

Observability of nonlinear systems with unmeasured inputs^{☆,☆☆}

K. Maes^{a,*}, M.N. Chatzis^b, G. Lombaert^a

^a*KU Leuven, Department of Civil Engineering, Leuven, Belgium*

^b*The University of Oxford, Department of Engineering Science, Oxford, United Kingdom*

Abstract

This paper presents a geometric algorithm to investigate the theoretical observability of nonlinear systems with partially measured inputs and outputs. The algorithm is based on Lie algebra and applies to systems whose state and measurement equations are analytical and affine in all inputs. It investigates whether the system satisfies a necessary observability condition that is named the Observability Rank Condition for systems with Direct Feedthrough (ORC-DF). The presented algorithm allows to assess the observability of the dynamic system states, the identifiability of constant-in-time parameters, and the ability to track unmeasured inputs, which is referred to as system invertibility. It is also shown how the developed methodology can be extended to investigate the observability of non-smooth systems that can be broken into different smooth branches, often encountered in mechanical applications related to sliding and damage. Possible applications are illustrated with several examples from structural engineering.

Keywords: system identification, geometric observability, observability, identifiability, invertibility

1. Introduction

Civil and mechanical engineering systems are often characterized by uncertainty in structural parameters, for example stiffness, as well as uncertainties in the inputs applied to the system and the corresponding system states. Sensors can be installed on the structure to reduce this uncertainty by means of system identification and structural health monitoring techniques [1, 2]. The emergence of such techniques in recent years has created a clear need for optimal sensor placement strategies [3, 4]. Most optimal sensor placement techniques that are presented in the literature aim at minimizing the uncertainty on the estimated quantities. However, before selecting an optimal sensor layout, one should deal with the observability of the system [5]. This is a theoretical property of the dynamic system that determines whether or not the unknown quantities (states, inputs, and parameters) can be estimated under the specific set of measurements assumed. If a state, an input, or a parameter

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*Corresponding author. Tel.: +32 (0) 16 32 25 73.

Email address: kristof.maes@kuleuven.be (K. Maes)

is unobservable, it cannot be expected that any system identification method would be successful at estimating their value properly.

The joint estimation of parameters and states generally results in augmenting the state vector to include both the dynamic states and unknown parameters of the system [6]. This state augmentation results in a nonlinear augmented system, even if the underlying dynamic system is linear. Hence, it can be argued that all problems where parameters, states, and inputs of the system are estimated require the use of observability tools for nonlinear systems.

Whereas the observability of time-invariant linear systems is well understood [7], the concept of observability of nonlinear systems remains a challenge in many engineering disciplines. In general, two separate approaches to deal with the observability of nonlinear systems are presented in the literature, which are referred to as geometric and algebraic approaches. The former are applicable to analytical systems, while the latter have been developed for the special case of polynomial and rational systems [8, 9, 10]. Most geometric observability methods build on the work of Hermann and Krener [11], which extends the observability test for linear systems to nonlinear analytical systems by developing the Observability Rank Condition (ORC) based on Lie algebra [12, 13]. The ORC has applications in many fields, including civil and mechanical engineering [6], chemistry [14] and biology [15, 16].

The previously discussed observability methods assume all system inputs to be measured and therefore do not cover the ability to track the system states in presence of unmeasured inputs, nor the ability to track the unmeasured system inputs, referred to in the literature as system invertibility [17].

The conditions that lead to invertibility of time-invariant linear systems, i.e. linear systems with constant and known parameters, are well elaborated in the literature. Following the work of Sain and Massey in [17, 18], Maes et al. [19] have developed a practical implementation to design a sensor network for the inversion of structural dynamics systems, with special focus on modally reduced order models. Whereas [19] focuses on the case of instantaneous system inversion, i.e. inversion without any delay, an extension of the invertibility conditions for the case of delayed system inversion is provided in [20].

A recent extension of the ORC for investigating the observability of the states of systems with partially measured inputs and for investigating the ability to observe unmeasured inputs and their derivatives with respect to time was suggested by Martinelli in [21, 22]. The resulting method is named the Extended ORC or EORC. A common assumption of both the ORC and EORC is that the measurements are not a function of the inputs. However, that assumption is often not satisfied for engineering systems where the measurements can be affected by a direct feedthrough, as is for example the case when accelerations are measured in structural dynamics.

This paper builds on previous works [6, 11, 21] to present an algorithm that studies the observability of nonlinear analytical systems with partially measured inputs and outputs affected by a direct feedthrough. The presented method serves as an extension of both the ORC and the EORC. It investigates whether the system satisfies an analytical observability condition that is named the Observability Rank Condition for systems with Direct Feedthrough (ORC-DF). For a system to satisfy the ORC-DF is necessary when performing so-called joint input-state-parameter estimation [23, 24, 25]. When used in absence of unmeasured inputs, however, the ORC-DF is still applicable and it extends the regular ORC to cases where the system response is directly affected by the (measured) inputs. Although the presented method assumes analytical systems, it is shown how the method can be extended to investigate the observability of non-smooth systems that can

be broken into different smooth branches, which are often encountered in applications related to sliding and damage [26, 27]. The method is used to study the observability of suitably chosen examples from structural engineering.

2. Methodology

Consider a system driven by n_u measured inputs u_i , n_p unmeasured inputs p_i , and n_y measured outputs y_i , as schematically represented in Fig. 1. Without loss of generality, the dynamic behavior of this system can be mathematically described by the following nonlinear continuous-time state-space representation:

$$\dot{\mathbf{x}}_t = \mathbf{E}(\mathbf{x}_t, \boldsymbol{\theta}, \mathbf{u}, \mathbf{p}) \quad (1)$$

$$\mathbf{y} = \mathbf{H}(\mathbf{x}_t, \boldsymbol{\theta}, \mathbf{u}, \mathbf{p}) \quad (2)$$

where \mathbf{x}_t is the state vector of size n_{st} , \mathbf{u} is a vector of n_u measured inputs, \mathbf{p} is a vector of n_p unmeasured inputs, $\boldsymbol{\theta}$ is the vector of n_t parameters, \mathbf{y} is the output vector of size n_y , and \mathbf{E} and \mathbf{H} are nonlinear vector functions of dimension n_{st} and n_y , respectively.

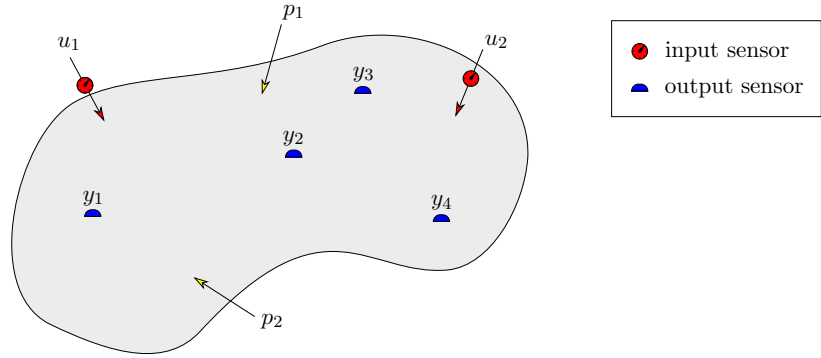


Fig. 1: Schematic representation of a dynamic system with measured inputs u_i , unmeasured inputs p_i , and measured outputs y_i .

Under the assumption that the system parameters are constant in time ($\dot{\boldsymbol{\theta}} = \mathbf{0}$), system (1–2) can be rewritten in augmented form:

$$\dot{\mathbf{x}} = \mathbf{e}(\mathbf{x}, \mathbf{u}, \mathbf{p}) \quad (3)$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{p}) \quad (4)$$

with

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_t \\ \boldsymbol{\theta} \end{bmatrix} \text{ and } \mathbf{e}(\mathbf{x}, \mathbf{u}, \mathbf{p}) = \begin{bmatrix} \mathbf{E}(\mathbf{x}_t, \boldsymbol{\theta}, \mathbf{u}, \mathbf{p}) \\ \mathbf{0} \end{bmatrix}. \quad (5)$$

In the latter state-space representation, the parameters are treated as states of the augmented system. In order to distinguish between the representation of the system in Eqs. (1–2) and Eqs. (3–4), they are referred to in the following as the underlying dynamic system and its augmented form, respectively. It should be pointed out that even if the underlying dynamic system is linear, its

augmented form will generally be nonlinear due to potential products between the augmented states [6].

Unless stated otherwise, it is assumed in the remainder of this paper that the vector functions \mathbf{e} and \mathbf{h} in Eqs. (3–4) are analytical, that is, infinitely differentiable. In addition, one special category of analytical systems is considered, that of affine-input nonlinear systems which may be written in the following form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{j=1}^{n_u} \mathbf{g}_{uj}(\mathbf{x})u_j + \sum_{j=1}^{n_p} \mathbf{g}_{pj}(\mathbf{x})p_j \quad (6)$$

$$\mathbf{y} = \mathbf{h}_0(\mathbf{x}) + \sum_{j=1}^{n_u} \mathbf{h}_{uj}(\mathbf{x})u_j + \sum_{j=1}^{n_p} \mathbf{h}_{pj}(\mathbf{x})p_j \quad (7)$$

where \mathbf{x} is the state vector of size $n_s = n_{s_t} + n_t$, u_j , $j = 1, \dots, n_u$, are the measured inputs, p_j , $j = 1, \dots, n_p$, are the unmeasured, and hence unknown, inputs, \mathbf{f} , \mathbf{g}_{uj} , and \mathbf{g}_{pj} are nonlinear vector functions of dimension n_s , and \mathbf{h}_0 , \mathbf{h}_{uj} , and \mathbf{h}_{pj} are nonlinear vector functions of dimension n_y . The vector of measured inputs is defined as $\mathbf{u} = [u_1, u_2, \dots, u_{n_u}]^T$, and the vector of unmeasured inputs as $\mathbf{p} = [p_1, p_2, \dots, p_{n_p}]^T$. The system described by Eqs. (6–7) is denoted by \mathcal{S} . Table 1 compares the restrictions to the system that are assumed in the development of the ORC and EORC rank conditions as well as the ORC-DF rank condition presented in this paper.

	ORC [12, 13]	EORC [21]	ORC-DF
Measured inputs	$n_u \geq 0$	$n_u \geq 0$	$n_u \geq 0$
Unmeasured inputs	$n_p = 0$	$n_p \geq 0$	$n_p \geq 0$
Direct feedthrough	$\mathbf{h}_{uj}(\mathbf{x}) = \mathbf{0}$	$\mathbf{h}_{uj}(\mathbf{x}) = \mathbf{h}_{pj}(\mathbf{x}) = \mathbf{0}$	-

Table 1: Restrictions to the system assumed in the development of different observability rank conditions.

This section develops a step-wise observability algorithm to investigate whether the states \mathbf{x} (including the dynamic states \mathbf{x}_t and parameters $\boldsymbol{\theta}$) and the unmeasured inputs \mathbf{p} can be observed from the measured response \mathbf{y} in presence of measured and hence known inputs \mathbf{u} . Before the algorithm is formulated, the underlying theory is presented. The derivation starts with an augmentation of the system, which enables a significant simplification of the following steps. Next, the Lie derivative operator is introduced. After its definition, it is shown how the sequential application of the Lie derivative to the measurement equations allows to investigate the observability of a system in presence of unknown inputs, leading to the formulation of the observability algorithm.

Similarly to what was done in [21], a further augmented system \mathcal{S}^l is defined by extending the state vector of the system \mathcal{S} to include the unmeasured inputs p_j and their time derivatives up to order l :

$$\mathbf{x}^l = [\mathbf{x}^T, \mathbf{p}^T, \dot{\mathbf{p}}^T, \dots, \mathbf{p}^{(l)T}]^T \quad (8)$$

where $\mathbf{p}^{(l)} = \frac{d^l \mathbf{p}}{dt^l}$. The augmented state vector \mathbf{x}^l is of size $n_{s^l} = n_s + (l+1)n_p$. The dynamics of

the system \mathcal{S}^l are described by the following equations:

$$\dot{\mathbf{x}}^l = \mathbf{f}^l(\mathbf{x}^l) + \sum_{j=1}^{n_u} \mathbf{g}_{uj}^l(\mathbf{x})u_j + \sum_{j=1}^{n_p} \mathbf{S}_j^l p_j^{(l+1)} \quad (9)$$

$$\mathbf{y} = \mathbf{h}_0(\mathbf{x}) + \sum_{j=1}^{n_u} \mathbf{h}_{uj}(\mathbf{x})u_j + \sum_{j=1}^{n_p} \mathbf{h}_{pj}(\mathbf{x})p_j \quad (10)$$

where

$$\mathbf{f}^l(\mathbf{x}^l) = \begin{bmatrix} \mathbf{f}(\mathbf{x}) + \sum_{j=1}^{n_p} \mathbf{g}_{pj}(\mathbf{x})p_j \\ \dot{\mathbf{p}} \\ \ddot{\mathbf{p}} \\ \vdots \\ \mathbf{p}^{(l)} \\ \mathbf{0}_{n_p \times 1} \end{bmatrix}, \quad \mathbf{g}_{uj}^l(\mathbf{x}) = \begin{bmatrix} \mathbf{g}_{uj}(\mathbf{x}) \\ \mathbf{0}_{(l+1)n_p \times 1} \end{bmatrix}, \quad \mathbf{S}_j^l = \begin{bmatrix} \mathbf{0}_{(n_s + ln_p + j - 1) \times 1} \\ 1 \\ \mathbf{0}_{(n_p - j) \times 1} \end{bmatrix} \quad (11)$$

with $\mathbf{0}_{a \times b}$ a matrix of dimensions a by b containing zeros. Note that the measurement equation has not changed due to the state augmentation. In addition, as mentioned in [21], the system \mathcal{S}^l still has n_u known inputs and n_p unknown inputs, which now coincide with the $(l+1)^{\text{th}}$ order time derivatives of the unknown input vector \mathbf{p} . It should further be noted that the systems described by Eqs. (6–7) and Eqs. (9–11) represent the same relation between the inputs \mathbf{u} and \mathbf{p} and the output \mathbf{y} .

Definition 2.1. The Lie derivative of a vector of n_o scalar functions $\mathbf{\Omega} = [o_1(\mathbf{x}^l), \dots, o_{n_o}(\mathbf{x}^l)]^T$ along a vector field $\mathbf{\kappa}^l = [\kappa_1, \dots, \kappa_{n_{sl}}]^T$ is defined as:

$$L_{\mathbf{\kappa}^l}(\mathbf{\Omega}) = d^l \mathbf{\Omega} \cdot \mathbf{\kappa} \quad (12)$$

where $d^l \mathbf{\Omega}$ is defined as:

$$d^l \mathbf{\Omega} = \begin{bmatrix} \nabla o_1(\mathbf{x}^l) \\ \vdots \\ \nabla o_{n_o}(\mathbf{x}^l) \end{bmatrix} \quad (13)$$

and ∇ denotes the gradient of a scalar function with respect to the vector \mathbf{x}^l and \cdot in Eq. (12) stands for the post-multiplication of the matrix $d^l \mathbf{\Omega}$ with the vector $\mathbf{\kappa}^l$.

Definition 2.2. The m^{th} order Lie derivative of a vector of n_o scalar functions $\mathbf{\Omega}$ along m vector fields $\mathbf{\kappa}_1^l, \dots, \mathbf{\kappa}_m^l$ of size n_{sl} , with $m \leq l$, is defined as:

$$L_{\mathbf{\kappa}_m^l} \dots L_{\mathbf{\kappa}_1^l}(\mathbf{\Omega}) = L_{\mathbf{\kappa}_m^l} \circ \dots \circ L_{\mathbf{\kappa}_1^l}(\mathbf{\Omega}) \quad (14)$$

where \circ denotes the successive application of the Lie derivation operator. The zero-order Lie derivative equals the vector $\mathbf{\Omega}$ itself.

Note that the first order Lie derivative ($m = 1$) equals the Lie derivative of the vector $\mathbf{\Omega}$ along the vector field $\mathbf{\kappa}_1^l$, as given in Definition 2.1.

Proposition 2.1. *The m^{th} order Lie derivative of the output vector \mathbf{y} along m vector fields $\boldsymbol{\mu}_1^l, \dots, \boldsymbol{\mu}_m^l$ that belong to the set of vector fields $\{\mathbf{f}^l, \mathbf{g}_{u1}^l, \dots, \mathbf{g}_{u n_u}^l\}$ is independent of the time derivatives of the unmeasured inputs of any order $q > m$.*

Proof. The proof is given in Appendix A. \square

In what follows, we will consider the case of piecewise constant measured inputs \mathbf{u} , defined in the following equation:

$$u_j(t) = u_j^i \quad \text{for } t \in [\bar{t}_{i-1}, \bar{t}_i)$$

where $\bar{t}_i = \bar{t}_0 + t_1 + \dots + t_i$ for any $i > 0$, and t_1, \dots, t_i represent i infinitesimally small time intervals. As will be shown later, considering a piecewise constant input allows deriving a straightforward and stepwise method to investigate the theoretical observability of a system. If a system is observable for piecewise constant inputs \mathbf{u} , it must also be observable if the inputs are any bounded function [28]. Meanwhile, one could argue that by assuming $t_1, \dots, t_i \rightarrow 0$, every input u_j could be considered piecewise constant. In addition to the assumption of piecewise constant measured inputs, assume a piecewise constant $(l+1)^{\text{th}}$ order time derivative of the unmeasured inputs \mathbf{p} :

$$p_j^{(l+1)}(t) = p_{j,l+1}^i \quad \text{for } t \in [\bar{t}_{i-1}, \bar{t}_i)$$

Without loss of generality, we consider the case where $l = \infty$ in the remainder of the manuscript. Next, consider the vector fields $\boldsymbol{\theta}_i^l$, given by:

$$\boldsymbol{\theta}_i^l = \mathbf{f}^l(\mathbf{x}^l) + \sum_{j=1}^{n_u} \mathbf{g}_{uj}^l(\mathbf{x}) u_j^i + \sum_{j=1}^{n_p} \mathbf{S}_j^l p_{j,l+1}^i \quad (15)$$

The vector fields $\boldsymbol{\theta}_i^l$ play an important role in the evolution of the state \mathbf{x}^l of the system \mathcal{S}^l over time. Indeed, for any time interval $t \in [\bar{t}_{i-1}, \bar{t}_i)$, the time history of the state \mathbf{x}^l is described by the flow Φ_i^l , which depends directly on the vector field $\boldsymbol{\theta}_i^l$ [13]:

$$\begin{cases} \frac{\partial \Phi_i^l(\mathbf{x}^l(\bar{t}_{i-1}), t)}{\partial t} = \boldsymbol{\theta}_i^l(\Phi_i^l(\mathbf{x}^l(\bar{t}_{i-1}), t)) \\ \mathbf{x}^l(\bar{t}_i) = \Phi_i^l(\mathbf{x}^l(\bar{t}_{i-1}), t = \bar{t}_i) \end{cases} \quad (16)$$

For a given initial condition $\mathbf{x}^l(\bar{t}_{i-1})$ and given piecewise constant inputs u_j^i and $p_{j,l+1}^i$, Eq. (16) can be solved to obtain the state $\mathbf{x}^l(t) = \Phi_i^l(\mathbf{x}^l(\bar{t}_{i-1}), t)$ for any time t in the interval $[\bar{t}_{i-1}, \bar{t}_i)$. By successive evaluation of the flow Φ_i^l at time $\bar{t}_1, \bar{t}_2, \dots, \bar{t}_k$, with $k \leq l$, one obtains:

$$\begin{aligned} \mathbf{x}^l(\bar{t}_1) &= \Phi_1^l(\mathbf{x}^l(\bar{t}_0), t_1) \\ \mathbf{x}^l(\bar{t}_2) &= \Phi_2^l(\mathbf{x}^l(\bar{t}_1), t_2) = \Phi_2^l \circ \Phi_1^l(\mathbf{x}^l(\bar{t}_0), t_1, t_2) \\ &\vdots \\ \mathbf{x}^l(\bar{t}_k) &= \Phi_k^l \circ \dots \circ \Phi_1^l(\mathbf{x}^l(\bar{t}_0), t_1, t_2, \dots, t_k) \end{aligned}$$

Under the assumption of piecewise constant measured inputs and a piecewise constant $(l+1)^{\text{th}}$ order time derivative of the unmeasured inputs, the following equality holds for every continuously differentiable function $\alpha(\mathbf{x}^l)$ (see also [13]):

$$\frac{\partial \alpha(\Phi_k^l \circ \dots \circ \Phi_1^l(\mathbf{x}^l(\bar{t}_0), t_1, t_2, \dots, t_k))}{\partial t_1 \dots \partial t_k} \Big|_{t_1, \dots, t_k \rightarrow 0} = L_{\boldsymbol{\theta}_1^l} \dots L_{\boldsymbol{\theta}_k^l} \alpha(\mathbf{x}^l) \Big|_{t=\bar{t}_0} \quad (17)$$

Lemma 2.1. *Two state vectors \mathbf{x}_a^l and \mathbf{x}_b^l that describe the initial condition of the system \mathcal{S}^l at time $t = \bar{t}_0$ lead to the same output \mathbf{y} at time $t = \bar{t}_k$ for any possible piecewise constant inputs \mathbf{u} , if and only if:*

$$L_{\nu_k^{k-1}} \dots L_{\nu_2^1} L_{\nu_1^0}(\mathbf{h}_{\text{xp}}(\mathbf{x}^0)) \Big|_{\mathbf{x}^k = \mathbf{x}_a^k} = L_{\nu_k^{k-1}} \dots L_{\nu_2^1} L_{\nu_1^0}(\mathbf{h}_{\text{xp}}(\mathbf{x}^0)) \Big|_{\mathbf{x}^k = \mathbf{x}_b^k} \quad (18)$$

$$L_{\nu_k^{k-1}} \dots L_{\nu_2^1} L_{\nu_1^0}(\mathbf{h}_{\text{uj}}(\mathbf{x})) \Big|_{\mathbf{x}^k = \mathbf{x}_a^k} = L_{\nu_k^{k-1}} \dots L_{\nu_2^1} L_{\nu_1^0}(\mathbf{h}_{\text{uj}}(\mathbf{x})) \Big|_{\mathbf{x}^k = \mathbf{x}_b^k}, \quad j = 1, \dots, n_{\text{u}} \quad (19)$$

where $\mathbf{h}_{\text{xp}}(\mathbf{x}^0) = \mathbf{h}_0(\mathbf{x}) + \sum_{j=1}^{n_{\text{p}}} \mathbf{h}_{\text{pj}}(\mathbf{x}) p_j$ and considering all possible combinations of vector fields ν_q^{q-1} belonging to the set $\{\mathbf{f}_{\text{xp}}^{q-1}, \mathbf{g}_{\text{u1}}^{q-1}, \dots, \mathbf{g}_{\text{un}_{\text{u}}}^{q-1}\}$, with $\mathbf{f}_{\text{xp}}^{q-1} = \mathbf{f}^{q-1} + \sum_{j=1}^{n_{\text{p}}} \mathbf{S}_j^{q-1} p_j^{(q)}$.

Proof. The proof is given in Appendix B. \square

Definition 2.3. *Following Hermann and Krener [11], a system \mathcal{S} is classified as observable in presence of unmeasured inputs if the initial state \mathbf{x}^0 at time $t = \bar{t}_0$ can be separated locally from its neighbors based on the system output \mathbf{y} for $t \in [\bar{t}_0, \infty)$. This imposes that the only initial state \mathbf{x}_b^0 in the direct closed neighborhood of a given initial state \mathbf{x}_a^0 that leads to the same output \mathbf{y} for $t \in [\bar{t}_0, \infty)$ for every possible input \mathbf{u} satisfies $\mathbf{x}_b^0 = \mathbf{x}_a^0$.*

Definition 2.4. *The states \mathbf{x}^k of the augmented system \mathcal{S}^k are classified as k -row observable if their value at time $t = \bar{t}_0$ can be separated locally from its neighbors based on the output \mathbf{y} at $k+1$ consecutive times $t = \bar{t}_0, \bar{t}_1, \dots, \bar{t}_k$, meanwhile considering every possible input \mathbf{u} over the considered time interval.*

Definitions 2.3 and 2.4 relate to the definition of local weak observability for systems with known inputs by Hermann and Krener in [11], which imposes that the system response must allow to distinguish a given state \mathbf{x}_0 from its neighbors $\mathbf{x}_0 + \Delta \mathbf{x}_0$, with $\Delta \mathbf{x}_0$ any infinitesimal vector. In the work of Hermann and Krener [11], a further distinction is made between local weak observability, weak observability, and observability. From a practical point of view and for the purposes of this paper, this distinction is not considered here.

Lemma 2.2. *Consider the vector field Ω_k , which is recursively defined as:*

$$\Omega_{k+1} = \begin{bmatrix} \Omega_k \\ \Delta \Omega_{k+1} \end{bmatrix}, \quad \Delta \Omega_{k+1} = \begin{bmatrix} L_{\mathbf{f}_{\text{xp}}^k}(\Delta \Omega_k) \\ L_{\mathbf{g}_{\text{u1}}^k}(\Delta \Omega_k) \\ \vdots \\ L_{\mathbf{g}_{\text{un}_{\text{u}}}^k}(\Delta \Omega_k) \end{bmatrix}, \quad \Omega_0 = \begin{bmatrix} \mathbf{h}_{\text{xp}}(\mathbf{x}^0) \\ \mathbf{h}_{\text{u1}}(\mathbf{x}) \\ \vdots \\ \mathbf{h}_{\text{un}_{\text{u}}}(\mathbf{x}) \end{bmatrix}, \quad \Delta \Omega_0 = \Omega_0 \quad (20)$$

All the states \mathbf{x}^k of the augmented system \mathcal{S}^k are k -row observable if $\text{rank}(\mathbf{d}^k \Omega_k) = n_{\text{s}} + (k+1)n_{\text{p}}$, where $\mathbf{d}^k \Omega_k$ denotes the Jacobian matrix of Ω_k with respect to the state vector \mathbf{x}^k and $n_{\text{s}} + (k+1)n_{\text{p}}$ equals the total number of states of the equivalent augmented system \mathcal{S}^k .

Proof. For a state vector \mathbf{x}^k to be observable, every set of output functions \mathbf{y} at $k+1$ consecutive times $\bar{t}_0, \bar{t}_1, \dots, \bar{t}_k$ should correspond to a finite set of states \mathbf{x}^k for every possible input \mathbf{u} (see Lemma 2.1). In other words, for every \mathbf{x}_a^k , one should obtain a finite set of solutions \mathbf{x}_b^k satisfying $\Omega_k(\mathbf{x}_a^k) = \Omega_k(\mathbf{x}_b^k)$, where $\mathbf{x}_b^k = \mathbf{x}_a^k$ is the only solution in the direct closed neighborhood of \mathbf{x}_a^k . Under the assumption that all functions in the vector $\Omega_k(\mathbf{x}^k)$ are continuously differentiable, the inverse function theorem [29] ensures that this holds if $\text{rank}(\mathbf{d}^k \Omega_k) = n_{\text{s}} + (k+1)n_{\text{p}}$. \square

If $\text{rank}(\mathbf{d}^k \boldsymbol{\Omega}_k) < n_s + (k+1)n_p$, the state vector \mathbf{x}^k is not k -row observable, implying that not all of the states \mathbf{x} , inputs \mathbf{p} and their derivatives $\dot{\mathbf{p}}, \ddot{\mathbf{p}}, \dots, \mathbf{p}^{(k)}$ can be independently observed. In this case, it is desirable from an identification point of view to investigate the partial observability for each of the states of the system \mathcal{S}^k separately, i.e. whether each of the states is k -row observable or not. In line with the method used to investigate partial observability in case of the ORC discussed in [6], the following corollary is derived.

Corollary 2.1. *Let $\mathbf{d}^k \boldsymbol{\Omega}_k^m$ denote the matrix that is obtained after removing the m^{th} column from the matrix $\mathbf{d}^k \boldsymbol{\Omega}_k$. If and only if $\text{rank}(\mathbf{d}^k \boldsymbol{\Omega}_k^m) = \text{rank}(\mathbf{d}^k \boldsymbol{\Omega}_k) - 1$, the m^{th} state of \mathbf{x}^k is k -row observable.*

Lemma 2.3. *If a state of \mathbf{x}^k is k -row observable, it will also be a q -row observable state of \mathbf{x}^q for any $q \geq k$.*

Proof. The proof is given in Appendix C. □

Definition 2.5. *A system \mathcal{S} satisfies the Observability Rank Criterion for systems with Direct Feedthrough (ORC-DF) if, a value k can be found such that the states \mathbf{x} and unmeasured inputs \mathbf{p} are k -row observable.*

Lemma 2.4. *If a system \mathcal{S} satisfies the ORC-DF, it is observable in presence of unmeasured inputs.*

Proof. If the states \mathbf{x} and inputs \mathbf{p} are k -row observable, it follows directly from Lemma 2.3 that they remain observable for $k \rightarrow \infty$, i.e. when considering the system output \mathbf{y} for $t \in [\bar{t}_0, \infty)$. □

Taking into account the aforementioned considerations, the following algorithm can be applied to verify the observability of the system \mathcal{S} in presence of unmeasured inputs.

Algorithm 2.1.

1. *Starting point:* $k = 0$, $\mathbf{x}^0 = [\mathbf{x}_t^T \quad \boldsymbol{\theta}^T \quad \mathbf{p}^T]^T = [\mathbf{x}^T \quad \mathbf{p}^T]^T$.
2. *Compose* $\Delta \boldsymbol{\Omega}_0 = [\mathbf{h}_{\text{xp}}(\mathbf{x}^0)^T \quad \mathbf{h}_{\text{u1}}(\mathbf{x})^T \quad \dots \quad \mathbf{h}_{\text{un}_u}(\mathbf{x})^T]^T$ and $\boldsymbol{\Omega}_0 = \Delta \boldsymbol{\Omega}_0$.
3. *Calculate* $\mathbf{d}^0 \boldsymbol{\Omega}_0 = \frac{\partial \boldsymbol{\Omega}_0}{\partial \mathbf{x}^0}$.
4. $k = k + 1$.
5. *Calculate* $\Delta \boldsymbol{\Omega}_k = \left[L_{\mathbf{f}_{\text{xp}}^{k-1}}(\Delta \boldsymbol{\Omega}_{k-1})^T \quad L_{\mathbf{g}_{\text{u1}}^{k-1}}(\Delta \boldsymbol{\Omega}_{k-1})^T \quad \dots \quad L_{\mathbf{g}_{\text{un}_u}^{k-1}}(\Delta \boldsymbol{\Omega}_{k-1})^T \right]^T$.
6. *Compose* $\mathbf{x}^k = [\mathbf{x}^{k-1T} \quad \mathbf{p}^{(k)T}]^T$.
7. *Calculate* $\mathbf{d}^k \Delta \boldsymbol{\Omega}_k = \frac{\partial \Delta \boldsymbol{\Omega}_k}{\partial \mathbf{x}^k}$.
8. *Calculate* $\mathbf{d}^k \boldsymbol{\Omega}_k = [\mathbf{d}^{k-1} \boldsymbol{\Omega}_{k-1} \quad \mathbf{0}] \cup \mathbf{d}^k \Delta \boldsymbol{\Omega}_k$.
9. *Optional step:* eliminate dependent rows from $\mathbf{d}^k \boldsymbol{\Omega}_k$.
10. If $\text{rank}(\mathbf{d}^k \boldsymbol{\Omega}_k) = n_s + (k+1)n_p$, *end*.
11. *Investigate partial observability for the m -th state of \mathbf{x}^k .*
 - a. *Starting point:* $m = 1$.

- b. If the m^{th} state is $(k-1)$ -row observable, it is also k -row observable. Go to step f.
 - c. Compose $d^k \mathbf{\Omega}_k^m$ by removing the m^{th} column from $d^k \mathbf{\Omega}_k$.
 - d. The m^{th} state is k -row observable if and only if $\text{rank}(d^k \mathbf{\Omega}_k^m) < \text{rank}(d^k \mathbf{\Omega}_k)$.
 - e. If $m = n_s + (k-1)n_p$, end.
 - f. $m=m+1$ and go to step b.
12. If the states \mathbf{x} and \mathbf{p} are k -row observable, end.
13. Go to step 4.

As far as the rank of $d^k \mathbf{\Omega}_k$ in step 10 of Algorithm 2.1 is concerned, one is interested in the generic rank and not in reduced ranks that might occur at certain values of the states corresponding to singularities. Such singular points are generally more interesting for control applications. Furthermore, step 9 of the algorithm is optional in the sense that if the dependent rows are not eliminated, the result of the algorithm does not change, but redundant calculations are performed in future steps. This is due to the fact that the dependent rows do not provide any additional information.

If Algorithm 2.1 has come to an end, the system \mathcal{S} is found to be observable in accordance with Lemma 2.4. This means that the constant-in-time parameters $\boldsymbol{\theta}$ can be identified and that the dynamic state variables \mathbf{x}_t and unmeasured inputs \mathbf{p} can be tracked over time. To understand the latter, it is important to see that the time \bar{t}_0 at which the initial conditions are defined can be shifted to any time of interest t . In addition, if one is able to observe the unmeasured inputs \mathbf{p} at an infinite number of infinitesimally closely spaced points in time $t, t+t_1, t+t_2, \dots$, with $t_i \rightarrow 0$, one could also argue that all its higher time derivatives $\dot{\mathbf{p}}, \ddot{\mathbf{p}}, \dots$ can be calculated as well. Therefore, if an unmeasured input \mathbf{p} is observable, so are all its higher derivatives $\dot{\mathbf{p}}, \ddot{\mathbf{p}}, \dots$.

The examples given in Section 4 show that Algorithm 2.1 succeeds in identifying systems that are in fact observable. For practical reasons, however, one might stop at a value of k for which none of the stopping criteria in step 10 or step 12 of Algorithm 2.1 is satisfied yet, i.e. the states \mathbf{x} and inputs \mathbf{p} are not found to be observable. At that point, it cannot be shown that the k -row unobservable states would remain unobservable in future steps. However, the states found to be observable are guaranteed to remain observable (see Lemma 2.3). The authors are studying the idea of a more general stopping criterion. For now, the reader is referred to Section 4.2 for a practical guideline on how to determine the maximum value of k in the case of an apparently unobservable system.

In the application of Algorithm 2.1, the Lie derivatives and, therefore, the gradients, should all be computed analytically by means of symbolic operations. An illustration of these operations for a small-scale system with only two dynamic states is included in Appendix D. Due to the computational cost of symbolic operations [31, 32] the suggested algorithm, as well as the original ORC, can be limited by the available RAM of standard desktop PC for systems with an augmented state of the order of a few dozens of DOFs. To apply the algorithm to more demanding cases, it is necessary to seek more efficient methods of calculating Lie Derivatives. The authors are currently working on an extension of the ideas suggested by Sedoglavic in [10] for the case of the original ORC with polynomial nonlinearities to the new EORC-DF algorithm suggested in this paper. Such ideas allow for an exact calculation of Lie Derivatives for specific realizations of the states and are expected to result in applying the suggested algorithm to systems with a large number of augmented states.

4. Examples

4.1. Application of the ORC-DF to a linear system

In this section, the ORC-DF is applied to a linear system with known time-invariant parameters, for which the augmented states match the dynamic states, i.e., $\mathbf{x} = \mathbf{x}_t$, and no parameters $\boldsymbol{\theta}$ are to be identified. For such a system, the dynamic behavior can generally be described by the following state-space equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_u\mathbf{u} + \mathbf{B}_p\mathbf{p} \quad (22)$$

$$\mathbf{y} = \mathbf{G}\mathbf{x} + \mathbf{J}_u\mathbf{u} + \mathbf{J}_p\mathbf{p} \quad (23)$$

where \mathbf{A} , \mathbf{B}_u , \mathbf{B}_p , \mathbf{G} , \mathbf{J}_u , and \mathbf{J}_p represent constant and known system matrices.

For the considered case of a linear system with known time-invariant parameters, it is easily verified that – after elimination of zero rows – the Jacobian matrix $\mathbf{d}^k\boldsymbol{\Omega}_k$ obtained from using Eq. (20) for any $k \geq 0$ is given by:

$$\mathbf{d}^k\boldsymbol{\Omega}_k = \begin{bmatrix} \boldsymbol{\mathcal{O}}_k & \boldsymbol{\mathcal{H}}_k \end{bmatrix} \quad (24)$$

where $\boldsymbol{\mathcal{O}}_k \in \mathbb{R}^{(k+1)n_y \times n_s}$ and $\boldsymbol{\mathcal{H}}_k \in \mathbb{R}^{(k+1)n_y \times (k+1)n_p}$ denote the extended observability matrix and the Toeplitz matrix, respectively, given by [20]:

$$\boldsymbol{\mathcal{O}}_k = \begin{bmatrix} \mathbf{G} \\ \mathbf{GA} \\ \mathbf{GA}^2 \\ \vdots \\ \mathbf{GA}^k \end{bmatrix}, \quad \boldsymbol{\mathcal{H}}_k = \begin{bmatrix} \mathbf{J}_p & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{GB}_p & \mathbf{J}_p & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{GAB}_p & \mathbf{GB}_p & \mathbf{J}_p & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{GA}^{k-1}\mathbf{B}_p & \mathbf{GA}^{k-2}\mathbf{B}_p & \mathbf{GA}^{k-3}\mathbf{B}_p & \cdots & \mathbf{J}_p \end{bmatrix} \quad (25)$$

The first n_s columns of the matrix $\mathbf{d}^k\boldsymbol{\Omega}_k$, contained in the matrix $\boldsymbol{\mathcal{O}}_k$, are related to the dynamic system states \mathbf{x} . The remaining $(k+1)n_p$ columns of the matrix $\mathbf{d}^k\boldsymbol{\Omega}_k$, contained in the matrix $\boldsymbol{\mathcal{H}}_k$, are related to the system inputs \mathbf{p} and their time derivatives $\dot{\mathbf{p}}, \dots, \mathbf{p}^{(k)}$.

Firstly, it is observed that the Jacobian matrix $\mathbf{d}^k\boldsymbol{\Omega}_k$ is independent from the matrices \mathbf{B}_u and \mathbf{J}_u and, therefore, that the observability of a linear system in absence of unknown parameters is not affected by the measured inputs \mathbf{u} . This is due to the fact that the terms $\mathbf{B}_u\mathbf{u}$ and $\mathbf{J}_u\mathbf{u}$ in Eqs. (22) and (23) are independent from the dynamic states \mathbf{x} and the unmeasured inputs \mathbf{p} , and therefore do not appear in the Jacobian matrix $\mathbf{d}^k\boldsymbol{\Omega}_k$. Secondly, the states \mathbf{x} and unmeasured inputs \mathbf{p} are k -row observable if and only if the first $n_s + n_p$ columns of the Jacobian matrix $\mathbf{d}^k\boldsymbol{\Omega}_k$ are independent from each other and the remaining rows of the Jacobian matrix. This is equivalent to imposing the following two rank criteria (see also Corollary 2.1):

$$\text{rank}(\boldsymbol{\mathcal{O}}_k) = n_s \quad \text{and} \quad \text{rank}(\boldsymbol{\mathcal{H}}_k^{1,\dots,n_p}) = \text{rank}(\boldsymbol{\mathcal{H}}_k) - n_p \quad (26)$$

where $\boldsymbol{\mathcal{H}}_k^{1,\dots,n_p}$ is obtained from $\boldsymbol{\mathcal{H}}_k$ by removing the first block column (n_p columns). The first rank criterion is the well-known observability rank criterion for linear systems [5], which ensures the observability of the system in absence of unmeasured inputs. The second rank criterion can be further elaborated, taking into account the following equality:

$$\boldsymbol{\mathcal{H}}_k^{1,\dots,n_p} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\mathcal{H}}_{k-1} \end{bmatrix} \quad (27)$$

and, consequently:

$$\text{rank}(\mathcal{H}_k^{1,\dots,n_p}) = \text{rank}(\mathcal{H}_{k-1}) \quad (28)$$

leading to:

$$\text{rank}(\mathcal{H}_{k-1}) = \text{rank}(\mathcal{H}_k) - n_p \quad (29)$$

This is the rank criterion that is known in the literature as the invertibility criterion for linear systems [17]. Finally, note that the invertibility criterion presented in [17] and therefore the ORC-DF do not impose the uniqueness of the estimated inputs and the stability of the system inversion [19].

4.2. A nonlinear two-DOF system

In this example, the observability of the nonlinear two-DOF system shown in Fig. 2 is examined in presence of a single unmeasured input. The displacement, velocity, and acceleration of mass m_j are denoted as x_j , \dot{x}_j , \ddot{x}_j , respectively. An unmeasured force F_{p2} is applied at mass m_2 . A measured force F_{u1} is applied at mass m_1 . The stiffness of the nonlinear spring that connects mass m_1 to the fixed support is displacement-proportional and given by $k_1 + \delta_{k1}x_1$. The spring with stiffness k_2 and the dampers with damping constants c_1 and c_2 are linear. The quantities to be identified are the dynamic states of the system x_1 , \dot{x}_1 , x_2 , and \dot{x}_2 , the parameters of the nonlinear spring k_1 and δ_{k1} , the mass m_2 , and the input F_{p2} . For presentation purposes, the remaining parameters are assumed to be known. It is assumed that displacement x_1 and acceleration \ddot{x}_2 are measured. Note that all the parameters of the system are time-invariant, despite the fact that the overall stiffness of the nonlinear spring depends on the state.

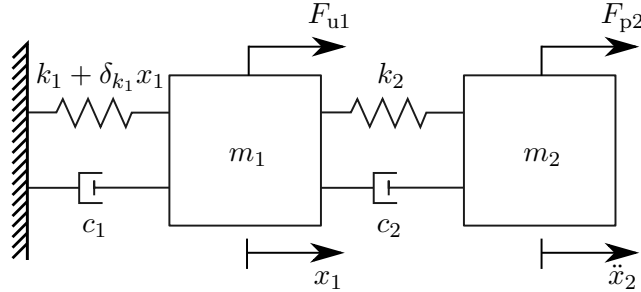


Fig. 2: Model of the two-DOF system in Example 4.2.

Taking into consideration that the parameters k_1 and δ_{k1} are time-invariant, the augmented state-space equations of the system can be written as follows:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \\ k_1 \\ \delta_{k1} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ -(k_1 + \delta_{k1}x_1)x_1 + k_2(x_2 - x_1) - c_1\dot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1) + F_{u1}/m_1 \\ (k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) + F_{p2})/m_2 \\ 0 \\ 0 \end{bmatrix} \quad (30)$$

$$\mathbf{y} = \begin{bmatrix} x_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ (k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) + F_{p2})/m_2 \end{bmatrix} \quad (31)$$

Writing Eqs. (30–31) in the form (6–7), with $n_u = 1$, $n_p = 1$, $u_1 = F_{u1}$, and $p_1 = F_{p2}$, yields the following expressions for the nonlinear vector functions \mathbf{f} , \mathbf{g}_{u1} , \mathbf{g}_{p1} , \mathbf{h}_0 , \mathbf{h}_{u1} , and \mathbf{h}_{p1} :

$$\mathbf{f} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ (-(k_1 + \delta_{k_1}x_1)x_1 + k_2(x_2 - x_1) - c_1\dot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1))/m_1 \\ (k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2))/m_2 \\ 0 \\ 0 \end{bmatrix} \quad (32)$$

$$\mathbf{g}_{u1} = [0 \ 0 \ 1/m_1 \ 0 \ 0 \ 0]^T, \quad \mathbf{g}_{p1} = [0 \ 0 \ 0 \ 1/m_2 \ 0 \ 0]^T \quad (33)$$

$$\mathbf{h}_0 = \begin{bmatrix} x_1 \\ (k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2))/m_2 \end{bmatrix}, \quad \mathbf{h}_{u1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{h}_{p1} = \begin{bmatrix} 0 \\ 1/m_2 \end{bmatrix} \quad (34)$$

Two different cases are considered to investigate the influence of known inputs on the system observability. In the first case, it is assumed that F_{u1} is a sufficiently exciting input that is measured. In the second case, there is no measured input applied ($F_{u1} = 0$). In both cases, the unmeasured force F_{p2} is applied to the second mass. Figure 3 summarizes the observability in the two considered cases. Figure 3a shows the evolution of $\text{rank}(\mathbf{d}\mathbf{\Omega}_k)$ with k . Figure 3b presents the k -row observability of the states. The results are shown for steps k from 1 to 8. Note, however, that Algorithm 2.1 would stop the rank computation for $k = 4$ in the case with known input F_{u1} , where all states of the augmented state vector \mathbf{x}^4 are found to be observable. For the case where $F_{u1} = 0$, however, the rank computation was manually terminated at step $k = 8$. At this step, all states are found to be observable, except for the mass m_2 , the unmeasured input F_{p2} , and all its higher derivatives. As from $k = 5$, the rank condition $n_s + (k+1)n_p$ (Fig. 3, black dotted line) and $\text{rank}(\mathbf{d}\mathbf{\Omega}_k)$ (Fig. 3, blue dashed line) increase with the same rate when k is further increased, leading to two parallel lines. Although one cannot exclude the possibility that the unobservable states will become observable at some point, the authors believe that the occurrence of these parallel lines is an indication that the unobservable states and inputs of this system will remain unobservable. From a practical point of view, this means that the rank computation for this example can be terminated at $k = 6$.

This example shows that, similarly to the regular ORC for systems in absence of unmeasured inputs, the observability is affected by the measured inputs [6]. This is depicted by the presence of the Lie derivatives along the vector fields $\mathbf{g}_{u1}^k, \dots, \mathbf{g}_{u_{n_u}}^k$ in $\Delta\mathbf{\Omega}_{k+1}$ (see Eq. (C.1)). In general, it can be stated that measured and hence known inputs improve observability.

4.3. A linear three-DOF system with unknown parameters

In this example, the observability of an idealized shear building model representing a three-storey building is examined. The model and its corresponding mass, damping, and stiffness matrices are shown in Figure 4. The displacement, velocity, and acceleration of floor j are denoted as x_j , \dot{x}_j , and \ddot{x}_j , respectively, with 1 corresponding to the bottom floor and 3 the top. An unmeasured force F_{p3} is applied at the top floor. Every floor j ($j = 1, 2, 3$) has a mass m_j and is connected with two linear springs with stiffness k_j and two linear dashpots with damping constant c_j to floor $j - 1$. The quantities to be identified are the dynamic states of the system x_j , \dot{x}_j ($j = 1, 2, 3$), the spring constants k_1 , k_2 , and k_3 , the damping constants c_1 , c_2 , and c_3 , and the force applied at the third floor F_{p3} , which acts as an unmeasured input of the system. The masses m_1 , m_2 , and

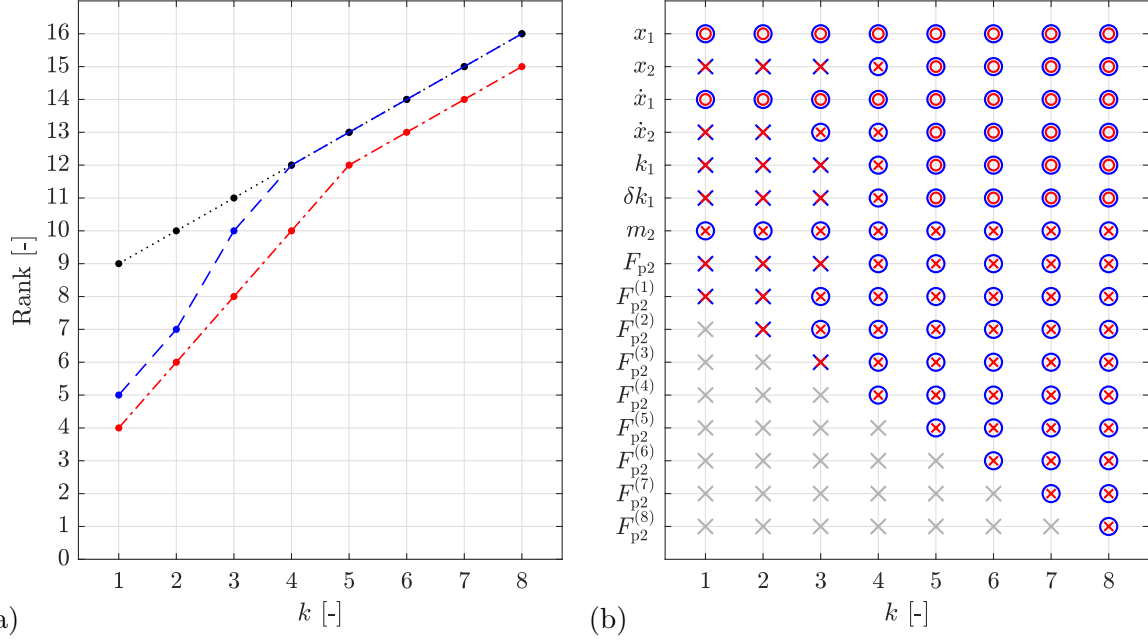


Fig. 3: (a) Rank of the matrix $d^k \Omega_k$ for the case with a measured input F_{u1} (blue dashed line), for the case with no F_{u1} applied (red dash-dotted line), and comparison to the number of states $n_s + (k+1)n_p$ of the augmented system \mathcal{S}^k (black dotted line) as a function of step number k . (b) observable states (circles) and k -row unobservable states (x-marks) for the case with known input F_{u1} (blue) and the case with $F_{u1} = 0$ (red), as a function of step number k .

m_3 are assumed known. The measurements consist of the acceleration at the second floor \ddot{x}_2 and the displacement at the top floor x_3 .

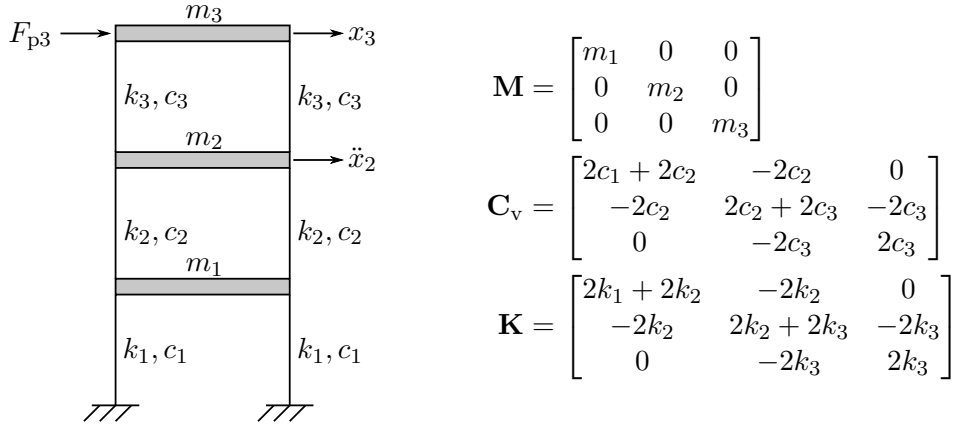


Fig. 4: Three-DOF shear building model considered in Example 4.3 and its mass, damping, and stiffness matrices.

In this example, the underlying dynamic system is linear and its dynamic behavior can generally be described by the following state-space equations:

$$\dot{\mathbf{x}}_t = \mathbf{A}_t(\boldsymbol{\theta})\mathbf{x}_t + \mathbf{B}_{ut}(\boldsymbol{\theta})\mathbf{u} + \mathbf{B}_{pt}(\boldsymbol{\theta})\mathbf{p} \quad (35)$$

$$\mathbf{y} = \mathbf{G}_t(\boldsymbol{\theta})\mathbf{x}_t + \mathbf{J}_u(\boldsymbol{\theta})\mathbf{u} + \mathbf{J}_p(\boldsymbol{\theta})\mathbf{p} \quad (36)$$

where the state vector \mathbf{x}_t is given by:

$$\mathbf{x}_t = [x_1 \ x_2 \ x_3 \ \dot{x}_1 \ \dot{x}_2 \ \dot{x}_3]^T \quad (37)$$

The expressions for the system matrices \mathbf{A}_t , \mathbf{B}_{ut} , \mathbf{B}_{pt} , \mathbf{G}_t , \mathbf{J}_u , and \mathbf{J}_p that apply to the considered case of a lumped mass system and can also be applied in the more general case of distributed mass systems are given by:

$$\mathbf{A}_t = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}_v \end{bmatrix}, \quad \mathbf{B}_{ut} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{S}_u \end{bmatrix}, \quad \mathbf{B}_{pt} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{S}_p \end{bmatrix} \quad (38)$$

$$\mathbf{G}_t = [\mathbf{S}_d - \mathbf{S}_a\mathbf{M}^{-1}\mathbf{K} \quad \mathbf{S}_v - \mathbf{S}_a\mathbf{M}^{-1}\mathbf{C}_v], \quad \mathbf{J}_u = \mathbf{S}_a\mathbf{M}^{-1}\mathbf{S}_u, \quad \mathbf{J}_p = \mathbf{S}_a\mathbf{M}^{-1}\mathbf{S}_p \quad (39)$$

where $\mathbf{M} \in \mathbb{R}^{n_{\text{dof}} \times n_{\text{dof}}}$, $\mathbf{C}_v \in \mathbb{R}^{n_{\text{dof}} \times n_{\text{dof}}}$, and $\mathbf{K} \in \mathbb{R}^{n_{\text{dof}} \times n_{\text{dof}}}$ are respectively the mass matrix, the viscous damping matrix, and the stiffness matrix of the structure, with n_{dof} the number of degrees of freedom in the model. The matrices $\mathbf{S}_u \in \mathbb{R}^{n_{\text{dof}} \times n_u}$ and $\mathbf{S}_p \in \mathbb{R}^{n_{\text{dof}} \times n_p}$ are selection matrices specifying the locations of the measured and the unmeasured forces, respectively. The matrices $\mathbf{S}_a \in \mathbb{R}^{n_y \times n_{\text{dof}}}$, $\mathbf{S}_v \in \mathbb{R}^{n_y \times n_{\text{dof}}}$, and $\mathbf{S}_d \in \mathbb{R}^{n_y \times n_{\text{dof}}}$ are selection matrices indicating the degrees of freedom corresponding to the acceleration, velocity and displacement or strain measurements, respectively.

In the considered example where the output vector is defined as $\mathbf{y} = [\ddot{x}_2 \ x_3]^T$, $n_u = 0$, and $n_p = 1$ with $p_1 = F_{p3}$, the system matrices \mathbf{B}_{pt} , \mathbf{G}_t , and \mathbf{J}_p simplify to:

$$\mathbf{B}_{pt} = [0 \ 0 \ 0 \ 0 \ 0 \ 1/m_3]^T \quad (40)$$

$$\mathbf{G}_t = \begin{bmatrix} 2k_2/m_2 & -(2k_2 + 2k_3)/m_2 & 2k_3/m_2 & 2c_2/m_2 & -(2c_2 + 2c_3)/m_2 & 2c_3/m_2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (41)$$

$$\mathbf{J}_p = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (42)$$

Augmenting the state vector \mathbf{x}_t with the vector of unmeasured parameters $\boldsymbol{\theta}$, with $\dot{\boldsymbol{\theta}} = \mathbf{0}$, allows rewriting system (35–36) in its augmented form:

$$\dot{\mathbf{x}} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x} + \mathbf{B}_u(\boldsymbol{\theta})\mathbf{u} + \mathbf{B}_p(\boldsymbol{\theta})\mathbf{p} \quad (43)$$

$$\mathbf{y} = \mathbf{G}(\boldsymbol{\theta})\mathbf{x} + \mathbf{J}_u(\boldsymbol{\theta})\mathbf{u} + \mathbf{J}_p(\boldsymbol{\theta})\mathbf{p} \quad (44)$$

which is a special case of the affine-input state-space representation of the system in Eqs. (6–7), with:

$$\mathbf{f}(\mathbf{x}) = \mathbf{A}(\boldsymbol{\theta})\mathbf{x}, \quad \mathbf{g}_{uj}(\mathbf{x}) = \mathbf{B}_{u,j}(\boldsymbol{\theta}), \quad \mathbf{g}_{pj}(\mathbf{x}) = \mathbf{B}_{p,j}(\boldsymbol{\theta}) \quad (45)$$

$$\mathbf{h}_0(\mathbf{x}) = \mathbf{G}(\boldsymbol{\theta})\mathbf{x}, \quad \mathbf{h}_{uj}(\mathbf{x}) = \mathbf{J}_{u,j}(\boldsymbol{\theta}), \quad \mathbf{h}_{pj}(\mathbf{x}) = \mathbf{J}_{p,j}(\boldsymbol{\theta}) \quad (46)$$

where $\mathbf{X}_{\alpha,j}$ denotes the j^{th} column of a matrix \mathbf{X}_α . The matrices \mathbf{A} , \mathbf{B}_u , \mathbf{B}_p , and \mathbf{G} in Eqs. (43–44) are given by:

$$\mathbf{A}(\boldsymbol{\theta}) = \begin{bmatrix} \mathbf{A}_t(\boldsymbol{\theta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_u(\boldsymbol{\theta}) = \begin{bmatrix} \mathbf{B}_{ut}(\boldsymbol{\theta}) \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_p(\boldsymbol{\theta}) = \begin{bmatrix} \mathbf{B}_{pt}(\boldsymbol{\theta}) \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{G}(\boldsymbol{\theta}) = [\mathbf{G}_t(\boldsymbol{\theta}) \quad \mathbf{0}] \quad (47)$$

Figure 5 summarizes the results of Algorithm 2.1 for the considered example. The system is observable in presence of unmeasured inputs, since the states \mathbf{x} and input \mathbf{p} become observable at $k \geq 10$. For a given step $k \geq 10$, however, the highest derivative of the input $F_{p3}^{(k)}$ is not k -row observable. This is due to the fact that the input does not directly affect the measurements ($\mathbf{h}_{p3}(\mathbf{x}) = \mathbf{0}$). However, if an input is found to be observable, its higher order derivatives will also become observable in future steps of applying the algorithm. This allows stopping the algorithm at a step where all the dynamic states, parameters and inputs are found to be observable.

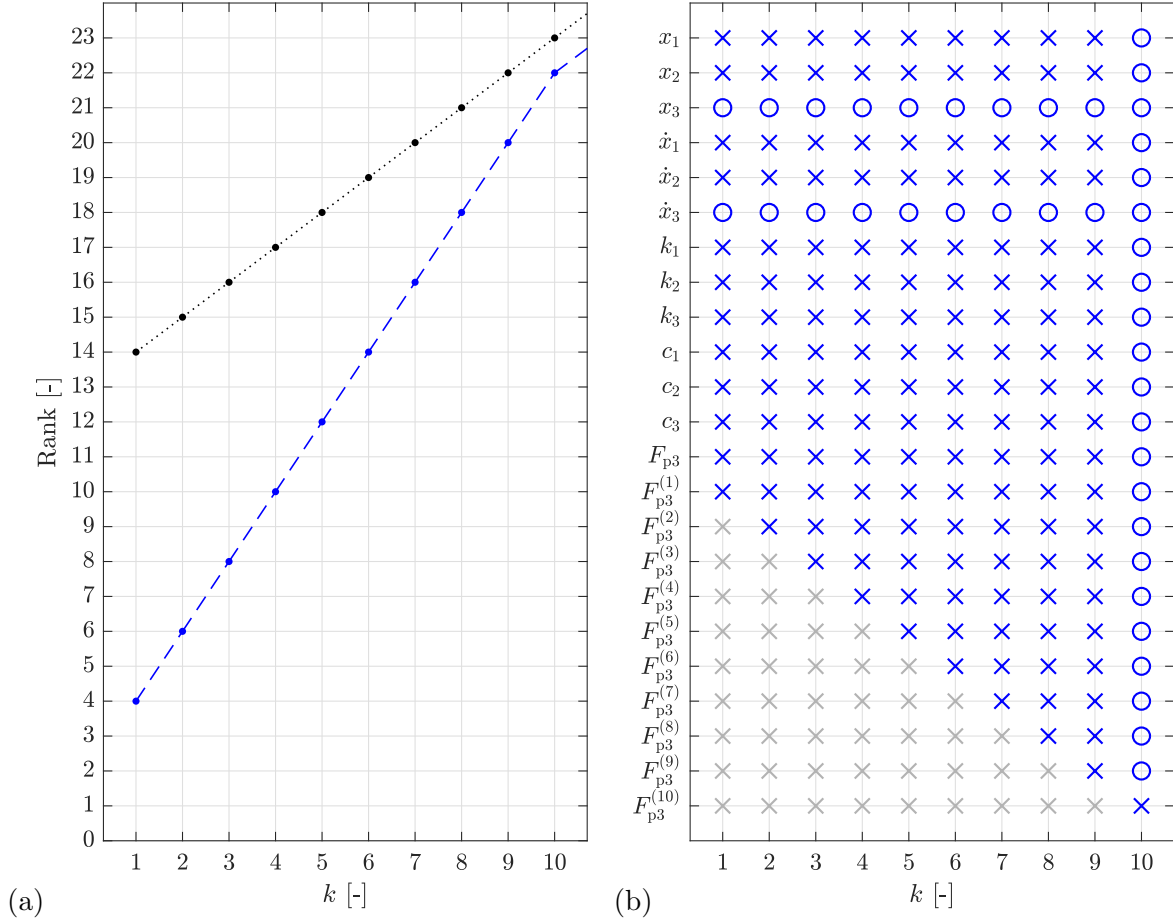


Fig. 5: (a) Rank of the matrix $d^k \Omega_k$ (blue dashed line) and comparison to the number of states $n_s + (k+1)n_p$ of the augmented system \mathcal{S}^k (black dotted line) as a function of the step number k . (b) observable states (circles) and k -row unobservable states (\times -marks) as a function of the step number k .

The same procedure is applied for the sixteen cases where different combinations of measured and unmeasured inputs are considered (Table 2). In all of these cases, for reasons of comparing to previous works in the literature [30, 6], the damping constants c_1 , c_2 , and c_3 are known and assumed to be zero, while the parameters to be identified are the masses m_1 , m_2 , and m_3 and the stiffness constants k_1 , k_2 , and k_3 . In all cases, the measurement consists of the displacement at the top floor x_3 . The rank computation is performed for k from 1 to 14. Table 2 lists the observable and unobservable states at step $k = 14$. For the unmeasured inputs, the observability of the time derivatives is not considered. In any case, if an input is observable, all of its higher time derivatives

are observable as well (Section 2).

For the four cases where no unmeasured inputs are considered ($F_{pi} = 0$, $i = 1, 2, 3$), the ORC-DF reduces to the regular ORC and the results are found to be in line with those presented in [30, 6], where the case of a linear three-DOF system with unknown mass and stiffness parameters was previously considered. Furthermore, it is found that the observability of the dynamic system states, inputs, and parameters heavily depends on the considered combination of measured and unmeasured inputs. For only two of the sixteen considered cases (cases (3,1) and (3,4)), all states are observable, meaning that the parameters can be identified, and that the dynamic system states and unmeasured inputs can be tracked over time. In line with what was previously observed in Example 4.2, it is seen from the results presented in Table 2 that measured and hence known inputs generally improve observability. Similarly, it is observed that unmeasured inputs tend to compromise the observability of the dynamic system states and the parameters. This is for example evidenced by comparing the observability results for the four cases where $F_{u3} \neq 0$, $F_{u1} = F_{u2} = 0$. Finally, only two cases are identified where the unmeasured input is observable (cases (1,1) and (3,1)). For the remaining cases, input observability requires a different sensor configuration, i.e., a different sensor location, a different sensor type, and/or a different number of sensors.

	$F_{p1} \neq 0,$ $F_{p2} = F_{p3} = 0$	$F_{p2} \neq 0,$ $F_{p1} = F_{p3} = 0$	$F_{p3} \neq 0,$ $F_{p1} = F_{p2} = 0$	$F_{pi} = 0,$ $i = 1, 2, 3$
$F_{u1} \neq 0,$ $F_{u2} = F_{u3} = 0$	O : $x_3, \dot{x}_3, k_1, F_{p1}$ U : $x_1, x_2, \dot{x}_1, \dot{x}_2,$ m_1, m_2, m_3, k_2, k_3	O : x_3, \dot{x}_3, k_1 U : $x_1, x_2, \dot{x}_1, \dot{x}_2,$ $m_1, m_2, m_3, k_2, k_3,$ F_{p2}	O : x_3, \dot{x}_3, k_1 U : $x_1, x_2, \dot{x}_1, \dot{x}_2,$ $m_1, m_2, m_3, k_2, k_3,$ F_{p3}	O : x_3, \dot{x}_3, k_1 U : $x_1, x_2, \dot{x}_1, \dot{x}_2,$ m_1, m_2, m_3, k_2, k_3
$F_{u2} \neq 0,$ $F_{u1} = F_{u3} = 0$	O : x_3, \dot{x}_3 U : $x_1, x_2, \dot{x}_1, \dot{x}_2,$ $m_1, m_2, m_3, k_1, k_2,$ k_3, F_{p1}	O : x_3, \dot{x}_3 U : $x_1, x_2, \dot{x}_1, \dot{x}_2,$ $m_1, m_2, m_3, k_1, k_2,$ k_3, F_{p2}	O : x_3, \dot{x}_3 U : $x_1, x_2, \dot{x}_1, \dot{x}_2,$ $m_1, m_2, m_3, k_1, k_2,$ k_3, F_{p3}	O : x_3, \dot{x}_3 U : $x_1, x_2, \dot{x}_1, \dot{x}_2,$ $m_1, m_2, m_3, k_1, k_2,$ k_3
$F_{u3} \neq 0,$ $F_{u1} = F_{u2} = 0$	O : $x_1, x_2, x_3, \dot{x}_1,$ $\dot{x}_2, \dot{x}_3, m_1, m_2,$ $m_3, k_1, k_2, k_3, F_{p1}$ U : -	O : $x_2, x_3, \dot{x}_2, \dot{x}_3,$ $m_1, m_2, m_3, k_1, k_2,$ k_3 U : x_1, \dot{x}_1, F_{p2}	O : $x_3, \dot{x}_3, m_1, m_2,$ m_3, k_1, k_2, k_3 U : $x_1, x_2, \dot{x}_1, \dot{x}_2,$ F_{p3}	O : $x_1, x_2, x_3, \dot{x}_1,$ $\dot{x}_2, \dot{x}_3, m_1, m_2,$ m_3, k_1, k_2, k_3 U : -
$F_{ui} = 0,$ $i = 1, 2, 3$	O : x_3, \dot{x}_3 U : $x_1, x_2, \dot{x}_1, \dot{x}_2,$ $m_1, m_2, m_3, k_1, k_2,$ k_3, F_{p1}	O : x_3, \dot{x}_3 U : $x_1, x_2, \dot{x}_1, \dot{x}_2,$ $m_1, m_2, m_3, k_1, k_2,$ k_3, F_{p2}	O : x_3, \dot{x}_3 U : $x_1, x_2, \dot{x}_1, \dot{x}_2,$ $m_1, m_2, m_3, k_1, k_2,$ k_3, F_{p3}	O : x_3, \dot{x}_3 U : $x_1, x_2, \dot{x}_1, \dot{x}_2,$ $m_1, m_2, m_3, k_1, k_2,$ k_3

Table 2: Overview of the partially observable (O) and k -row unobservable (U) states at step $k = 14$ for sixteen cases, considering different combinations of measured and unmeasured inputs.

4.4. The effect of non differentiable state equations

In this example, the nonlinear two-DOF system shown in Figure 6 is considered, where the mass m_1 is connected to a fixed support with a nonlinear Bouc-Wen element of stiffness k_1 and a linear damper with damping coefficient c_1 , while the spring with stiffness k_2 and the damper with damping coefficient c_2 that connect mass m_2 to m_1 are linear.

As in Example 4.2, an unmeasured force F_{p2} is applied at mass m_2 and a known force F_{u1} is applied at mass m_1 . The parameters to be identified are the stiffness k_1 and the parameters of the Bouc-Wen element, β , γ , and ν . For presentation purposes, the remaining parameters are assumed

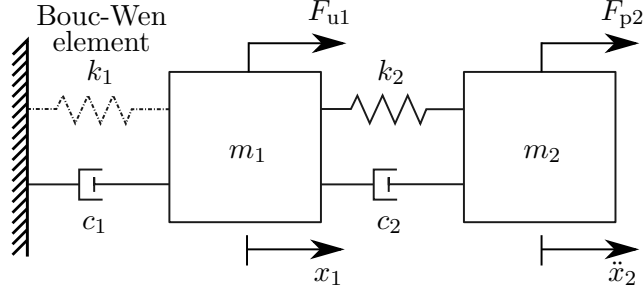


Fig. 6: Model of the two-DOF system with Bouc-Wen element in Example 4.4.

to be known. The equations of the underlying system are given by:

$$\begin{aligned} m_1 \ddot{x}_1 + c_1 \dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1 r + k_2(x_1 - x_2) &= F_{u1} \\ m_2 \ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) &= F_{p2} \end{aligned} \quad (48)$$

$$\dot{r} = \dot{x}_1 - \beta |\dot{x}_1| |r|^{\nu-1} - \gamma \dot{x}_1 |r|^\nu \quad (49)$$

where r is the elastic displacement of the spring. The specific form of the Bouc-Wen model corresponds to a zero post-yield stiffness. It is straightforward to bring Eq. (48) into the augmented state-space form (Eqs. (6–7)). In the following, it is assumed that displacement x_1 and acceleration \ddot{x}_2 are measured, leading to the following definition of the output vector:

$$\mathbf{y} = \begin{bmatrix} x_1 \\ (k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) + F_{p2})/m_2 \end{bmatrix} \quad (50)$$

The system in Eq. (48) is not analytical because the state equations are not differentiable, due to the presence of the absolute value operators. However, it can be expressed in smooth branches depending on the realization of the state and therefore belongs to the previously considered category of piecewise-smooth systems. The ORC-DF algorithm can be applied to study the observability for each of the smooth branches of the non-analytical function \dot{r} , which are defined as:

$$\begin{aligned} \text{(A)} : \dot{r} &= \dot{x}_1 - \beta \dot{x}_1 r^\nu - \gamma \dot{x}_1 r^\nu, \quad \text{for } \dot{x}_1 > 0 \text{ and } r > 0 \\ \text{(B)} : \dot{r} &= \dot{x}_1 + \beta \dot{x}_1 r^\nu - \gamma \dot{x}_1 r^\nu, \quad \text{for } \dot{x}_1 < 0 \text{ and } r > 0 \\ \text{(C)} : \dot{r} &= \dot{x}_1 + \beta \dot{x}_1 (-r)^\nu - \gamma \dot{x}_1 (-r)^\nu, \quad \text{for } \dot{x}_1 > 0 \text{ and } r < 0 \\ \text{(D)} : \dot{r} &= \dot{x}_1 - \beta \dot{x}_1 (-r)^\nu - \gamma \dot{x}_1 (-r)^\nu, \quad \text{for } \dot{x}_1 < 0 \text{ and } r < 0 \end{aligned} \quad (51)$$

Application of the ORC-DF for the system in Eq. (48) and considering each of the smooth branches in Eq. (51) separately reveals that the parameters β and γ are unobservable in all branches. This was previously reported in [6], where it is proposed to consider the following alternative parametrization of the non-analytical function \dot{r} :

$$\begin{aligned} \text{(A)} : \dot{r} &= \dot{x}_1 - \Delta_1 \dot{x}_1 r^\nu, \quad \text{for } \dot{x}_1 > 0 \text{ and } r > 0 \\ \text{(B)} : \dot{r} &= \dot{x}_1 + \Delta_2 \dot{x}_1 r^\nu, \quad \text{for } \dot{x}_1 < 0 \text{ and } r > 0 \\ \text{(C)} : \dot{r} &= \dot{x}_1 + \Delta_2 \dot{x}_1 (-r)^\nu, \quad \text{for } \dot{x}_1 > 0 \text{ and } r < 0 \\ \text{(D)} : \dot{r} &= \dot{x}_1 - \Delta_1 \dot{x}_1 (-r)^\nu, \quad \text{for } \dot{x}_1 < 0 \text{ and } r < 0 \end{aligned} \quad (52)$$

with $\Delta_1 = \beta + \gamma$ and $\Delta_2 = \beta - \gamma$.

Considering each of the smooth branches in Eq. (52) together with the system of equations (48) results in the conclusion that each of these systems becomes observable according to the ORC-DF, i.e., all the states x_1 , x_2 , \dot{x}_1 , \dot{x}_2 , the parameters k_1 , ν , and Δ_1 or Δ_2 , and the unmeasured input F_{p2} are observable. If Δ_1 is identified from either branch A or D and Δ_2 from branch B or C, then β and γ can also be identified by solving the system $\Delta_1 = \beta + \gamma$ and $\Delta_2 = \beta - \gamma$. It should be mentioned that it is not possible to identify β and γ separately if the system remains in one of the branches. Therefore, the identification of β and γ requires a sufficiently rich input to ensure that the system state reaches at least one combination of branches which allows to identify β and γ separately, i.e. $\{A,B\}$, $\{A,C\}$, $\{D,B\}$, or $\{D,C\}$.

In a final calculation, it is investigated how the observability of the system changes when the displacement measurement x_1 is replaced by an acceleration measurement \ddot{x}_1 . In this case, the states x_1 and x_2 are found to be unobservable for the four smooth branches, whereas the states \dot{x}_1 and \dot{x}_2 , the parameters k_1 , ν , and Δ_1 or Δ_2 , and the unmeasured input F_{p2} are observable. This result is in line with the result obtained in [6], where it was found that measuring a displacement of a point is more likely to result in an observable system than measuring the velocity or acceleration of the same point.

5. Conclusions

In this work, a new geometric method is developed to investigate the observability of nonlinear systems driven by both measured and unmeasured inputs. The method is an extension of the existing ORC and EORC methods for the case where the system output is affected by a direct feedthrough from the system input. This extension is needed for applications in structural dynamics, where accelerations are commonly used and, as such, a direct feedthrough is inevitable. A new analytical observability condition is derived based on Lie algebra, named the Observability Rank Condition for systems with Direct Feedthrough (ORC-DF). It is proven that analytical systems which satisfy this rank condition are observable in presence of unknown inputs. This implies that its dynamic system states are observable, its parameters are identifiable, and its inputs can be tracked over time. Several examples are given to demonstrate the applicability of the developed method to engineering systems. In extension to applications for analytical systems, it is also demonstrated how the developed method can be applied for the individual smooth branches of piecewise-smooth systems. This permits the detection of unobservable parameters in each branch and, in certain cases, the deduction of observability properties of the overall system.

The presented algorithm makes the assumption that the measurements are continuous over time. This assumption is rarely met in practice. The observability of the continuous-time system is a necessary but not sufficient condition for the observability of the equivalent discrete-time system. The authors are working towards an extension of the presented algorithm investigating the observability of continuous-time systems with discrete-time measurements.

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Appendix A. Proof of Proposition 2.1

The proof is an extension of the proof provided in [21] for the case where $\mathbf{h}_{uj}(\mathbf{x}) \neq \mathbf{0}$ and $\mathbf{h}_{pj}(\mathbf{x}) \neq \mathbf{0}$, and proceeds by induction on m . For $m = 0$, the zero-order Lie derivative of the output vector, i.e., \mathbf{y} given in Eq. (7), is independent of $p_j^{(q)}$ for $q > 0$. We now prove that if Proposition 2.1 holds for $m - 1$, it also holds for m . The m^{th} order Lie derivative of the output vector is obtained from the $(m - 1)^{\text{th}}$ order Lie derivative as:

$$L_{\boldsymbol{\mu}_m^l} \dots L_{\boldsymbol{\mu}_1^l}(\mathbf{y}) = d^l(L_{\boldsymbol{\mu}_{m-1}^l} \dots L_{\boldsymbol{\mu}_1^l}(\mathbf{y})) \cdot \boldsymbol{\mu}_m^l \quad (\text{A.1})$$

If the $(m - 1)^{\text{th}}$ order Lie derivative is independent of $p_j^{(q)}$, $j = 1, \dots, n_p$, for $q > m - 1$, it can be represented by a vector field $\boldsymbol{\gamma}(\mathbf{x}, \mathbf{p}, \dot{\mathbf{p}}, \dots, \mathbf{p}^{(m-1)})$. The gradient of this vector field with respect to \mathbf{x}^k is given by:

$$d^l \boldsymbol{\gamma} = \begin{bmatrix} \frac{\partial \boldsymbol{\gamma}}{\partial \mathbf{x}} & \frac{\partial \boldsymbol{\gamma}}{\partial \mathbf{p}} & \dots & \frac{\partial \boldsymbol{\gamma}}{\partial \mathbf{p}^{(m-1)}} & \mathbf{0}_{n_y \times (l-m+2)n_p} \end{bmatrix} \quad (\text{A.2})$$

The product of the gradient $d^l \boldsymbol{\gamma}$ with $\boldsymbol{\mu}_m^l = \{\mathbf{f}^l, \mathbf{g}_{u1}^l, \dots, \mathbf{g}_{un_u}^l\}$ depends at most on $p_j^{(m)}$. This concludes the proof.

Appendix B. Proof of Lemma 2.1

The proof is an extension of the proof provided in [13] for the case with unmeasured inputs and where the system output is affected by a direct feedthrough from measured and unmeasured inputs. If two initial states \mathbf{x}_a^l and \mathbf{x}_b^l , $l = \infty$, produce the same output \mathbf{y} at time $t = \bar{t}_k$ for any possible piecewise constant inputs, we must have:

$$\mathbf{y}(\mathbf{x}_a^l, t_1, t_2, \dots, t_k) = \mathbf{y}(\mathbf{x}_b^l, t_1, t_2, \dots, t_k) \quad (\text{B.1})$$

Eq. (B.1) has to hold for all possible t_1, t_2, \dots, t_k . From this, we deduce that the following equality should also hold:

$$\left. \frac{\partial \mathbf{y}(\mathbf{x}_a^l, t_1, t_2, \dots, t_k)}{\partial t_1 \partial t_2 \dots \partial t_k} \right|_{t_1, \dots, t_k \rightarrow 0} = \left. \frac{\partial \mathbf{y}(\mathbf{x}_b^l, t_1, t_2, \dots, t_k)}{\partial t_1 \partial t_2 \dots \partial t_k} \right|_{t_1, \dots, t_k \rightarrow 0} \quad (\text{B.2})$$

Taking into account Eq. (17) yields:

$$L_{\boldsymbol{\theta}_1^l} \dots L_{\boldsymbol{\theta}_k^l} \mathbf{y} \Big|_{\mathbf{x}^l = \mathbf{x}_a^l} = L_{\boldsymbol{\theta}_1^l} \dots L_{\boldsymbol{\theta}_k^l} \mathbf{y} \Big|_{\mathbf{x}^l = \mathbf{x}_b^l} \quad (\text{B.3})$$

or,

$$L_{\boldsymbol{\theta}_1^l} \dots L_{\boldsymbol{\theta}_k^l} \left(\mathbf{h}_{xp}(\mathbf{x}^0) + \sum_{j=1}^{n_u} \mathbf{h}_{uj}(\mathbf{x}) u_j^{k+1} \right) \Big|_{\mathbf{x}^l = \mathbf{x}_a^l} = \dots \\ L_{\boldsymbol{\theta}_1^l} \dots L_{\boldsymbol{\theta}_k^l} \left(\mathbf{h}_{xp}(\mathbf{x}^0) + \sum_{j=1}^{n_u} \mathbf{h}_{uj}(\mathbf{x}) u_j^{k+1} \right) \Big|_{\mathbf{x}^l = \mathbf{x}_b^l} \quad (\text{B.4})$$

with $\mathbf{h}_{\text{xp}}(\mathbf{x}^0) = \mathbf{h}_0(\mathbf{x}) + \sum_{j=1}^{n_p} \mathbf{h}_{pj}(\mathbf{x})p_j$. Note that the vector fields $\boldsymbol{\theta}_i^l$, depend on $u_1^i, \dots, u_{n_u}^i$, for $i = 1, \dots, k$. Imposing that Eq. (B.4) must hold for all possible choices of $u_j^i \in \mathbb{R}$ enables to break this equation down into the following set of equalities:

$$L_{\boldsymbol{\mu}_k^l} \dots L_{\boldsymbol{\mu}_1^l}(\mathbf{h}_{\text{xp}}(\mathbf{x}^0)) \Big|_{\mathbf{x}^l = \mathbf{x}_a^l} = L_{\boldsymbol{\mu}_k^l} \dots L_{\boldsymbol{\mu}_1^l}(\mathbf{h}_{\text{xp}}(\mathbf{x}^0)) \Big|_{\mathbf{x}^l = \mathbf{x}_b^l} \quad (\text{B.5})$$

$$L_{\boldsymbol{\mu}_k^l} \dots L_{\boldsymbol{\mu}_1^l}(\mathbf{h}_{uj}(\mathbf{x})) \Big|_{\mathbf{x}^l = \mathbf{x}_a^l} = L_{\boldsymbol{\mu}_k^l} \dots L_{\boldsymbol{\mu}_1^l}(\mathbf{h}_{uj}(\mathbf{x})) \Big|_{\mathbf{x}^l = \mathbf{x}_b^l}, \quad j = 1, \dots, n_u \quad (\text{B.6})$$

where $\boldsymbol{\mu}_1^l, \dots, \boldsymbol{\mu}_k^l$ represent all possible vector fields belonging to the set $\{\mathbf{f}^l, \mathbf{g}_{u1}^l, \dots, \mathbf{g}_{un_u}^l\}$. Applying the result of Proposition 2.1, it easily derived that Eqs. (B.5) and (B.6) are equivalent to:

$$L_{\boldsymbol{\nu}_k^{q-1}} \dots L_{\boldsymbol{\nu}_1^0}(\mathbf{h}_{\text{xp}}(\mathbf{x}^0)) \Big|_{\mathbf{x}^k = \mathbf{x}_a^k} = L_{\boldsymbol{\nu}_k^{q-1}} \dots L_{\boldsymbol{\nu}_1^0}(\mathbf{h}_{\text{xp}}(\mathbf{x}^0)) \Big|_{\mathbf{x}^k = \mathbf{x}_b^k} \quad (\text{B.7})$$

$$L_{\boldsymbol{\nu}_k^{q-1}} \dots L_{\boldsymbol{\nu}_1^0}(\mathbf{h}_{uj}(\mathbf{x})) \Big|_{\mathbf{x}^k = \mathbf{x}_a^k} = L_{\boldsymbol{\nu}_k^{q-1}} \dots L_{\boldsymbol{\nu}_1^0}(\mathbf{h}_{uj}(\mathbf{x})) \Big|_{\mathbf{x}^k = \mathbf{x}_b^k}, \quad j = 1, \dots, n_u \quad (\text{B.8})$$

when considering all possible vector fields $\boldsymbol{\nu}_q^{q-1}$ belonging to the set $\{\mathbf{f}_{\text{xp}}^{q-1}, \mathbf{g}_{u1}^{q-1}, \dots, \mathbf{g}_{un_u}^{q-1}\}$, with $\mathbf{f}_{\text{xp}}^{q-1} = \mathbf{f}^{q-1} + \sum_{j=1}^{n_p} \mathbf{S}_j^{q-1} p_j$.

Appendix C. Proof of Lemma 2.3

The proof proceeds by induction. We prove that the k -row observable states of the vector \mathbf{x}_k are also $(k+1)$ -row observable states of the vector \mathbf{x}_{k+1} . Assume that the system \mathcal{S} has b k -row unobservable states and $n_s + (k+1)n_p - b$ k -row observable states. For this system, the matrix $\mathbf{d}^k \boldsymbol{\Omega}_k$ contains $n_s + (k+1)n_p - b$ linearly independent columns, corresponding to the observable states. From Eq. (20) and Proposition 2.1, the following relation between the matrices $\mathbf{d}^k \boldsymbol{\Omega}_k$ and $\mathbf{d}^{k+1} \boldsymbol{\Omega}_{k+1}$ is derived:

$$\mathbf{d}^{k+1} \boldsymbol{\Omega}_{k+1} = \begin{bmatrix} \mathbf{d}^k \boldsymbol{\Omega}_k & \mathbf{0} \\ \mathbf{d}^{k+1} \Delta \boldsymbol{\Omega}_{k+1} \end{bmatrix} \quad (\text{C.1})$$

Due to the presence of the zero matrix which is added to the first rows corresponding to $\mathbf{d}^k \boldsymbol{\Omega}_k$, it can be concluded that the columns of matrix $\mathbf{d}^{k+1} \boldsymbol{\Omega}_{k+1}$ corresponding to the $n_s + (k+1)n_p - b$ observable states of \mathbf{x}^k are still linearly independent and therefore that the k -row observable states of the vector \mathbf{x}_k constitute a subset of the $(k+1)$ -row observable states of \mathbf{x}^{k+1} .

Appendix D. Illustration of the symbolic operations in Algorithm 2.1

The steps of Algorithm 2.1 are illustrated in the following lines using an example. The dynamic system under consideration is the following:

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = -\exp(-kz_1)z_2/m - k/m z_1 + 1/m p \quad (\text{D.1})$$

$$\dot{k} = 0, \quad \dot{m} = 0 \quad (\text{D.2})$$

A single measurement $y = z_1$ is considered. The input consist of a single unknown force p ($n_p = 1$, $n_u = 0$). For this system, $\mathbf{x} = [z_1 \quad z_2 \quad k \quad m]^T$, $\mathbf{f}(\mathbf{x}) = [z_2 \quad -\exp(-kz_1)z_2/m - k/m z_1 \quad 0 \quad 0]^T$, $\mathbf{g}_p(\mathbf{x}) = [0 \quad 1/m \quad 0 \quad 0]^T$, $\mathbf{h}_0(\mathbf{x}) = z_1$, and $\mathbf{h}_p(\mathbf{x}) = 0$.

Applying the steps of the Algorithm 2.1 for the first iteration yields:

1. $k = 0$, $\mathbf{x}^0 = [z_1 \ z_2 \ k \ m \ p]^T$
2. $\Delta\mathbf{\Omega}_0 = z_1$, $\mathbf{\Omega}_0 = z_1$.
3. $d^0\mathbf{\Omega}_0 = [1 \ 0 \ 0 \ 0 \ 0]$
4. $k = 1$
5. $\mathbf{f}_{xp}^0 = [z_2 \ -\exp(-kz_1)z_2/m - k/m \ z_1 + 1/m \ p \ 0 \ 0 \ \dot{p}]^T$, $\Delta\mathbf{\Omega}_1 = z_2$
6. $\mathbf{x}^1 = [z_1 \ z_2 \ k \ m \ p \ \dot{p}]^T$
7. $d^1\Delta\mathbf{\Omega}_1 = [0 \ 1 \ 0 \ 0 \ 0 \ 0]$
8. $d^1\mathbf{\Omega}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$
9. All rows of $d^1\mathbf{\Omega}_1$ are independent.
10. $\text{rank}(d^1\mathbf{\Omega}_1) = 2$, $n_s + (k+1)n_p = 6 \rightarrow \text{rank}(d^1\mathbf{\Omega}_1) < n_s + (k+1)n_p$.
11. States z_1 and z_2 are k -row observable, whereas k , m , p , and \dot{p} are not.
12. States \mathbf{x} and \mathbf{p} are not k -row observable.

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