k : [a](y) = 0\}$, namely the zero set of the principal part of a .

Remark 7.2.6. By Lemma 7.2.4, we see that $Z([a])$ is definable with a parameter for t and also uniformly in a .

Remark 7.2.7. Let $k((t^{\mathbb{Q}}))$ be the Hahn field with residue field k and value group \mathbb{Q} . Let $a = \sum_{q \in A} c_q t^q \in k((t^{\mathbb{Q}}))$, where $A \subseteq \mathbb{Q}$ is well-ordered. We write $p_a(X) = \sum_{n \in A \cap \mathbb{Z} \leq 0} c_q X^q$ and see that

$$\text{res}\left(\frac{a}{1-ty}\right) = p_a(y)$$

is definable in L_{res} , via the same calculation as in Lemma 7.2.4. Similarly, for the field $P(k) = \bigcup_{n \in \mathbb{N}} k((t^{1/n}))$ of Puiseux series over k .

7.2.8 Infinite residue field

Let k be an *infinite* field.

Undecidability of $(k((t)), \text{res})$

We isolate the following elementary fact from algebra to pinpoint exactly where the proof of Theorem 7.2.11 fails for the case where k is finite:

Lemma 7.2.9. Let k be any *infinite* field and $f(X) \in k[X]$. Then $f(X) \equiv 0$ if and only if $f(k) = \{0\}$.

Proof. Any non-zero polynomial over any field has finitely many roots. □

Fact 7.2.10 (Denef). Let $R = k[t]$, where k is any field. Then Hilbert's tenth problem over R with coefficients in $\mathbb{Z}[t]$ is unsolvable.

Proof. See [Den79b] and [Den78]. □

Theorem 7.2.11. Let k be an *infinite* field. Then $(k((t)), \text{res})$ is $\exists\forall$ -undecidable in $L_{\text{res}, t}$.

Proof. Let $p_1(X_1, \dots, X_m, T), \dots, p_n(X_1, \dots, X_m, T) \in \mathbb{Z}[X_1, \dots, X_m, T]$, where $n, m \in \mathbb{N}$. The system

$$p_1(X_1, \dots, X_m, T) = \dots = p_n(X_1, \dots, X_m, T) = 0$$

has a solution in $k[T]$ if and only if there exist $a_1, \dots, a_m \in k((t))$ such that

$$p_1(\lfloor a_1 \rfloor, \dots, \lfloor a_m \rfloor, t^{-1}) = \dots = p_n(\lfloor a_1 \rfloor, \dots, \lfloor a_m \rfloor, t^{-1}) = 0$$

By Lemma 7.2.9, this is also equivalent to the existence of $a_1, \dots, a_m \in k((t))$ such that

$$p_1(\lfloor a_1 \rfloor(y), \dots, \lfloor a_m \rfloor(y), y) = \dots = p_n(\lfloor a_1 \rfloor(y), \dots, \lfloor a_m \rfloor(y), y) = 0$$

for all $y \in k$. The latter condition is expressible in L_{res} by Lemma 7.2.4 and Lemma 7.2.2. We conclude from Fact 7.2.10. □

Remark 7.2.12. Using Remark 7.2.7, the proof of Theorem 7.2.11 works verbatim for the Hahn field $k((t^{\mathbb{Q}}))$ or the Puiseux series $P(k)$ over an infinite field k .

A natural example from complex analysis

Consider the field of Laurent series $\mathbb{C}((t))$ over the complex numbers \mathbb{C} . In complex analysis, one defines the *residue* of a complex function $f \in \mathbb{C}((t))$ at an isolated singularity $a \in \mathbb{C}$, denoted by $\text{Res}(f, a)$ or $\text{Res}_a(f)$. Numerically, if $f = \sum_{i=-n}^{\infty} c_i(t-a)^i$, we have that $\text{Res}(f, a) = c_{-1}$.

Corollary 7.2.13. *We have that $(\mathbb{C}((t)), \text{Res}_0)$ is $\exists\forall$ -undecidable in $L_{\text{Res}_0, t}$.*

Proof. Note that $\text{Res}_0(a) = \text{res}(t \cdot a)$ and conclude from Theorem 7.2.11. \square

7.2.14 Eliminating t

We now give a different proof of the undecidability of $(k((t)), \text{res})$, where k is an infinite field. The proof discussed here does not require a parameter for t and even shows that $(\mathbb{N}, +, \cdot)$ is interpretable without parameters. On the other hand, this seems to come at the expense of increasing the quantifier complexity, although we have not tried to fine-tune the argument.

Interpreting finite subsets of k

Recall that *weak* monadic second-order logic is the fragment of second-order logic where second-order quantification is restricted to quantification over *finite* subsets. We will eventually interpret the weak monadic second-order theory of $(k, +, \cdot)$ in $(k((t)), \text{res})$. We start with the following:

Lemma 7.2.15. For every $a \in k((t))$ and $b \in k[[t]]$, we have that the set

$$S_{a,b} = \{\beta \in k : \text{res}\left(\frac{a}{1-b\beta}\right) = 0\}$$

is either finite or equal to k .

Proof. Let $n \in \mathbb{N}$ be such that $ab^n \in tk[[t]]$. We then have that

$$\frac{a}{1-b\beta} \equiv a(1 + b\beta + b^2\beta^2 + \dots + b^n\beta^n) \pmod{tk[[t]]}$$

and therefore

$$\text{res}\left(\frac{a}{1-b\beta}\right) = 0 \iff p_{a,b}(\beta) = 0$$

where $p_{a,b}(X) = \sum_{k=0}^n \text{res}(ab^k) \cdot X^k \in k[X]$. If $p_{a,b}(X) \equiv 0$, then $S_{a,b} = k$ and otherwise $S_{a,b}$ is finite. \square

Remark 7.2.16. When $b = t$, the set $S_{a,t}$ has a nice description due to Lemma 7.2.4, namely

$$S_{a,t} = Z(\lfloor a \rfloor) = \{y \in k : \lfloor a \rfloor(y) = 0\}$$

In words, this is simply the set of zeros of the principal part of $\lfloor a \rfloor$.

Lemma 7.2.17. The weak monadic second-order theory of $(k, +, \cdot)$ is interpretable (without parameters) in $(k((t)), \text{res})$.

Proof. By Lemma 7.2.2, we have that the first-order theory $(k, +, \cdot)$ is \emptyset -interpretable in $(k((t)), \text{res})$. By Lemma 7.2.15, we have a uniformly \emptyset -definable family

$$\{S_{a,b} : a \in k((t)) \text{ and } b \in k[[t]]\}$$

of finite subsets of k together with k itself. Note that every finite subset $S \subseteq k$ arises as $S_{a,b}$, for some $a \in k((t))$ and $b \in k[[t]]$. Indeed, if $a = \prod_{s \in S} (t^{-1} - s)$ and $b = t$, then $S_{a,b} = Z(\lfloor a \rfloor) = S$ (see Remark 7.2.16). It follows that the weak monadic second-order theory of $(k, +, \cdot)$ is interpretable. \square

Lemma 7.2.18. Let k be an infinite field. Then the weak monadic second-order theory of $(k, +, \cdot)$ interprets $(\mathbb{N}, +, \cdot)$. In particular, it is undecidable.

Proof. See §1 [Che82]. \square

Theorem 7.2.19. Let k be an infinite field. Then $(k((t)), \text{res})$ interprets $(\mathbb{N}, +, \cdot)$. In particular, $(k((t)), \text{res})$ is undecidable in L_{res} .

Proof. Clear from Lemma 7.2.17 and Lemma 7.2.18. \square

Remark 7.2.20. When k is an infinite, perfect field of positive characteristic, one can also deduce Theorem 7.2.19 directly from Lemma 7.2.2 and Cherlin's result saying that $k((t))$ is undecidable with a predicate for $k \subseteq k((t))$ (see §2 [Che82]).

7.2.21 Defining $k[t]$

We now prove Proposition 7.2.26, which shows that $k[t] \subseteq k((t))$ is definable in $L_{\text{res},t}$.

Lemma 7.2.22. We have that $k[t] = \{x \in k((t)) : \exists y(y^2 = 1 + tx^2)\}$ if $\text{char}(k) \neq 2$ and $k[t] = \{x \in k((t)) : \exists y(y^3 = 1 + tx^3)\}$ if $\text{char}(k) = 2$.

Proof. This is a standard application of Hensel's Lemma. \square

Lemma 7.2.23. Let k be any field. For $a \in k((t))$, we have $\deg(\lfloor a \rfloor) = -va$.

Proof. Clear. \square

Lemma 7.2.24. Let k be an infinite field.

(a) The set $\text{Div} = \{(a, b) \in k((t))^2 : [a] \mid [b] \text{ in } k[t^{-1}]\}$ is definable in $L_{\text{res}, t}$.

(b) The set $\text{Irred} = \{a \in k((t)) : [a] \text{ is irreducible in } k[t^{-1}]\}$ is definable in $L_{\text{res}, t}$.

(c) The set $\text{Split} = \{a \in k((t)) : [a] \text{ splits over } k\}$ is definable in $L_{\text{res}, t}$.

Proof. (a) Since k is infinite, we have $[a] \mid [b]$ if and only if there exists $c \in k((t))$ such that $[b](y) = [a](y) \cdot [c](y)$, for all $y \in k$. The latter is expressible in $L_{\text{res}, t}$ by Lemma 7.2.4.

(b) We have that $[a]$ is irreducible in $k[t^{-1}]$ if and only if there is no $b \in k((t))$ such that $[b] \mid [a]$ and $1 < \deg([b]) < \deg([a])$. The latter is expressible in $L_{\text{res}, t}$ by part (a) and Lemma 7.2.23.

(c) Note that $\text{Split} = \{a \in k((t)) : \text{for all } b \in \text{Irred we have } \deg(b) = 1\}$. We conclude from (b) and Lemma 7.2.23. \square

Lemma 7.2.25. Let k be an infinite field. The subset $\text{Pow} = \bigcup_{n \in \mathbb{N}} t^{-n} + tk[[t]]$ of $k((t))$ is definable in $L_{\text{res}, t}$.

Proof. By Lemma 7.2.24 and Remark 7.2.6, we see that $\bigcup_{n \in \mathbb{N}} k^\times \cdot t^{-n} + tk[[t]] = \{a \in \text{Split} : Z([a]) = \{0\}\}$ is definable in $L_{\text{res}, t}$. Finally, note that

$$\text{Pow} = \left\{ a \in \bigcup_{n \in \mathbb{N}} k^\times \cdot t^{-n} + tk[[t]] : [a](1) = 1 \right\}$$

The latter is again definable by Lemma 7.2.4. \square

Proposition 7.2.26. Let k be an infinite field. The subring $k[t] \subseteq k[[t]]$ is definable in $L_{\text{res}, t}$.

Proof. The key is the following:

Claim: We have that

$$k[t] = \left\{ a \in k[[t]] : \exists n \in \mathbb{N} \text{ such that } \text{res}(a \cdot b) = 0 \text{ for all } b \in \bigcup_{N > n} t^{-N} + tk[[t]] \right\}$$

Proof. Suppose $a \in k[t]$ and write $a = c_0 + c_1 \cdot t + \dots + c_n \cdot t^n$. Let $b \in k((t))$ be of the form $b = t^{-N} + \epsilon$, for some $N > n$ and $\epsilon \in tk[[t]]$. It follows that

$$\text{res}(a \cdot b) = \text{res}(a \cdot t^{-N}) + \text{res}(a \cdot \epsilon) = \text{res}(a \cdot t^{-N}) = c_0 \cdot \text{res}(t^{-N}) + \dots + c_n \cdot \text{res}(t^{n-N}) = 0$$

since $N > n$.

Conversely, suppose $a \in k[[t]]$ and let $n \in \mathbb{N}$ be such that $\text{res}(a \cdot b) = 0$, for all $b \in \bigcup_{N > n} t^{-N} + tk[[t]]$. It follows that $\text{res}(a \cdot t^{-N}) = 0$ for all $N \geq n$ and therefore a is an element of $k[t]$ of degree at most n . \square *Claim*

By Lemma 7.2.25, the set $\bigcup_{n \in \mathbb{N}} t^{-n} + tk[[t]]$ is definable in $L_{\text{res}, t}$. By Lemma 7.2.22, the valuation ring $k[[t]]$ is definable in the language L_{rings} with a parameter for t . Finally Lemma 7.2.23 allows us to quantify over $\deg(b)$, whence $k[t]$ is definable in $L_{\text{res}, t}$ by the Claim. \square

7.2.27 Cherlin's Problem 3A

Finally, we would like to bring to the reader's attention the following interesting problem, which was originally posed by Cherlin and is very much in the spirit of the present chapter. Recall that the *angular component* map ac on $k((t))$ picks out the coefficient of the leading term:

$$\text{ac}(f) := \alpha_{i_0} \quad \text{if } f = \sum_{i \geq i_0} \alpha_i t^i, \alpha_{i_0} \neq 0, \quad \text{ac}(0) := 0.$$

When k is a *finite* field, the map $\text{ac} : k((t)) \rightarrow k$ is definable in L_{rings} with a parameter for t (see §5.1 [Che82]).

Problem 7.2.28 (Problem 3A [Che82]). Let k be an infinite field of positive characteristic. Is $(k((t)), \text{ac})$ undecidable?

Cherlin [Che82] shares his premonition that this should be the case. For now, we simply note that the angular component was encoded in $(k((t)), \text{res})$ all along:

Observation 7.2.29. Let k be any field. The angular component $\text{ac} : k((t)) \rightarrow k$ is definable in $L_{\text{res}, t}$.

Proof. If k is finite, this is standard and does not require the total residue map (see §5.1 [Che82]). Suppose k is infinite. Note that

$$\text{ac}(x) = 1 \iff \text{there is } a \in \text{Pow} \text{ such that } vx = va \text{ and } v(x - a) > va$$

The latter is expressible by Lemma 7.2.25 and Lemma 7.2.22. Finally, for $x \in k((t))$ and $\alpha \in k$, we have that $\text{ac}(x) = \alpha$ if and only if $\text{ac}(\frac{x}{v(\alpha)}) = 1$. The latter is expressible in $L_{\text{res}, t}$ by Lemma 7.1.11. \square

Bibliography

- [Abh56] Shreeram Abhyankar. “Two notes on formal power series”. In: *Proceedings of the American Mathematical Society Vol. 7, No. 5, pp. 903-905* (1956).
- [Abh66] Shreeram Abhyankar. *Resolution of singularities of embedded algebraic surfaces*. Academic Press, New York, 1966.
- [AF16] Sylvy Anscombe and Arno Fehm. “The existential theory of equicharacteristic henselian valued fields”. In: *Algebra & Number Theory Volume 10, Number 3 (2016), 665-683*. (2016).
- [AJ19] Sylvy Anscombe and Franziska Jahnke. “The model theory of Cohen rings”. In: [arXiv: 1904.08297 \[math.LO\]](https://arxiv.org/abs/1904.08297) (2019).
- [AK16] Sylvy Anscombe and Franz-Viktor Kuhlmann. “Notes on extremal and tame valued fields”. In: *The Journal of Symbolic Logic , Volume 81 , Issue 2 , pp. 400-416* (2016).
- [AK65] James Ax and Simon Kochen. “Diophantine problems over local fields II”. In: *Amer. J. Math.* 87 (1965).
- [AK66] James Ax and Simon Kochen. “Diophantine problems over local fields III”. In: *Annals of Mathematics, Second Series, Vol. 83, No. 3, pp. 437-456* (1966).
- [Ax68] James Ax. “The Elementary Theory of Finite Fields”. In: *Annals of Mathematics , Second Series, Vol. 88, No. 2 , pp. 239-271* (1968).
- [Bas78] Serban A. Basarab. “Some model theory for henselian valued fields”. In: *Journal of Algebra* (1978).
- [Bas91] Serban A. Basarab. “Relative elimination of quantifiers for Henselian valued fields”. In: *Annals of Pure and Applied Logic* 53 ((1991)).
- [BDL80] J. Becker, J. Denef, and L. Lipshitz. “Further remarks on the elementary theory of formal power series rings”. In: *Model Theory of Algebra and Arithmetic, Pacholski et al. eds., LNM 834, Springer-Verlag NY, pp 1-9* (1980).
- [BDLD79] J. Becker, J. Denef, L. Lipshitz, and L. van den Dries. “Ultraproducts and Approximation in Local Rings I”. In: *Inventiones math.* 51,189-203 (1979).
- [BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron Models*. *Ergeb. Math. Grenz.*, 21, Springer, New York-Heidelberg-Berlin, 1990.

- [Bou06] Nicolas Bourbaki. *Algèbre Commutative. Chapitres 5 à 7*. Springer, 2006.
- [Bél99] Luc Bélair. “Types dans les corps valués munis d’applications coefficients”. In: *Illinois J. Math.* 43(2):410-425 (1999).
- [CDLM13] Raf Cluckers, Jamshid Derakhshan, Eva Leenknegt, and Angus Macintyre. “Uniformly defining valuation rings in Henselian valued fields with finite or pseudo-finite residue fields”. In: *Ann. Pure Appl. Logic* 164, 12, 1236-1246. (2013).
- [CDM81] Gregory Cherlin, Lou van den Dries, and Angus Macintyre. “Decidability and undecidability theorems for PAC-fields”. In: *Bull. Amer. Math. Soc. (N.S.)* 4(1): 101-104 (1981).
- [CF67] J.W.S Cassels and A. Fröhlich. *Algebraic Number Theory*. Proceedings of an Instructional Conference Organized by the London Mathematical Society, 1967.
- [CH99] Zoé Chatzidakis and Ehud Hrushovski. “Model Theory of Difference Fields”. In: *Transactions of the American Mathematical Society* 351.8 (1999), pp. 2997–3071. ISSN: 00029947. URL: <http://www.jstor.org/stable/118013> (visited on 04/19/2022).
- [Che82] Gregory Cherlin. “Undecidability of rational function fields in nonzero characteristic”. In: *Logic Colloq., no. 82, North-Holland, Amsterdam*. (1982).
- [CLO04] David Cox, John Little, and Donal O’Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2004.
- [Con16] Brian Conrad. “From normal crossings to strict normal crossings”. In: *available at <http://math.stanford.edu/~conrad/249BW17Page/handouts/crossings.pdf>* ((2016)).
- [CP08] Vincent Cossart and Olivier Piltant. “Resolution of singularities of threefolds in positive characteristic. I.: Reduction to local uniformization on Artin-Schreier and purely inseparable coverings”. In: *Journal of Algebra Volume 320, Issue 3, Pages 1051-1082* (2008).
- [CP09] Vincent Cossart and Olivier Piltant. “Resolution of singularities of threefolds in positive characteristic. II”. In: *Journal of Algebra Volume 321, Issue 7, Pages 1836-1976* (2009).
- [CP19] Vincent Cossart and Olivier Piltant. “Resolution of singularities of arithmetical threefolds”. In: *Journal of Algebra Volume 529, 1 July 2019, Pages 268-535* (2019).
- [Cut09] Steven Dale Cutkosky. “Resolution of Singularities for 3-folds in positive characteristic”. In: *American J. of Math.* 131, 59-127. (2009).

- [Del84] Pierre Deligne. “Les corps locaux de caractéristique p , limites de corps locaux de caractéristique 0”. In: *Représentations des groupes réductifs sur un corps local*, - Hermann Paris (1984).
- [Den16] Jan Denef. “Geometric proofs of Ax-Kochen and Ershov”. In: *American Journal of Mathematics* 138(1):181-199 ((2016)).
- [Den78] Jan Denef. “The Diophantine Problem for Polynomial Rings of Positive Characteristic”. In: *Transactions of the American Mathematical Society* (1978).
- [Den79a] Jan Denef. “The Diophantine problem for polynomial rings of positive characteristic”. In: *Logic Colloq., no. 78, North-Holland, Amsterdam* (1979).
- [Den79b] Jan Denef. “The Diophantine Problem for Polynomial Rings and rings of rational functions”. In: *Logic Colloquium, North-Holland Publishing Company* (1979).
- [Der20] Jamshid Derakhshan. “Model Theory of Adeles and Number Theory”. In: arXiv: 2007.09237 [math.LO] (2020).
- [DM16] Jamshid Derakhshan and Angus Macintyre. “Model completeness for henselian fields with finite ramification valued in a \mathbb{Z} -group”. In: arXiv: 1603.08598 [math.LO] (2016).
- [DM22] Jamshid Derakhshan and Angus Macintyre. “Model theory of adèles I”. In: *Ann. Pure Appl. Logic* 173.3 (2022), Paper No. 103074, 43. ISSN: 0168-0072. DOI: 10.1016/j.apal.2021.103074. URL: <https://doi.org/10.1016/j.apal.2021.103074>.
- [Dri12] Lou van den Dries. *Lectures on the Model Theory of Valued Fields*. Ed. by Carlo Toffalori H. Dugald Macpherson. Chapter 5 in the book "Model Theory in Algebra, Analysis and Arithmetic", Cetraro, Italy, 2012.
- [Dri99] Lou van den Dries. “On the elementary theory of rings of Witt vectors with a multiplicative set of representatives for the residue field”. In: *Manuscripta Mathematica*, 98(2) (1999).
- [DS03] Jan Denef and Hans Schoutens. “On the Decidability of the Existential Theory of $\mathbb{F}_p[[t]]$ ”. In: *Fields Institute Communications Volume 00, 0000* (2003).
- [End72] Otto Endler. *Valuation theory*. Berlin ; New York: Springer-Verlag, 1972.
- [EP05] Antonio J. Engler and Alexander Prestel. *Valued Fields*. Springer Monographs in Mathematics, 2005.
- [Ers65] Ju.L. Ershov. “On elementary theories of local fields”. In: *Algebra i Logika* 4, No. 2, 5-30 (1965).

- [FF14] Laurent Fargues and Jean-Marc Fontaine. “Vector bundles on curves and p -adic Hodge theory”. In: *Chapter 2 in "Automorphic forms and Galois representations"* (2014).
- [FF19] Laurent Fargues and Jean-Marc Fontaine. “Courbes et fibrés vectoriels en théorie de Hodge p -adique”. In: (2019). URL: <https://webusers.imj-prg.fr/~laurent.fargues>.
- [Fle11] Joseph Flenner. “Relative decidability and definability in henselian valued fields”. In: *J. Symbolic Logic* 76(4): 1240-1260 (2011).
- [FW79] Jean-Marc Fontaine and Jean-Pierre Wintenberger. “Extensions algébriques et corps des normes des extensions APF des corps locaux”. French. In: *C. R. Acad. Sci., Paris, Sér. A* 288 (1979), pp. 441–444. ISSN: 0366-6034.
- [GR03] Ofer Gabber and Lorenzo Ramero. *Almost ring theory*. Lecture Notes in Mathematics, vol. 1800, Berlin: Springer-Verlag, 2003.
- [Gro65] Alexander Grothendieck. *Éléments de géométrie algébrique : IV. Étude locale des schémas et des morphismes de schémas, Seconde partie*. Publications mathématiques de l’ I.H.É.S., tome 24, p. 5-231, 1965.
- [GTZ88] Patrizia Giannia, Barry Trager, and Gail Zacharias. “Gröbner bases and primary decomposition of polynomial ideals”. In: *Journal of Symbolic Computation Volume 6, Issues 2-3, Pages 149-167* ((1988)).
- [Hir64] Heisuke Hironaka. “Resolution of Singularities of an Algebraic Variety Over a Field of Characteristic Zero: I”. In: *Annals of Mathematics, Second Series, Vol. 79, No. 1* (1964).
- [HK09] Ehud Hrushovski and David Kazhdan. “Motivic Poisson summation”. In: *Mosc. Math. J., Volume 9, Number 3, Pages 569-623* (2009).
- [Hod93] Wilfrid Hodges. *Model theory*. Cambridge University Press, 1993.
- [Jon96] Aise Johan de Jong. “Smoothness, semi-stability and alterations”. In: *Publications mathématiques de l’I.H.É.S., tome 83, p. 51-93* ((1996)).
- [Kar20] Konstantinos Kartas. “Decidability via the tilting correspondence”. In: [arXiv: 2001.04424 \[math.LO\]](https://arxiv.org/abs/2001.04424) (2020).
- [Kar21a] Konstantinos Kartas. “An undecidability result for the asymptotic theory of p -adic fields”. In: [arXiv:2105.03771 \[math.LO\]](https://arxiv.org/abs/2105.03771) (2021).
- [Kar21b] Konstantinos Kartas. “Diophantine problems over tamely ramified fields”. In: [arXiv: 2103.14646 \[math.AG\]](https://arxiv.org/abs/2103.14646) (2021).
- [Kar22] Konstantinos Kartas. “Valued fields with a total residue map”. In: [arXiv:2203.02374 \[math.LO\]](https://arxiv.org/abs/2203.02374) (2022).
- [Kaz86] D. Kazhdan. “Representations of groups over close local fields”. In: *Journal d’ Analyse Mathématique volume 47, pages 175-179* (1986).
- [Ked06] Kiran S. Kedlaya. “Finite automata and algebraic extensions of function fields”. In: *J. Théor. Nombres Bordeaux, 18(2):379-420* (2006).

- [Kei70] H. Jerome Keisler. “Ultraproducts and Saturated Models”. In: *Journal of Symbolic Logic* 35.4 (1970), pp. 584–585. DOI: 10.2307/2271469.
- [KK09] Franz-Viktor Kuhlmann and Hagen Knaf. “Every place admits local uniformization in a finite extension of the function field”. In: *Advances in Mathematics* (2009).
- [KL15] Kiran S. Kedlaya and Ruochuan Liu. “Relative p -adic Hodge theory: Foundations”. In: *Astérisque* (2015).
- [Koc74] Simon Kochen. “The model theory of local fields”. In: *ISILC Logic Conference (Proc. Internat. Summer Inst. and Logic Colloq., Kiel, 1974)*, pp. 384–425. *Lecture Notes in Math., Vol. 499*, Springer, Berlin, 1975. (1974).
- [Koe18] Jochen Koenigsmann. “Decidability in local and global fields”. In: *Proc. Int. Cong. of Math. Rio de Janeiro, Vol. 1 (45-60)* (2018).
- [KR21] Franz-Viktor Kuhlmann and Anna Rzepka. “The valuation theory of deeply ramified fields and its connection with defect extensions”. In: arXiv: 1811.04396v3[math.AC] (2021).
- [Kra56] M. Krasner. “Approximation des corps valués complets de caractéristique $p = 0$ par ceux de caractéristique 0”. In: *Colloque d’Algebre Superleure Bruxelles* (1956).
- [KT18] Kiran S. Kedlaya and M Temkin. “Endomorphisms of power series fields and residue fields of Fargues-Fontaine curves”. In: *Proceedings of the American Mathematical Society*, (2018).
- [Kuh] Franz-Viktor Kuhlmann. *Book on Valuation Theory (in preparation)*.
- [Kuh00] Franz-Viktor Kuhlmann. *Valuation Theoretic and Model Theoretic Aspects of Local Uniformization*. Resolution of Singularities pp 381–456, 2000.
- [Kuh10] Franz-Viktor Kuhlmann. “Elimination of ramification I: the generalized stability theorem”. In: *Trans. Amer. Math. Soc.* 362.11 (2010), pp. 5697–5727. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-2010-04973-6. URL: <https://doi.org/10.1090/S0002-9947-2010-04973-6>.
- [Kuh11] Franz-Viktor Kuhlmann. “The Defect”. In: *Commutative algebra - Noetherian and non-Noetherian perspectives, 277-318*, Springer, New York (2011).
- [Kuh16a] Franz-Viktor Kuhlmann. “The Algebra and Model Theory of Tame Valued fields”. In: *Journal für die reine und angewandte Mathematik* (2016).
- [Kuh16b] Franz-Viktor Kuhlmann. “The Algebra and Model Theory of Tame Valued fields”. In: *Journal für die reine und angewandte Mathematik* (2016).

- [Kuh94] Franz-Viktor Kuhlmann. “Quantifier elimination for henselian fields relative to additive and multiplicative congruences”. In: *Israel Journal of Mathematics volume 85, pages 277-306* (1994).
- [Lan94] Serge Lang. *Algebraic Number Theory*. Springer, Graduate Texts in Mathematics book series (GTM, volume 110), 1994.
- [Lee20] Junguk Lee. “Hyperfields, truncated DVRs, and valued fields”. In: *Journal of Number Theory, Volume 212, July 2020, Pages 40-71* (2020).
- [Lis21] Victor Lisinski. “Decidability of equal characteristic tame Hahn fields in the language L_t ”. In: [arXiv:2108.04132\[math.LO\]](https://arxiv.org/abs/2108.04132) (2021).
- [Liu06] Qing Liu. “Algebraic Geometry and Arithmetic Curves”. In: *Oxford University Press* ((2006)).
- [LL21] Junguk Lee and Wan Lee. “On the structure of certain valued fields”. In: *Annals of Pure and Applied Logic Volume 172, Issue 4* (2021).
- [Lur18] Jacob Lurie. “The Fargues-Fontaine curve”. In: (2018). URL: <https://www.math.ias.edu/~lurie/205.html>.
- [Mac86] Angus Macintyre. “Twenty Years of p -adic Model Theory”. In: *Logic Colloquium '84 Elsevier Science Publishers B. K (North-Holland)* (1986).
- [Mar02] David Marker. *Model Theory : An Introduction*. Springer-Verlag New York, 2002.
- [Mat70] Yuri Matiyasevich. “Enumerable sets are Diophantine”. In: *Dokl. Akad. Nauk SSSR 191, 279- 282*. (1970).
- [MB11] Laurent Moret-Bailly. “An extension of Greenberg’s theorem to general valuation rings”. In: *Manuscripta mathematica* (2011).
- [Mor17] Matthew Morrow. “The Fargues-Fontaine curve and diamonds.” In: *Séminaire Bourbaki* (2017).
- [Nic13] Johannes Nicaise. “Geometric criteria for tame ramification”. In: *Mathematische Zeitschrift volume 273, pages 839-868* (2013).
- [OV82] MA de Orozco and WY Vélez. “The lattice of subfields of a radical extension”. In: *Journal of Number Theory* (1982).
- [Pap94] Christos H. Papadimitriou. *Computational Complexity*. Addison-Wesley Publishing Company, Inc., 1994.
- [Phe87] Thanases Pheidas. “An Undecidability Result for Power Series Rings of Positive Characteristic. II”. In: *Proceedings of the American Mathematical Society, Vol. 100, pp. 526-530* (1987).
- [RZ60] Abraham Robinson and Elias Zakon. “Elementary properties of ordered abelian groups”. In: *Transactions of the American Mathematical Society* (1960).

- [Sca99] Thomas Scanlon. “Quantifier Elimination for the relative Frobenius”. In: *Valuation Theory and Its Applications Volume II, Conference proceedings of the International Conference on Valuation Theory (Saskatoon)* (1999).
- [Sch] Peter Scholze. “How many untilts?” In: (). URL: <https://mathoverflow.net/q/390522>.
- [Sch12a] Peter Scholze. “Perfectoid Spaces”. In: *Publ. Math. Inst. Hautes Études Sei.* 116 (2012).
- [Sch12b] Peter Scholze. “Perfectoid Spaces: A survey”. In: *Current Developments in Mathematics* (2012).
- [Sch14] Peter Scholze. “Perfectoid Spaces and their Applications”. In: *Proceedings of the ICM* (2014).
- [Sei78] Abraham Seidenberg. “Constructions in a Polynomial Ring Over the Ring of Integers”. In: *American Journal of Mathematics*, Vol. 100, pp. 685-703 (1978).
- [Ser80] Jean-Pierre Serre. *Local Fields*. New York: Springer-Verlag, 1980.
- [Spi20] Mark Spivakovsky. *Resolution of Singularities: An Introduction*. Chapter 3 of the book: Handbook of Geometry and Topology of Singularities I, Springer International Publishing, 2020.
- [Stacks] The Stacks Project Authors. *Stacks Project*. <https://stacks.math.columbia.edu>.
- [Tem08a] Michael Temkin. “Desingularization of quasi-excellent schemes in characteristic zero”. In: *Advances in Mathematics* 219, pp. 488-522 (2008).
- [Tem08b] Michael Temkin. “Inseparable local uniformization”. In: *Journal of Algebra*, Volume 373, 1 January 2013, Pages 65-119 (2008).
- [Tem11] Michael Temkin. “Absolute desingularization in characteristic zero”. In: *Motivic integration and its interactions with model theory and non-archimedean geometry*, Volume II London Math. Soc. Lecture Note Ser. 384 213-250 (2011).
- [TZ12] Katrin Tent and Martin Ziegler. *A Course in Model Theory*. Lecture Notes in Logic. Cambridge University Press, 2012. DOI: 10.1017/CB09781139015417.
- [Was97] Lawrence C. Washington. *Introduction to Cyclotomic Fields*. Springer Science & Business Media, Graduate texts in mathematics, 1997.
- [Wei90] V. Weispfenning. “Existential equivalence of ordered abelian groups with parameters”. In: *Arch. Math. Logic* 29.4 (1990), pp. 237-248. ISSN: 0933-5846. DOI: 10.1007/BF01651327. URL: <https://doi.org/10.1007/BF01651327>.
- [Zar40] Oscar Zariski. “Local uniformization on algebraic varieties”. In: *Ann. of Math.*, 2, 41 (4): 852-896 (1940).

- [Zie72] Martin Ziegler. *Die elementare Theorie der henselschen Körper*. Dissertation thesis, 1972.