The Oseen-Frank Theory of Liquid Crystals

Lu Liu

The Queen's College

University of Oxford

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Abstract

This thesis concerns the minimization of the Oseen-Frank bulk free energy. The structure is as following: in Chapter 1, we will give a brief introduction to the Oseen-Frank theory and the Landau-de Gennes theory. Also we will introduce some established results related to the two theories. In Chapter 2, we define first in Section 2.1 the degree for a vector field $\mathbf{n} \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$, and then in Section 2.2 the degree for an $H^{1/2}$ vector field from a Lipschitz boundary $\partial\omega$ to $\mathbb{S}^1$. In Section 2.3, we prove the fact that vector fields subject to some given boundary conditions and degree constraints in a given exterior N−connected domain can be written explicitly, and the result is stated in Proposition 2.3.6. We began in Chapter 3 by focusing on the existence and uniqueness of minimizers for a modified one-constant Oseen-Frank energy subject to some prescribed boundary conditions in a 2D circular domain $\Omega = \{x \in \mathbb{R}^2 \,|\, 0 < a < |x| < \infty\}$, and derive some 'nice' asymptotic behaviour at ∞ of the minimizer. In Chapter 4, we make the problem studied in Chapter 3 more complicated by adding more 'holes' in the domain. Then by introducing the homotopy classes for vector fields subject to prescribed boundary conditions, we prove that there exists a unique minimizer for a modified one-constant Oseen-Frank energy in each homotopy class and we still have the 'nice' asymptotic behaviour of the minimizer in each homotopy class. Also, there exists a minimizer in the union of these homotopy classes, although this minimizer may not be unique. Then in Chapter 5, we work with line fields on the boundary in the given exterior N-connected Lipschitz domain. By introducing an auxiliary vector field and modifying the definition of homotopy classes in Chapter 4, we prove the uniqueness of the minimizing line field in each homotopy class. Also the result proved in Chapter 5 applies to both orientable and non-orientable line fields on the boundaries, and when the given boundary line field is orientable, it is equivalent to our result proved in Chapter 4. Finally, in Chapter 6, we study the 2D non-equal constant case (i.e. without assuming $k_1 = k_2 = k_3$, and $k_4 = 0$) in a one-circular domain. In particular, in Section 6.1 we assume the vector fields are radius-independent and derive the minimizers to the Oseen-Frank bulk free energy subject to prescribed degree constraint on the circular boundary. Then in Section 6.2

we show that the result proved in Section 6.1 will also hold when one of the Frank elastic constant is zero (degenerate case), but will be in a different function space $\mathbf{W}^{1,1}(\Omega;\mathbb{S}^1)$ instead of $\mathbf{H}^1(\Omega;\mathbb{S}^1)$. At last in Section 6.3 we will show that the minimizers derived in Section 6.1 and Section 6.2 are in general not minimizers for the radius-dependent case.

Contents

List of Figures

Chapter 1

Introduction

1.1 The mathematics of liquid crystals

Liquid crystals are of many different types, three main classes being nematics, cholesterics and smectics. Many liquid crystals consist of rod-like molecules. Unlike isotropic fluids for which there is no orientational or positional order (see Figure 1.1), the molecules of liquid crystals can arrange themselves in different phases depending on the nature of the molecules, the interactions between them and the temperature.

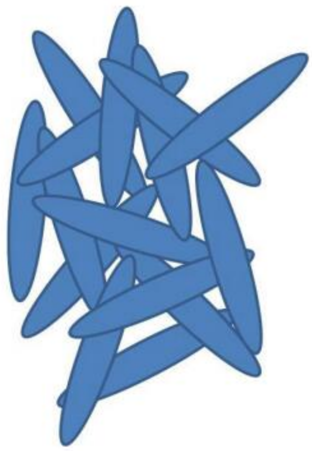

Figure 1.1: Isotropic fluid.

For the nematic phase, there is orientational but no positional order; while for the smectic A phase and smectic C phase, there is orientational and some positional order (See Figure 1.2 and Figure 1.3 for illustration). In the cholesteric phase there is orientational but no positional order; however, there is a macroscopic positional order coming from a twisted ground state with length-scale characteristic. The molecules have time-varying orientations due to thermal motion. The most commonly studied nematic phase

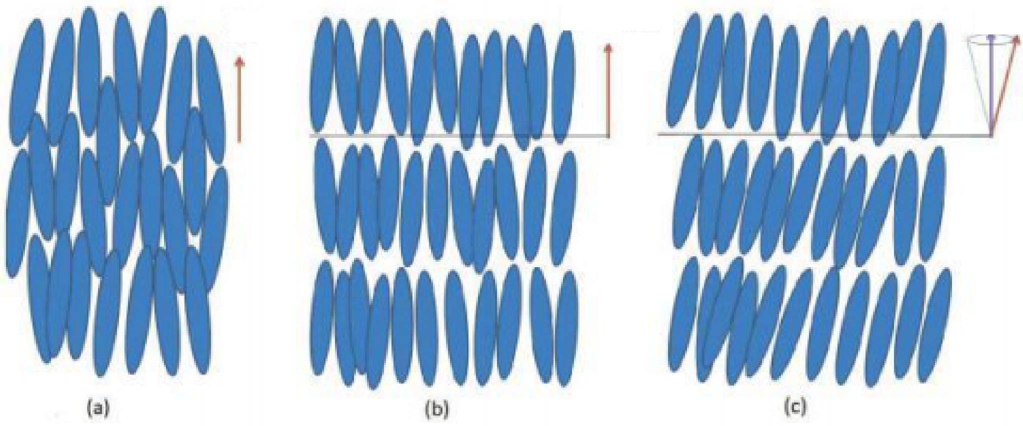

Figure 1.2: (a) Nematic phase; (b) Smectic A phase; (c) Smectic C phase. As can be seen from this figure, the nematic phase has orientational order but has no positional order, whereas there is both orientational and positional order for the smectic A and smectic C phases.

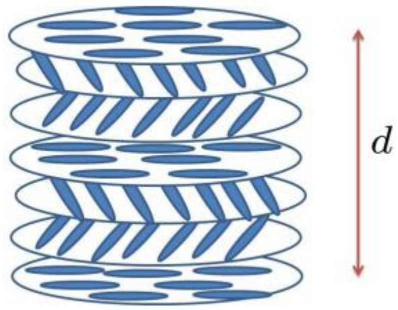

Figure 1.3: Cholesteric phase. There is orientational but no positional order for cholesteric liquid crystals. However there is a macroscopic positional order coming from a twisted ground state with length-scale characteristic, i.e. the pitch d of the twisted ground state is at a length-scale much greater than molecular.

typically forms on cooling through a critical temperature by a phase transformation from a high temperature isotropic phase. For nematics in general as illustrated in Figure 1.4, when $\theta > \theta_c$, the material is in the isotropic phase, when $\theta^* < \theta < \theta_c$, the material is in the nematic phase, while when $\theta < \theta^*$, the material is in some other liquid crystal or solid phase which depends on the material.

In this thesis, we only study nematic liquid crystals. The challenge of describing nematic liquid crystals by a model that is both comprehensive and simple enough to manipulate efficiently has led to the existence of several major competing theories. One of the most simple and successful is the Oseen-Frank theory that describes nematics using a relatively simple description, namely a unit vector field **n**, the *director*, representing the mean orientation of the molecules. But the Oseen-Frank theory has the deficiency of ignoring the physical statistical head-to-tail symmetry of the rod-like molecules. A more complex theory was proposed by de Gennes and uses matrix-valued functions (Q-

Figure 1.4: For nematics in general, when $\theta < \theta_{\rm m}$, the material is in other liquid crystal or solid phase; $\theta_{\rm m} < \theta < \theta_{\rm c}$, the material is in nematic phase; $\theta > \theta_{\rm c}$, the material is in isotropic phase, where $\theta_{\rm m}$ and $\theta_{\rm c}$ are some threshold temperatures in phase transitions which depend on the given material.

tensors). In the simplest constrained case of uniaxial Q-tensors with a constant scalar order parameter, these Q-tensors can be interpreted as line fields described by a pair of antipodal vector fields $\{\mathbf{n}, -\mathbf{n}\}$, so that the Landau - de Gennes theory respects the head-to-tail symmetry. But an orientability problem arises, i.e. it may not be possible to 'orient' a line field in the domain Ω so that it becomes a vector field without changing its regularity class. To deal with this orientability problem, Ball and Zarnescu studied in [BZ08] the differences between these two theories and established when the more complicated, but physically more realistic, theory of de Gennes can be replaced by the simpler Oseen-Frank theory, and when this cannot be done. They have shown that for simply-connected domains and in the natural energy class $\mathbf{H}^1$ the constrained Landau - de Gennes and Oseen-Frank theories coincide. It is also shown in [BZ08] that, under suitable assumptions, the orientability of a line field on a 2D bounded domain can be determined simply by the orientability of the line field on the boundary of the domain.

1.2 The Oseen-Frank model

The idea of the Oseen-Frank model is to associate to each point in the macroscopic physical domain Ω a director describing the preferred direction of the molecules at the point. The Oseen-Frank model takes this director to be a unit vector $\mathbf{n}(x) \in \mathbb{S}^2$, $x \in \Omega$. The Oseen-Frank free energy density for a nematic liquid crystal is given by (see [Ste04a, Chapter 2]):

$$\begin{aligned} F(\mathbf{n}, \nabla\mathbf{n}) = &\frac{1}{2}k_1(\operatorname{div}\mathbf{n})^2 + \frac{1}{2}k_2(\mathbf{n}\cdot\nabla\wedge\mathbf{n})^2 + \frac{1}{2}k_3|\mathbf{n}\wedge\nabla\wedge\mathbf{n}|^2 \\ &+ \frac{1}{2}(k_2 + k_4)(\operatorname{tr}(\nabla\mathbf{n})^2) - (\operatorname{div}\mathbf{n})^2), \end{aligned} \tag{1.2.1}$$

where the constants k_i, $i = 1, 2, 3$ are often referred to as the Frank elastic constants (or moduli) and are dependent on the particular liquid crystal. The terms involving the constants k_1, k_2, k_3 in the energy density correspond respectively to the change of nematic energy caused by splay, twist and bend reorientations of the director $\mathbf{n}$. The last term with coefficient $k_2 + k_4$ is called the saddle-splay term, and $k_2 + k_4$ the saddle-splay constant, see also [Ste04a, Chapter 2].

Note that k_i, $i = 1, 2, 3$ are assumed to satisfy the following conditions:

$$\begin{aligned} &k_1 > 0, \quad k_2 > 0, \quad k_3 > 0 \\ &k_2 > |k_4|, \quad k_1 > |k_1 - k_2 - k_4| \end{aligned} \tag{1.2.2}$$

which are referred to as Ericksen inequalitites (see [Eri66]). The Ericksen inequalities are necessary and sufficient condition for $F(\mathbf{n}, \nabla\mathbf{n})$ to satisfy

$$F(\mathbf{n}, \nabla\mathbf{n}) \geq \alpha\,|\nabla\mathbf{n}|^2$$

for some $\alpha > 0$.

The saddle-splay term in the free energy density $F(x, \mathbf{n}, \nabla\mathbf{n})$ is often omitted since we have the identity

$$\operatorname{tr}(\nabla\mathbf{n})^2 - (\operatorname{div}\mathbf{n})^2 = \nabla\cdot((\mathbf{n}\cdot\nabla)\mathbf{n} - (\nabla\cdot\mathbf{n})\mathbf{n}), \tag{1.2.3}$$

which after integration over Ω becomes

$$\int_{\partial\Omega} ((\mathbf{n}\cdot\nabla)\mathbf{n} - (\nabla\cdot\mathbf{n})\mathbf{n})\cdot\, d\mathbf{S}.$$

The value of the integral depends on what boundary data is prescribed. If $\mathbf{n}$ is given on $\partial\Omega$ then the integral is constant. But e.g. for planar degenerate boundary data, the saddle-splay term cannot be ignored. Moreover, in the two-dimensional case where

$\mathbf{n}(x) = (n_1(x_1, x_2), n_2(x_1, x_2))$, namely $\mathbf{n}(x) \in \mathbb{S}^1$, the expression in (1.2.3) becomes zero, i.e.

$$\mathrm{tr}(\nabla\mathbf{n})^2 - (\mathrm{div}\,\mathbf{n})^2 \equiv 0, \quad x \in \Omega, \tag{1.2.4}$$

regardless of the prescribed boundary data, and we also have

$$\mathbf{n} \cdot \nabla \wedge \mathbf{n} \equiv 0, \quad x \in \Omega. \tag{1.2.5}$$

The physical configuration corresponding to the planar vector field $\mathbf{n}(x) : \Omega \subset R^2 \mapsto \mathbb{S}^1$ can be considered as a layer of nematic liquid crystal material contained within a very thin membrane.

By applying the identity $|\nabla\mathbf{n}|^2 = \mathrm{tr}(\nabla\mathbf{n})^2 + (\mathbf{n} \cdot \nabla \wedge \mathbf{n})^2 + |\mathbf{n} \wedge \nabla \wedge \mathbf{n}|^2$ to (1.2.1), we have:

$$\begin{aligned} F(\mathbf{n}, \nabla\mathbf{n}) = &\frac{1}{2}k_3|\nabla\mathbf{n}|^2 + \frac{1}{2}(k_1 - k_3)(\mathrm{div}\,\mathbf{n})^2 + \frac{1}{2}(k_2 - k_3)(\mathbf{n} \cdot \nabla \wedge \mathbf{n})^2 \\ &+ \frac{1}{2}(k_2 + k_4 - k_3)(\mathrm{tr}(\nabla\mathbf{n})^2) - (\mathrm{div}\,\mathbf{n})^2). \end{aligned} \tag{1.2.6}$$

Note again that the last two terms in (1.2.6) are zero in the two-dimensional case.

The one-constant approximation

$$k = k_1 = k_2 = k_3 \quad \text{and} \quad k_4 = 0 \tag{1.2.7}$$

is often used for its mathematical simplicity. Some authors, in the one-constant approximation, equate the splay, twist, and bend constants and assume that $k_2 + k_4 = 0$ rather than $k_4 = 0$. In the case of strong anchoring (see [Vir95, Chapter 3]), this alternative approximation leads to equilibrium equations that are identical to those obtained using (1.2.7).

Under the one-constant approximation (1.2.7), the Oseen-Frank free energy density becomes

$$F(\mathbf{n}, \nabla\mathbf{n}) = \frac{1}{2}k\,|\nabla\mathbf{n}|^2,$$

and hence the bulk energy is given by

$$I(\mathbf{n}) = \frac{1}{2}\,k \int_\Omega |\nabla\mathbf{n}|^2 dx,$$

where for simplicity we often omit the constant k.

1.3 Minimization problem with specified boundary condition

Our main focus is the energy minimization problem on an open set $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary that is not necessarily bounded, i.e. find $\mathbf{n}$ that minimizes $I(\mathbf{n})$ subject to suitable boundary conditions, for example $\mathbf{n}|_{\partial\Omega} = \mathbf{n}_0$, where $\mathbf{n}_0$ is given. In the unbounded case the energy may be infinite. More specifically, for $F(x, \mathbf{n}, \nabla\mathbf{n})$ given by (1.2.6), our problem is to find the minimizer for the Oseen-Frank free energy:

$$I(\mathbf{n}) = \frac{1}{2}\int_\Omega F(x, \nabla\mathbf{n}, \nabla\mathbf{n})\,dx,$$

over all vector fields $\mathbf{n}$ in the admissible set

$$A_0 = \{\mathbf{n} \in H^1(\Omega; \mathbb{S}^2) \,|\, \mathbf{n}|_{\partial\Omega} = \mathbf{n}_0\}.$$

Then if $\mathbf{n}$ is a minimizer and $\mathbf{v} : \Omega \to \mathbb{R}^3$ is any smooth mapping with compact support, and $\epsilon > 0$ is sufficiently small,

$$\mathbf{n}_\epsilon(x) = \frac{\mathbf{n}(x) + \epsilon\mathbf{v}(x)}{|\mathbf{n}(x) + \epsilon\mathbf{v}(x)|}$$

satisfies $|\mathbf{n}_\epsilon(x)| = 1$ and $\mathbf{n}_\epsilon|_{\partial\Omega} = \mathbf{n}_0$. Hence formally we have

$$\frac{d}{d\epsilon} I(\mathbf{n}_\epsilon)\Big|_{\epsilon=0} = 0.$$

Noting that $\frac{d\mathbf{n}_\epsilon}{d\epsilon}\Big|_{\epsilon=0} = (\mathbf{1} - \mathbf{n}(x)\otimes\mathbf{n}(x))\,\mathbf{v}(x)$, we obtain for the general Oseen-Frank energy $I(\mathbf{n}) = \int_\Omega F(x, \mathbf{n}, \nabla\mathbf{n})\,dx$ that $\mathbf{n}$ is a weak solution of the Euler-Lagrange equation

$$(\mathbf{1} - \mathbf{n}\otimes\mathbf{n})\left(\operatorname{div}\frac{\partial F}{\partial \nabla\mathbf{n}} - \frac{\partial F}{\partial \mathbf{n}}\right) = \mathbf{0}, \tag{1.3.1}$$

which equivalently, can be written as

$$\operatorname{div}\frac{\partial F}{\partial \nabla\mathbf{n}} - \frac{\partial F}{\partial \mathbf{n}} = \lambda(x)\,\mathbf{n}, \tag{1.3.2}$$

where $\lambda(x)$ is a Lagrange multiplier corresponding to the unit vector constraint for $\mathbf{n}$.

It is also easy to verify that for the one-constant case that $\mathbf{n}$ is a weak solution of the Euler-Lagrange equation

$$\Delta\mathbf{n} + |\nabla\mathbf{n}|^2\,\mathbf{n} = \mathbf{0}, \tag{1.3.3}$$

i.e.

$$n_{i,jj} + (n_{j,k}\, n_{j,k})\, n_i = 0 \quad (i = 1, 2, 3).$$

In the rest of this section we will describe some known results on the minimization problem of the Oseen-Frank energy. The existence of minimizers for the energy functional $I(\mathbf{n})$ with prescribed boundary conditions is a straightforward application of the direct method of the calculus of variations, and is stated in the following theorem.

Theorem 1.3.1. *If* $\mathbf{n}_0 \in H^1(\Omega; \mathbb{S}^2)$, *then there exists a minimizer* $\mathbf{n}^*$ *that minimizes* $I(\mathbf{n})$ *over all* $\mathbf{n}$ *in the admissible set* $A_0 = \{\mathbf{n} \in H^1(\Omega; \mathbb{S}^2) \,|\, \mathbf{n}|_{\partial\Omega} = \mathbf{n}_0\}$, *and* $\mathbf{n}^*$ *satisfies* (1.3.1), *namely the strong form of Euler-Lagrange equation of* $F(x, \mathbf{n}, \nabla\mathbf{n})$.

In the one-constant approximation (1.2.7), there are results that reveal deeper properties of the minimizers such as the results proved by Schoen & Uhlenbeck in [SU+82][SU+83] and by Brezis, Coron & Lieb in [BCL86] saying that any minimizer $\mathbf{n}^*$ of $I(\mathbf{n})$ has at most finitely many point singularities, and these singularities are rotated hedgehogs with degree (which will be defined in Chapter 2) ± 1. The result is stated in the following theorem.

Theorem 1.3.2. *In the one-constant case, any minimizer* $\mathbf{n}^*$ *in Theorem 1.3.1 is smooth except for a finite number of point defects located at points* $x(i) \in \Omega$, *and*

$$\mathbf{n}^*(x) \sim R(i)\,\frac{x - x(i)}{|x - x(i)|} \quad \text{as} \quad x \to x(i),$$

for some $R(i) \in O(3)$.

One can notice that the hedgehog $\hat{\mathbf{n}} : \mathbb{R}^3 \to \mathbb{S}^2$ which is defined by $\hat{\mathbf{n}} = \frac{x}{|x|}$ is in the Sobolev space $H^1(\Omega; \mathbb{S}^2)$ for a bounded domain Ω containing the origin and it plays an important role in the energy minimization problem of liquid crystals. Some interesting results on the energy minimization properties of the hedgehog are stated below.

Theorem 1.3.3. *(Brezis, Coron, Lieb [BCL86]) In the one-constant approximation, the hedgehog* $\hat{\mathbf{n}}$ *minimizes the Oseen-Frank free energy* $I(\mathbf{n})$ *subject to its own boundary conditions.*

There is also an elegant alternative proof of Theorem 1.3.3 due to F. H. Lin [Lin87]. Earlier, Hardt et al had shown in [HKL86b] that the degrees of these singularities are bounded by some universal constant and Cohen et al. [CHK+87] showed in their numerical study that singularities of degree two or more are unstable. These works as well as other studies such as [HKL86a] [HKL88a] [HKL88b] [WPC72] motivated the discovery of Theorem 1.3.2 and 1.3.3.

Theorem 1.3.3 can also be generalized to the non-equal constant case, with further assumption $k_1 \leq k_2$, and we show this in the following theorem which is due to [Ou92].

Theorem 1.3.4. *If* $k_1 \leq k_2$, *then the hedgehog* $\hat{\mathbf{n}}$ *is the unique minimizer of* $I(\mathbf{n})$ *in* $H^1(\Omega; \mathbb{S}^2)$, *subject to its own boundary conditions.*

Note that a simple proof due to J. Ball and E. Virga will appear in [BV].

Hélein [Hél87] has shown that when $8(k_1 - k_2) > k_3$ the second variation at $\hat{\mathbf{n}}$ can be negative, so that $\hat{\mathbf{n}}$ is not a minimizer. Moreover, results of Hélein [Hél87], Cohen & Taylor [CT90] and Kinderlehrer & Ou [KO92] show that the second variation $\delta^2 I(\hat{\mathbf{n}}) > 0$ if and only if

$$8(k_1 - k_2) \leq k_3.$$

The following result provides another deep insight with respect to the defects of the minimizers for Oseen-Frank free energy with general Frank constants.

Theorem 1.3.5. *(Hardt, Lin & Kinderlehrer [HKL86a]) Any minimizer* $\mathbf{n} \in H^1(\Omega; \mathbb{S}^2)$ *is analytic outside a closed set* χ *whose Hausdorff dimension is less than one.*

In the following chapters, we will consider a 2D exterior problem in the Oseen-Frank theory in a given domain $\Omega = \mathbb{R}^2 \setminus \cup_{j=1}^{N} \bar{\omega}_j$, where each of the $\{\omega_j\}_{j=1}^{N}$ is bounded, simply connected and has Lipschitz boundary, and also $\bar{\omega}_j$ are disjoint for $j = 1, ..., N$, i.e. $\bar{\omega}_i \cap \bar{\omega}_j = \emptyset$ for $i \neq j$, see Figure 1.5. We will be answering questions concerning the existence and uniqueness of minimizers satisfying the boundary conditions $\mathbf{n}|_{\partial\omega_j} = \mathbf{n}_j$, $j = 1, ..., N$ with each of the $\mathbf{n}_j$ having prescribed degree (which will be defined in Chapter 2). We will start by considering a single circular domain Ω in Chapter 3 and then study the case of general N-connected exterior domain Ω by introducing homotopy classes for vector fields. In Chapter 4, we will see that the minimizer $\mathbf{n}_m$ in each homotopy class C_m (which will be defined in Chapter 4) of a modified one-constant Oseen-Frank energy subject to prescribed boundary condition and given degree d_j on each of the boundaries $\partial\omega_j$ can be written in the form

$$\mathbf{n}_m(z) = \Pi_{j=1}^{N} h_j^{d_j}(z) \exp(i\phi_m)$$

where $h_j(z) = \frac{z-a_j}{|z-a_j|}$ for some $a_j \in \omega_j$, and ϕ_m satisfies

$$||\phi_m(x) - \beta||_{L^2(\mathbb{S}^1)} \leq C_0 \frac{1}{r} \quad \text{for} \quad r \geq r_0,$$

for some constant β, C_0, and some sufficiently large r_0. This describes the behaviour at ∞ of the minimizer $\mathbf{n}_m$ in each homotopy class C_m. Here we treat $\mathbf{n}$ as a vector field with

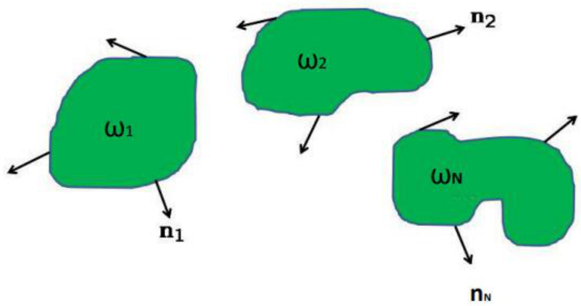

Figure 1.5: N-connected domain $\Omega = \mathbb{R}^2 \setminus \cup_{j=1}^{N} \bar{\omega}_j$.

range in the complex plane and $x = z$ as a point in the complex plane $\mathbb{C}$. Each $h_j(z)$ is in fact a hedgehog centred at $a_j \in \omega_j$, and we can somehow regard this result as being related to Theorem 1.3.2. Before beginning this analysis we will give a brief introduction to the Landau - de Gennes model as mentioned in Section 1.1, which is used in studying the minimization problem with non-orientable boundary conditions which will be discussed in Chapter 5.

1.4 An introduction to the Landau - de Gennes model

In this section we will give a brief introduction on the Landau - de Gennes model. Most of the content in this section can be found in [Bal17] as well as [BZ08].

The Landau - de Gennes model uses a tensor order parameter based on the probability distribution $\rho(\mathrm{x}, \mathbf{p})$ of molecular orientations $\mathbf{p} \in \mathbb{S}^2$ at a point x. Here $\mathbf{p}$ is parallel to the long axis of a molecule and we regard $\mathbf{p}$ and $-\mathbf{p}$ as being equivalent as was mentioned in Section 1.1. Two advantages over the Oseen-Frank model are that (i) it gives structure to defects, so that in particular they have finite energy, (ii) resolves the problem of orientability of the director.

Now consider the small ball $\mathbb{B}(\mathrm{x}, \delta) = \{\mathrm{z} \in \mathbb{R}^3 \,|\, |\mathrm{z} - \mathrm{x}| < \delta\}$ with centre x and small radius $\delta > 0$. We want δ to be smaller than macroscopic length scales but large enough to contain sufficiently many molecules for a statistical description to be possible. For example, if $\delta = 1\mu$m, then $\mathbb{B}(\mathrm{x}, \delta)$ would contain about a billion molecules. Then $\rho(\mathrm{x}, \mathbf{p})$ can be thought of as a smoothed out version of the probability of a molecule chosen at random from those in $\mathbb{B}(\mathrm{x}, \delta)$ having orientation $\mathbf{p}$.

Since $\rho(\mathrm{x}, \mathbf{p})$ is a probability distribution and $\pm\mathbf{p}$ are equivalent, $\rho(\mathrm{x}, \mathbf{p})$ satisfies

$$\begin{aligned} &\rho(\mathrm{x}, \mathbf{p}) \geq 0, \\ &\rho(\mathrm{x}, \mathbf{p}) = \rho(\mathrm{x}, -\mathbf{p}), \\ &\int_{\mathbb{S}^2} \rho(\mathrm{x}, \mathbf{p}) \, \mathrm{d}\mathbf{p} = 1, \end{aligned}$$

where $\mathbf{dp}$ denotes the area element on $\mathbb{S}^2$. Thus the first moment satisfies

$$\int_{\mathbb{S}^2} \mathbf{p} \, \rho(\mathrm{x}, \mathbf{p}) \, \mathrm{d}\mathbf{p} = -\int_{\mathbb{S}^2} \mathbf{p} \, \rho(\mathrm{x}, -\mathbf{p}) \, \mathrm{d}\mathbf{p} = \mathbf{0},$$

and the second moment

$$\mathrm{M}(\mathrm{x}) = \int_{\mathbb{S}^2} \mathbf{p} \otimes \mathbf{p} \, \rho(\mathrm{x}, \mathbf{p}) \, \mathrm{d}\mathbf{p}$$

is a symmetric second-order tensor which is positive definite, namely $\mathbf{Me} \cdot \mathbf{e} > 0$ for all $\mathbf{e} \in \mathbb{S}^2$. This is true since

$$\mathrm{M}(\mathrm{x})\mathbf{e} \cdot \mathbf{e} = \int_{\mathbb{S}^2} (\mathrm{p} \cdot \mathbf{e})^2 \rho(\mathrm{x}, \mathbf{p}) \, \mathrm{d}\mathbf{p} \geq 0,$$

and $\mathrm{M}(\mathrm{x}) \, \mathbf{e} \cdot \mathbf{e} = 0$ implies that $\rho(\mathrm{x}, \mathbf{p}) = 0$ for $\mathbf{p} \cdot \mathbf{e} \neq 0$, contradicting the fact $\int_{\mathbb{S}^2} \rho(\mathrm{x}, \mathbf{p}) \, \mathrm{d}\mathbf{p} = 1$.

Also, one can easily see that $\operatorname{tr} \mathrm{M}(\mathrm{x}) = 1$. The case when $\rho(\mathrm{x}, \mathbf{p}) = \frac{1}{4\pi}$ corresponds to an isotropic molecular distribution at x, with $\mathrm{M}(\mathrm{x}) = \frac{1}{3}\mathbf{I}$.

The de Gennes Q-tensor is then defined as

$$Q(x) = M(x) - \frac{1}{3}\mathbf{I} = \int_{\mathbb{S}^2} (\mathbf{p} \otimes \mathbf{p} - \frac{1}{3}\mathbf{I})\, \rho(x, \mathbf{p})\, d\mathbf{p}, \tag{1.4.1}$$

which measures the deviation of M(x) from its isotropic value, and it satisfies

$$\begin{aligned} &Q(x) = Q^T(x), \\ &\operatorname{tr} Q(x) = 0, \\ &\lambda_{\min}(Q(x)) > -\frac{1}{3}, \end{aligned}$$

where $\lambda_{\min}(Q(x))$ denotes the minimum eigenvalue of Q(x).

Following de Gennes we suppose that the free energy for a nematic at constant temperature is given by

$$I(Q) = \int_\Omega \psi(Q, \nabla Q)\, dx. \tag{1.4.2}$$

Note that if we write $H = \nabla Q$, where $Q = (Q_{ij})$, $H = (H_{ijk}) = (Q_{ij,k})$, then as a consequence of frame-indifference, $\psi = \psi(Q, H)$ should satisfy for any $R = (R_{ij}) \in O(3) = \{R \mid R^T R = \mathbf{I}\}$ the isotropy condition

$$\psi(Q^*, H^*) = \psi(Q, H)$$

where $Q^* = R_{ir} R_{js} Q_{rs}$, and $H^*_{ijk} = R_{ir} R_{js} R_{kt} H_{rst}$.

It is also usual to decompose the free energy density ψ as

$$\begin{aligned} \psi(Q, \nabla Q) &= \psi(Q, 0) + (\psi(Q, \nabla Q) - \psi(Q, 0)) \\ &= \psi_B(Q) + \psi_E(Q, \nabla Q), \end{aligned}$$

where $\psi_B(Q)$ and $\psi_E(Q)$ are respectively the bulk- and elastic-energy densities.

It is often assumed that the bulk-energy density has the form

$$\psi_B(Q) = a \operatorname{tr} Q^2 - \frac{2b}{3} \operatorname{tr} Q^3 + c \operatorname{tr} Q^4 \tag{1.4.3}$$

where $b > 0$, $c > 0$, and where a depends linearly on temperature. Moreover, if $a < \frac{b^2}{27c}$ then ψ_B is minimized by Q having the uniaxial form

$$Q = s\,(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}), \quad \mathbf{n} \in \mathbb{S}^2 \tag{1.4.4}$$

where

$$s = \frac{b + \sqrt{b^2 - 24ac}}{4c} > 0.$$

Also it is usually assumed that the elastic-energy density $\psi_E(Q, \nabla Q)$ is quadratic in ∇Q. Examples of isotropic functions quadratic in ∇Q are the invariants $I_i = I_i(Q, \nabla Q)$, namely

$$\begin{aligned} I_1 &= Q_{ij,k}\, Q_{ij,k} \\ I_2 &= Q_{ij,k}\, Q_{ik,k} \\ I_3 &= Q_{ik,j}\, Q_{ij,k} \\ I_4 &= Q_{lk}\, Q_{ij,l}\, Q_{ij,k}. \end{aligned}$$

The first three linearly independent invariants I_1, I_2, I_3 span the possible isotropic quadratic functions of ∇Q. The invariant I_4 is one of six possible linearly independent cubic terms that are quadratic in ∇Q. Note also that

$$I_2 - I_3 = (Q_{ij}\, Q_{ik,k})_{,j} - (Q_{ij}\, Q_{ik,j})_{,k}$$

is a null Lagrangian. Similar to (1.2.1) and (1.2.6), we assume that

$$\psi_E(Q, \nabla Q) = \Sigma_{i=1}^{4} L_i\, I_i, \tag{1.4.5}$$

where the L_i are constants related to the given materials. In addition, in a similar way to the one-constant approximation (1.2.7) of Oseen-Frank model, we have that if $L_2 = L_3 = L_4 = 0$, then $\psi_E = L_1\, |\nabla Q|^2$, the one-constant approximation of the Landau - de Gennes theory.

Now we show how the Oseen-Frank theory can be obtained from that of Landau - de Gennes (see [MN14]). Since ψ_B is minimized for uniaxial Q given by (1.4.4), in the limit of small elastic constants L_i (see [GJ15] for reference), we expect minimizers of $\mathbf{I}(Q)$ to be nearly uniaxial. This motivates the constrained theory in which we minimize $\mathbf{I}(Q)$ subject to the constraint

$$Q = s\,(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}), \quad \mathbf{n} \in \mathbb{S}^2$$

for fixed $s > 0$. Calculating the invariants in terms of $\mathbf{n}$ we find that

$$\begin{aligned} I_1 &= s^2 \left((\operatorname{div} \mathbf{n})^2 + |\mathbf{n} \wedge \nabla \wedge \mathbf{n}|^2 \right) \\ I_2 &= s^2 \left(|\mathbf{n} \wedge \nabla \wedge \mathbf{n}|^2 + \operatorname{tr}(\nabla \mathbf{n})^2 \right) \\ I_3 &= 2s^2 \left(\operatorname{tr}(\nabla \mathbf{n})^2 + (\mathbf{n} \cdot \nabla \wedge \mathbf{n})^2 + |\mathbf{n} \wedge \nabla \wedge \mathbf{n}|^2 \right) \\ I_4 &= 2s^3 \left(\frac{2}{3}|\mathbf{n} \wedge \nabla \wedge \mathbf{n}|^2 - \frac{1}{3}\operatorname{tr}(\nabla \mathbf{n})^2 - \frac{1}{3}|\mathbf{n} \cdot \nabla \wedge \mathbf{n}|^2 \right). \end{aligned}$$

Letting

$$\begin{aligned} k_1 &= L_1 s^2 + L_2 s^2 + 2\,L_3 s^2 - \frac{2}{3} L_4 s^3 \\ k_2 &= 2\,L_3 s^2 - \frac{2}{3} L_4 s^3 \\ k_3 &= L_1 s^2 + L_2 s^2 + 2\,L_3 s^2 + \frac{4}{3} L_4 s^3 \\ k_4 &= L_2 s^2 \end{aligned}$$

we find that

$$W(\mathbf{n}, \nabla \mathbf{n}) = \psi_E(Q, \nabla Q).$$

Thus the Oseen-Frank elastic energy is formally the same as the constrained Landau-de Gennes elastic energy. On the other hand, although the energy density can be regarded as being the same in the two theories, in the constrained Landau-de Gennes theory there are more possibilities for energy minimization than in the Oseen-Frank theory, as there are more line fields than vector fields. In fact, Ball and Zarnescu studied in [BZ08] and [BZ11] the differences and the overlaps between the two models. In particular, in the case of non-zero constant scalar order parameter s the Landau-de Gennes theory is equivalent to that of Oseen-Frank when the director field is orientable. This will be discussed with more detail in Chapter 5 when we study the minimization problem with non-orientable boundary line fields.

Chapter 2

Degree theory and liftings

2.1 Degree for vector field $\mathbf{n} \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$

Sobolev maps between manifolds appear naturally in different contexts: harmonic maps, liquid crystals and the Ginzburg-Landau equation, etc. Given a continuous mapping $\mathbf{n}$ from one compact smooth $n-$dimensional manifold X to another compact smooth manifold Y of the same dimension, where that X and Y are manifolds without boundary and are oriented, the classical degree counts the "number of times" Y is covered by $\mathbf{n}(X)$, taking into account algebraic multiplicity. If $\mathbf{n} \in C^1(X, Y)$ and $y \in Y$ is a regular value of the map $\mathbf{n}$, i.e. $\mathbf{n}^{-1}(y)$ consists of a finite number of points $\{x_1, ..., x_k\}$ at each of which the Jacobian matrix of the map, $\nabla\mathbf{n}$, in terms of local coordinates with the given orientation, is nonsingular, then

$$\deg(\mathbf{n}, X, y) = \Sigma_j \operatorname{sgn} \det \nabla\mathbf{n}(x_j), \tag{2.1.1}$$

where it can be shown that the right-hand side is independent of the choice of the regular value y.

This above definition of the degree for differentiable maps extends to continuous maps $\mathbf{n} : X \mapsto Y$, since if $u, v \in C^1(X, Y)$, and are close to each other in the C^0 topology, then they have the same degree. Further, from [Nir74, p. 4] one can see that, for $\mathbf{n} \in C^1(X, Y)$, there is an integral formula for the degree using any smooth $n-$form ν on Y:

$$\int_X \nu \circ \mathbf{n} = \deg(\mathbf{n}, X, Y) \cdot \int_Y \nu,$$

which may be represented using local coordinates with given orientation by

$$\int_X f(\mathbf{n}) \det \nabla\mathbf{n}(x)\, dx_1 \wedge ... \wedge dx_n,$$

if the smooth n−form $\nu = f(y)\, dy_1 \wedge ... \wedge dy_n$. In particular, if X and Y are Riemannian manifolds, we have

$$\deg(\mathbf{n}, X, Y) = \frac{1}{|Y|} \int_X \det \nabla \mathbf{n}(x)\, d\sigma(x) \tag{2.1.2}$$

where $d\sigma$ is the volume element on X, and the Jacobian $\det \nabla \mathbf{n}(x)$ is computed by geodesic normal coordinates at x and geodesic normal coordinates at $\mathbf{n}(x)$. More specifically, when $X = \partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^{n+1}$, and $Y = \mathbb{S}^n$, consider a mapping $\mathbf{n} \in C^1(X, Y)$ and let $\widetilde{\mathbf{n}}$ be any C^1−extension of $\mathbf{n}$ inside Ω with values in $\mathbb{R}^{n+1}$, then the degree of $\mathbf{n}$ can be written as

$$\deg(\mathbf{n}, \partial\Omega, \mathbb{S}^n) = \frac{1}{|\mathbb{B}|} \int_\Omega \det \nabla \widetilde{\mathbf{n}}\, dx_1 ... dx_{n+1} \tag{2.1.3}$$

where $|\mathbb{B}|$ is volume of unit ball $\mathbb{B}$ in $\mathbb{R}^{n+1}$. By applying Green's formula and noticing the fact that $\det \nabla \mathbf{n}$ is a divergence form, one can easily see that (2.1.2) and (2.1.3) are equivalent. One may use (2.1.2) and (2.1.3) together with approximation by smooth maps to define the degree even if $\mathbf{n}$ is not continuous.

A significant fact is that the degree is invariant for maps in the same homotopy class. Also if the degree is not zero, then the map $\mathbf{n}$ is onto from X to Y.

Motivated by a problem concerning the Ginzburg-Landau equation (see [dMBGP91]), L. Boutet de Monvel and O. Gabber introduced a degree for $\mathbf{n} \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$, where $\mathbf{n}$ is treated as a vector field with range in the complex plane. This definition of degree is in the more familiar form, i.e. "change in argument":

$$\deg_{\mathbb{S}^1} \mathbf{n} = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{d\mathbf{n}}{\mathbf{n}} = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \overline{\mathbf{n}}\, d\mathbf{n} = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \overline{\mathbf{n}} \frac{\partial \mathbf{n}}{\partial \theta}\, d\theta, \tag{2.1.4}$$

where $\overline{\mathbf{n}}$ is the complex conjugate of $\mathbf{n}$. One can see that (2.1.4) is well defined due to the duality between $H^{1/2}$ and $H^{-1/2}$ $\Big($ that is $\overline{\mathbf{n}} \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ and $\frac{\partial \mathbf{n}}{\partial x} \in H^{1/2}(\mathbb{S}^1; \mathbb{R}^2)$ $\Big)$. Their definition of degree for $\mathbf{n} \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ can be found in the appendix of [dMBGP91] and we state the result in the following Theorem 2.1.2.

Remark 2.1.1. In the following context, when we treat $\mathbf{n}$ as a vector field with range in the complex plane $\mathbb{C}$, we usually do not distinguish the use of notation between $\mathbf{n}(x)$ and $\mathbf{n}(z)$, as well as between $\mathbb{R}^2$ and $\mathbb{C}$. Also, in the following Theorem 2.1.2, the given vector field $\mathbf{n}(x) \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ can be expressed explicitly in terms of $z \in \mathbb{S}^1 \subset \mathbb{C}$, i.e. we will write $\mathbf{n}(x)$ as $\mathbf{n}(z) = z^k \exp(i\phi)$. In most cases, the given unit vector field $\mathbf{n}$ is treated in the real sense when we apply integration with respect to the variable $x \in \Omega$, and it is treated in the complex sense when we multiply vectors. This occurs throughout the following chapters.

Theorem 2.1.2. *Let* $\mathbf{n}$ *be a function of Sobolev class* $H^{1/2}$ *from the circle* $\mathbb{S}^1$ *to itself. Then there exists an integer* k *and a real function* $\phi \in H^{1/2}(\mathbb{S}^1)$, *unique up to an integral multiple of* 2π, *such that* $\mathbf{n} = z^k \exp(i\phi)$. *The winding number* k *is given by*

$$k = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \mathbf{n}^{-1}\frac{\partial \mathbf{n}}{\partial \theta}\, d\theta. \tag{2.1.5}$$

Note that $\mathbf{n}^{-1} = \overline{\mathbf{n}} \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$ because $|\mathbf{n}| = 1$, and the integral is defined in the distribution sense since $\frac{\partial \mathbf{n}}{\partial \theta} \in H^{-1/2}(\mathbb{S}^1;R^2)$. A detailed proof can also be found in [dMBGP91].

As a consequence, the degree for the given vector field $\mathbf{n} \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$ can be defined as

$$\deg_{\mathbb{S}^1}\mathbf{n} = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \mathbf{n}^{-1}\frac{\partial \mathbf{n}}{\partial \theta}\, d\theta.$$

An alternative to defining the degree (winding number) for $\mathbf{n} \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$ as in Theorem 2.1.2, is to extend $\mathbf{n} \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$ to $\widetilde{\mathbf{n}} \in H^1(\mathbb{D};\mathbb{R}^2)$, and then apply formula (2.1.3), where $\mathbb{D}$ is the unit disk in $\mathbb{R}^2 \cong \mathbb{C}$,. This definition of the $H^{1/2}$ degree is also used in [BBH+94] by Bethuel, Brezis and Hélein. In the following proposition, we will sketch the idea of defining the degree (winding number) for a given vector field $\mathbf{n} \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$ by extending $\mathbf{n} \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$ inward (to some $\widetilde{\mathbf{n}} \in H^1(\mathbb{D};\mathbb{R}^2)$) using an argument kindly suggested by P. Mironescu.

Proposition 2.1.3. *For a given vector field* $\mathbf{n} \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$, *define the degree* $\deg_{\mathbb{S}^1}\mathbf{n}$ *to be:*

$$\deg_{\mathbb{S}^1}\mathbf{n} = \fint_{\mathbb{D}} \det \nabla\widetilde{\mathbf{n}}\, dx = \frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} \det \nabla\widetilde{\mathbf{n}}\, dx \tag{2.1.6}$$

for any $\widetilde{\mathbf{n}} \in H^1(\mathbb{D};\mathbb{R}^2))$ *with* $\widetilde{\mathbf{n}}|_{\mathbb{S}^1} = \mathbf{n}$. *The value of the degree* $\deg_{\mathbb{S}^1}\mathbf{n}$ *so defined is independent of the extension* $\widetilde{\mathbf{n}}$ *and equals the value in (2.1.5), i.e.*

$$\deg_{\mathbb{S}^1}\mathbf{n} = \fint_{\mathbb{D}} \det \nabla\widetilde{\mathbf{n}}\, dx = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \mathbf{n}^{-1}\frac{\partial \mathbf{n}}{d\theta}\, d\theta. \tag{2.1.7}$$

Proof. We prove the proposition in several steps.

Step 1

We prove in this step that $\fint_{\mathbb{D}} \det \nabla\widetilde{\mathbf{n}}\, dx$ is independent of the extension. For this, we need to show that given two vector fields $\widetilde{\mathbf{n}} \in H^1(\mathbb{D};\mathbb{R}^2)$ and $\widetilde{\mathbf{m}} \in H^1(\mathbb{D};\mathbb{R}^2)$ with $\widetilde{\mathbf{n}}|_{\mathbb{S}^1} = \widetilde{\mathbf{m}}|_{\mathbb{S}^1} = \mathbf{n}$, we have

$$\int_{\mathbb{D}} \det \nabla\widetilde{\mathbf{n}}\, dx = \int_{\mathbb{D}} \det \nabla\widetilde{\mathbf{m}}\, dx. \tag{2.1.8}$$

To show this, we can take two sequences $\{\widetilde{\mathbf{n}}^j\}_{j=1}^{\infty}$ and $\{\mathbf{w}^j\}_{j=1}^{\infty}$ satisfying $\widetilde{\mathbf{n}}^j \in C^1(\overline{\mathbb{D}};\mathbb{R}^2)$ and $\mathbf{w}^j \in C_0^{\infty}(\mathbb{D};\mathbb{R}^2)$, respectively, such that

$$\widetilde{\mathbf{n}}^j \to \widetilde{\mathbf{n}} \quad \text{in} \quad H^1(\mathbb{D};\mathbb{R}^2)$$
$$\mathbf{w}^j \to \widetilde{\mathbf{m}} - \widetilde{\mathbf{n}} \quad \text{in} \quad H^1(\mathbb{D};\mathbb{R}^2).$$

Therefore we have

$$\int_{\mathbb{D}} \det \nabla\widetilde{\mathbf{n}}^j\, dx = \int_{\mathbb{D}} \det \nabla(\widetilde{\mathbf{n}}^j + \mathbf{w}^j)\, dx,$$

which is a direct result from Theorem 2 of Chapter 8.1 in [Eva10]. (The determinant function $L(P) = \det P$, $P \in \mathbb{M}^{n\times n}$ is a null Lagrangian.)

Also since

$$\int_{\mathbb{D}} \det \nabla\widetilde{\mathbf{n}}^j\, dx \to \int_{\mathbb{D}} \det \nabla\widetilde{\mathbf{n}}\, dx$$
$$\int_{\mathbb{D}} \det \nabla(\widetilde{\mathbf{n}}^j + \mathbf{w}^j)\, dx \to \int_{\mathbb{D}} \det \nabla\widetilde{\mathbf{m}}\, dx,$$

we have shown that

$$\int_{\mathbb{D}} \det \nabla\widetilde{\mathbf{n}}\, dx = \int_{\mathbb{D}} \det \nabla\widetilde{\mathbf{m}}\, dx,$$

which is exactly (2.1.8).

Since (2.1.8) holds, it will be sufficient to find some vector field $\mathbf{v} \in H^1(\mathbb{D};\mathbb{R}^2)$, which satisfies $\mathbf{v}|_{\mathbb{S}^1} = \mathbf{n}$ and to show that the integral $\fint_{\mathbb{D}} \det \nabla\mathbf{v}\, dx$ is an integer. We will construct such a vector field $\mathbf{v}$ in the following steps.

Step 2

Claim: Given any $\mathbf{v} \in H^1(\mathbb{D};\mathbb{R}^2)$, we have

$$\lim_{r\to 1} \mathbf{v}(r\,\theta) = \mathbf{v}(\theta) \quad \text{in} \quad H^{1/2}(\mathbb{S}^1;\mathbb{R}^2).$$

By trace theory there is a constant $C > 0$ such that for any $\mathbf{v} \in H^1(\mathbb{D}; \mathbb{R}^2)$

$$||\mathbf{v}(\theta)||_{H^{1/2}(\mathbb{S}^1;\mathbb{R}^2)} \leq C\, ||\mathbf{v}(x)||_{H^1(\mathbb{D};\mathbb{R}^2)}.$$

As a consequence, we have for any $r \in (0, 1)$ that

$$||\mathbf{v}(r\,\theta) - \mathbf{v}(\theta)||_{H^{1/2}(\mathbb{S}^1;\mathbb{R}^2)} \leq C\, ||\mathbf{v}(r\,x) - \mathbf{v}(x)||_{H^1(\mathbb{D};\mathbb{R}^2)}.$$

Also, given any $\epsilon > 0$ and by choosing smooth functions $\{\mathbf{v}_j\}_{j=1}^{\infty}$, such that

$$||\mathbf{v}_j(x) - \mathbf{v}(x)||_{H^1(\mathbb{D};\mathbb{R}^2)} < \frac{\epsilon}{3}$$

$$||\mathbf{v}_j(rx) - \mathbf{v}(rx)||_{H^1(\mathbb{D};\mathbb{R}^2)} < \frac{\epsilon}{3}$$

and for sufficiently large j independent of radius r, and so for such a j we can apply the bounded convergence theorem and have

$$||\mathbf{v}_j(rx) - \mathbf{v}_j(x)||_{H^1(\mathbb{D};\mathbb{R}^2)} \to 0 \quad \text{as} \quad r \to 1.$$

Therefore for any $\epsilon > 0$, there exists an r_0 close enough to 1 such that for $\forall r > r_0$, we have

$$\begin{aligned} ||\mathbf{v}(r\,x) - \mathbf{v}(x)||_{H^1(\mathbb{D};\mathbb{R}^2)} &\leq ||\mathbf{v}_j(rx) - \mathbf{v}(rx)||_{H^1(\mathbb{D};\mathbb{R}^2)} + ||\mathbf{v}_j(rx) - \mathbf{v}_j(x)||_{H^1(\mathbb{D};\mathbb{R}^2)} + ||\mathbf{v}_j(x) - \mathbf{v}(x)||_{H^1(\mathbb{D};\mathbb{R}^2)} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon, \end{aligned}$$

which proves

$$||\mathbf{v}(rx) - \mathbf{v}(x)||_{H^1(\mathbb{D};\mathbb{R}^2)} \to 0 \quad \text{as} \quad r \to 1,$$

and hence

$$||\mathbf{v}(r\,\theta) - \mathbf{v}(\theta)||_{H^{1/2}(\mathbb{S}^1;\mathbb{R}^2)} \to 0 \quad \text{as} \quad r \to 1.$$

This proves our claim.

Step 3

We now extend $\mathbf{n} \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ to the unit disk by its harmonic extension $\widetilde{\mathbf{n}} \in H^1(\mathbb{D}; \mathbb{R}^2)$.

Recall that we do not distinguish between $\mathbb{C}$ and $\mathbb{R}^2$, but still treat $\mathbf{n}$ and $\widetilde{\mathbf{n}}$ as vector fields with range in the complex plane $\mathbb{C}$.

The harmonic extension $\mathbf{n}$ is the unique weak solution of

$$\begin{cases} \Delta\widetilde{\mathbf{n}} = 0 \\ \widetilde{\mathbf{n}}|_{\mathbb{S}^1} = \mathbf{n}. \end{cases}$$

Using the regularity up to the boundary for second-order elliptic equations (see [Eva10, Theorem 6, Chapter 6.3]), we know that $\widetilde{\mathbf{n}}$ exists and is smooth in $\mathbb{D}$.

Step 4

Claim: $|\widetilde{\mathbf{n}}| \to 1$ uniformly as $|x| \to 1$.

By [dMBGP91, Proposition A. 2] we know that $H^{1/2}(\mathbb{S}^1) \subset \mathrm{VMO}$ (one can refer to [BN95] for the definition of VMO space). Thus we can assume $\mathbf{n}, \mathbf{m} \in \mathrm{VMO} \cap L^\infty(\mathbb{S}^1)$ and set $\mathbf{w} = \mathbf{n}\,\mathbf{m}$. Then by [Dou12, Lemma 6.44], their harmonic extensions to the unit disc $\mathbb{D}$, say $\widetilde{\mathbf{n}}$, $\widetilde{\mathbf{m}}$ and $\widetilde{\mathbf{w}}$, satisfy

$$\lim_{|x|\to 1} (\widetilde{\mathbf{m}}\,\widetilde{\mathbf{n}} - \widetilde{\mathbf{w}}) = 0. \tag{2.1.9}$$

Now take in the above $\mathbf{m} = \overline{\mathbf{n}}$ (complex conjugate) for some $\mathbf{n}$ such that $|\mathbf{n}| = 1$. Then $\mathbf{w} = 1$, and thus $\widetilde{\mathbf{w}} = 1$. On the other hand, $\widetilde{\mathbf{m}} = \overline{\widetilde{\mathbf{n}}}$, and therefore we have $\widetilde{\mathbf{m}}\,\widetilde{\mathbf{n}} = |\widetilde{\mathbf{n}}|^2$. As a consequence (2.1.9) becomes

$$\lim_{|x|\to 1} |\widetilde{\mathbf{n}}| = 1,$$

which proves our claim.

Step 5

We now define

$$v = \begin{cases} \frac{\widetilde{\mathbf{n}}}{|\widetilde{\mathbf{n}}|}, & |\widetilde{\mathbf{n}}| \geq \frac{1}{2} \\ 2\widetilde{\mathbf{n}}, & |\widetilde{\mathbf{n}}| < \frac{1}{2} \end{cases} \tag{2.1.10}$$

and we have $\mathbf{v} \in H^1(\mathbb{D}; \mathbb{R}^2)$ and $\mathbf{v}|_{\mathbb{S}^1} = \widetilde{\mathbf{n}}|_{\mathbb{S}^1} = \mathbf{n} \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$.

By step 3 and step 4, we can also easily observe that $\mathbf{v}$ is smooth and $|\mathbf{v}| = 1$ in an annulus

$$A = \{x \,|\, 0 < a < |x| < 1\}.$$

Therefore $\mathbf{v}(r_j\,\theta)$ is smooth and as a result, for $r_j < 1$ close to 1, we have

$$\fint_{\mathbb{D}} \det \nabla \mathbf{v}(r_j\,x)\,dx = \fint_{\mathbb{D}(r_j)} \det \nabla \mathbf{v}\,dx = \deg_{\mathbb{S}^1} \mathbf{v}(r_j\,\theta) \in \mathbb{Z}.$$

Further, since we have

$$\fint_{\mathbb{D}(r_j)} \det \nabla \mathbf{v}\,dx \to \fint_{\mathbb{D}} \det \nabla \mathbf{v}\,dx, \quad r_j \to 1, \tag{2.1.11}$$

we know that the integral $\fint_{\mathbb{D}} \det \nabla \mathbf{v}\,dx$ is an integer.

Also since

$$\fint_{\mathbb{D}(r_j)} \det \nabla \mathbf{v}\,dx = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \mathbf{v}^{-1}(r_j\,\theta) \frac{\partial \mathbf{v}}{\partial \theta}(r_j\,\theta)\,d\theta$$

by step 2 and (2.1.11) we have:

$$\fint_{\mathbb{D}} \det \nabla \mathbf{v}\,dx = \lim_{r_j \to 1} \frac{1}{2\pi i} \int_{\mathbb{S}^1} \mathbf{v}^{-1}(r_j\,\theta) \frac{\partial \mathbf{v}}{\partial \theta}(r_j\,\theta)\,d\theta = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \mathbf{n}^{-1} \frac{\partial \mathbf{n}}{d\theta}\,d\theta. \tag{2.1.12}$$

Hence, as a result, we can use v in (2.1.10) to define the degree of $\mathbf{n} \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$, i.e.

$$\deg_{\mathbb{S}^1} \mathbf{n} = \fint_{\mathbb{D}} \det \nabla \mathbf{v},$$

which is well defined and is an integer. By (2.1.12) we have

$$\deg_{\mathbb{S}^1} \mathbf{n} = \fint_{\mathbb{D}} \det \nabla \widetilde{\mathbf{n}}\,dx = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \mathbf{n}^{-1} \frac{\partial \mathbf{n}}{d\theta}\,d\theta,$$

which is exactly (2.1.7).

□

Remark 2.1.4. The above definition of degree make sense for maps in the class VMO: the closure in the BMO (= bounded mean oscillation) topology of smooth maps. L. Boutet de Monvel and O. Gabber made this interesting observation but did not establish the basic properties of VMO degree, such as stability under homotopy within VMO, surjectivity if deg ≠ 0, etc. VMO degree is defined via approximation instead of by an integral formula. More specifically, they pointed out that if $\mathbf{n} \in VMO(\mathbb{S}^1; \mathbb{S}^1)$ and

$$\bar{\mathbf{n}}_\epsilon(\theta) = \frac{1}{2\epsilon} \int_{\theta-\epsilon}^{\theta+\epsilon} \mathbf{n}(s)\,ds,$$

then $|\bar{\mathbf{n}}_\epsilon| \to 1$ uniformly in θ, where $\mathbf{n}$ need not be continuous. Then for ϵ sufficiently small,

$$\mathbf{n}_\epsilon(\theta) = \frac{\bar{\mathbf{n}}_\epsilon(\theta)}{|\bar{\mathbf{n}}_\epsilon(\theta)|}$$

has a well defined degree, which is independent of ϵ.

More results and properties of degree of VMO maps can be found in the exhaustive survey article by Petru Mironescu [Mir07], and in J. Bourgain, H. Brezis, P. Mironescu [BBM05]. Detailed proofs of the properties of the degree of VMO maps are given in articles by H. Brezis and L. Nirenberg, see [BN95], [BN96]. For those interested in topics in degree and lifting with respect to Sobolev maps on manifolds, one can refer to [Mir07], [Bre97] and [Bre06] for open problems in the topic.

2.2 Definition of degree in N-connected domain

In the last section, we have defined a degree for a given vector field $\mathbf{n} \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$. We now generalize this definition of degree to a more general domain. Consider a N-connected domain Ω generated by N Lipschitz Jordan curves, i.e. $\Omega = \mathbb{R}^2 \setminus \cup_1^N \bar{\omega}_j$, where each of $\{\omega_j\}_{j=1}^N$ is bounded, simply connected and Lipschitz. Also we assume that $\bar{\omega}_j$ are disjoint for $j = 1, ..., N$, i.e. $\bar{\omega}_i \cap \bar{\omega}_j = \emptyset$ for $i \neq j$. Then we have the following proposition.

Proposition 2.2.1. *For each* j *there exists a orientation-preserving bi-Lipschitz map* $\Psi_j : \overline{\mathbb{D}} \mapsto \bar{\omega}_j$, *such that* Ψ_j *maps the interior* $\mathbb{D}$ *of* $\overline{\mathbb{D}}$ *to* ω_j, *and maps the boundary* $\partial\mathbb{D}$ *to the boundary* $\partial\omega_j$, *where* $\mathbb{D}$ *is the unit disk in* $\mathbb{R}^2 \cong \mathbb{C}$.

Proof. By using Remark 5.3 in Ball and Zarnescu [BZ17], we know there exists an orientation-preserving bi-Lipschitz map $\phi_j : \bar{\omega} \mapsto \bar{\omega}_j$, where ω is a bounded domain of class C^∞, and $\bar{\omega} \subset \omega_j$. It is easily seen that ω is simply-connected. Thus by applying [BK87, Theorem A], which gives a boundary regularity result for the Riemann Mapping, we can derive a bi-Lipschitz map $\tilde{\phi}_j : \bar{\omega} \mapsto \overline{\mathbb{D}}$. Therefore we obtained our bi-Lipschitz map $\Psi_j = \phi_j \circ \tilde{\phi}_j^{-1}$ as required.

□

With Proposition 2.2.1, we can now define the degree d_j of a given vector field $\mathbf{n}_j$ on the boundary of each domain ω_j, i.e. $\deg_{\partial\omega_j}\mathbf{n}_j = d_j$ for $j = 1, ..., n$.

Definition 2.2.2. *For* $\mathbf{n}_j \in H^{1/2}(\partial\omega_j; \mathbb{S}^1)$, *we define the degree of* $\mathbf{n}_j$ *on the boundary* $\partial\omega_j$ *to be:*

$$\deg_{\partial\omega_j}(\mathbf{n}_j) = \deg_{\mathbb{S}^1} \mathbf{n}_j \circ \Psi_j.$$

We can see that $\mathbf{n} \circ \Psi_j$ is a map from $\mathbb{S}^1$ to $\mathbb{S}^1$, and we need to show that the degree of $\mathbf{n}_j$ in Definition 2.2.2 above is well defined. We need to prove the following proposition:

Proposition 2.2.3. *The degree is well defined in Definition 2.2.2 since we have:*

- $\mathbf{n}_j \circ \Psi_j \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$.
- $\deg_{\partial\omega_j} \mathbf{n}_j$ *does not depend on the particular orientation-preserving bi-Lipschitz map.*

Proof. To show that $\mathbf{n}_j \circ \Psi_j \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$, we need to prove that

$$I = \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|(\mathbf{n}_j \circ \Psi_j)(\theta_1) - (\mathbf{n}_j \circ \Psi_j)(\theta_2)|^2}{|\theta_1 - \theta_2|^2}\, d\theta_1\, d\theta_2 < \infty.$$

Recalling that Ψ_j is bi-lipschitz and changing variables to $t_1 = \Psi_j(\theta_1)$ and $t_2 = \Psi(\theta_2)$, we have that

$$\begin{aligned} I &= \int_{\partial\omega_1}\int_{\partial\omega_2} \frac{|(\mathbf{n}_j \circ \Psi_j)(\theta_1) - (\mathbf{n}_j \circ \Psi_j)(\theta_2)|^2}{|\Psi^{-1}(t_1) - \Psi^{-1}(t_2)|^2}\, \frac{d\theta_1}{dt_1}\,\frac{d\theta_2}{dt_2}\, dt_1\, dt_2 \\ &\leq C \int_{\partial\omega_1}\int_{\partial\omega_2} \frac{|(\mathbf{n}_j \circ \Psi_j)(\theta_1) - (\mathbf{n}_j \circ \Psi_j)(\theta_2)|^2}{|t_1 - t_2|^2}\, dt_1\, dt_2 < \infty, \end{aligned}$$

which implies that $\mathbf{n}_j \circ \Psi_j \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$.

Next we show that $\deg_{\partial\omega_j} \mathbf{n}_j$ does not depend on the particular orientation-preserving bi-Lipschitz map. Assuming now that we have two orientation-preserving bi-Lipschitz maps Ψ_j and $\widetilde{\Psi}_j$, we have $\mathbf{n}_j \circ \Psi_j \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ and $(\mathbf{n}_j \circ \Psi_j) \circ \widetilde{\Psi}_j^{-1} \circ \Psi_j \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$, where $\widetilde{\Psi}_j^{-1} \circ \Psi_j$ is not the identity map from $\mathbb{S}^1$ to $\mathbb{S}^1$ as Ψ_j and $\widetilde{\Psi}_j$ are different. Therefore to show that $\deg_{\partial\omega_j} \mathbf{n}_j$ does not depend on the particular orientation-preserving bi-Lipschitz map is equivalent to showing that if $\phi : \mathbb{S}^1 \mapsto \mathbb{S}^1$ is any bi-Lipschitz and orientation preserving map, and $\mathbf{n} \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ then we have:

$$\deg_{\mathbb{S}^1}(\mathbf{n} \circ \phi) = \deg_{\mathbb{S}^1} \mathbf{n},$$

which we will show by using the fact that $C^\infty(\mathbb{S}^1; \mathbb{S}^1)$ is dense in $H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$. Since there exists a sequence $\{\mathbf{u}^{(k)}\}_{k=1}^{\infty} \subset C^{\infty}(\mathbb{S}^1;\mathbb{S}^1)$ such that $\mathbf{u}^{(k)} \to \mathbf{n}$ as $k \to \infty$ in $H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$, we have

$$\begin{aligned}
\deg_{\mathbb{S}^1}(\mathbf{n}\circ\phi) &= \frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{1}{\mathbf{n}\circ\phi}\frac{\partial(\mathbf{n}\circ\phi)}{\partial\theta}\,d\theta \\
&= \lim_{k\to\infty}\frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{1}{\mathbf{u}^{(k)}\circ\phi}\frac{\partial(\mathbf{u}^{(k)}\circ\phi)}{\partial\theta}\,d\theta \\
&= \lim_{k\to\infty}\frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{1}{\mathbf{u}^{(k)}(\theta')}\,d(\mathbf{u}^{(k)}(\theta')) \\
&= \lim_{k\to\infty}\frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{1}{\mathbf{u}^{(k)}(\theta)}\,d(\mathbf{u}^{(k)}(\theta)) \\
&= \frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{1}{\mathbf{n}}\frac{\partial\mathbf{n}}{\partial\theta}\,d\theta \\
&= \deg_{\mathbb{S}^1}\mathbf{n},
\end{aligned}$$

where $\theta' = \phi(\theta)$ and this completes our proof.

□

Next we need to establish some important properties of the degree $\deg_{\partial\omega_j}\mathbf{n}_j$.

Proposition 2.2.4. *The degree of* $\mathbf{n}$ *on the boundary* $\partial\omega_j$ *is an integer and if* $\mathbf{n}, \widetilde{\mathbf{n}} \in H^{1/2}(\partial\omega_j, \mathbb{S}^1)$ *then*

$$\deg_{\partial\omega_j}(\mathbf{n}\cdot\widetilde{\mathbf{n}}) = \deg_{\partial\omega_j}\mathbf{n} + \deg_{\partial\omega_j}\widetilde{\mathbf{n}}.$$

where $\mathbf{n}\cdot\widetilde{\mathbf{n}}$ *stands for the product of* $\mathbf{n}$ *and* $\widetilde{\mathbf{n}}$ *as complex numbers.*

Proof. The fact that $\deg_{\partial\omega_j} n$ is an integer is proved by approximation of $H^{1/2}(\mathbb{S}^1, \mathbb{S})$ by maps in $C^{\infty}(\mathbb{S}^1, \mathbb{S}^1)$ as in the proof of Proposition 2.2.3. Also, since $\mathbf{n}\circ\Psi_j \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ and $\widetilde{\mathbf{n}}\circ\Psi_j \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$, we know that there exist two sequences $\{\mathbf{u}^{(k)}\}_{k=1}^{\infty} \subset C^{\infty}(\mathbb{S}^1, \mathbb{S}^1)$ and $\{\mathbf{v}^{(k)}\}_{k=1}^{\infty} \subset C^{\infty}(\mathbb{S}^1, \mathbb{S}^1)$ such that

$$\begin{aligned}
\mathbf{u}^{(k)} &\to \mathbf{n}\circ\Psi_j \quad \text{in} \quad H^{1/2}(\mathbb{S}^1, \mathbb{S}^1) \\
\mathbf{v}^{(k)} &\to \widetilde{\mathbf{n}}\circ\Psi_j \quad \text{in} \quad H^{1/2}(\mathbb{S}^1, \mathbb{S}^1).
\end{aligned}$$

From here we have

$$
\begin{aligned}
\deg_{\partial\omega_j}(\mathbf{n}\cdot\widetilde{\mathbf{n}}) &= \deg_{\mathbb{S}^1}(\mathbf{n}\circ\Psi_j\cdot\widetilde{\mathbf{n}}\circ\Psi_j)\\
&= \frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{1}{\mathbf{n}\circ\Psi_j\cdot\widetilde{\mathbf{n}}\circ\Psi_j}\frac{\partial\mathbf{n}\circ\Psi_j\cdot\widetilde{\mathbf{n}}\circ\Psi_j}{\partial\theta}\,d\theta\\
&= \lim_{k\to\infty}\frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{1}{\mathbf{u}^{(k)}\cdot\mathbf{v}^{(k)}}\frac{\partial\mathbf{u}^{(k)}\cdot\mathbf{v}^{(k)}}{\partial\theta}\,d\theta\\
&= \lim_{k\to\infty}\Big(\frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{1}{\mathbf{u}^{(k)}}\frac{\partial\mathbf{u}^{(k)}}{\partial\theta}\,d\theta+\frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{1}{\mathbf{v}^{(k)}}\frac{\partial\mathbf{v}^{(k)}}{\partial\theta}\,d\theta\Big)\\
&= \deg_{\mathbb{S}^1}\mathbf{n}\circ\Psi_j+\deg_{\mathbb{S}^1}\widetilde{\mathbf{n}}\circ\Psi_j\\
&= \deg_{\partial\omega_j}\mathbf{n}+\deg_{\partial\omega_j}\widetilde{\mathbf{n}},
\end{aligned}
\tag{2.2.1}
$$

which completes our proof.

□

We also have the following result.

Proposition 2.2.5. *Given* $a \in \omega_j$*, we have* $\deg_{\partial\omega_j}\frac{z-a}{|z-a|} = 1$*; otherwise, if* $a \notin \bar{\omega}_j$*, we have* $\deg_{\partial\omega_j}\frac{z-a}{|z-a|} = 0$.

Proof. If $a \in \omega_j$, we have

$$\frac{1}{2\pi i}\int_{\partial\omega_j}\frac{dz}{z-a} = 1. \tag{2.2.2}$$

For $z \in \partial\omega_j$, put $z - a = r(\theta)\,e^{i\phi(\theta)}$. Then we have that

$$
\begin{aligned}
1 &= \frac{1}{2\pi i}\int_{\partial\omega_j}\frac{dz}{z-a}\\
&= \frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{1}{r(\theta)e^{i\phi(\theta)}}(r'(\theta)e^{i\phi(\theta)}+ir(\theta)\phi'(\theta)e^{i\phi(\theta)})\,d\theta\\
&= \frac{1}{2\pi i}\int_{\mathbb{S}^1}\Big(\frac{r'(\theta)}{r(\theta)}+i\phi'(\theta)\Big)d\theta\\
&= \frac{1}{2\pi i}\log\frac{r(2\pi)}{r(0)}+\frac{1}{2\pi}(\phi(2\pi)-\phi(0))\\
&= \frac{1}{2\pi}(\phi(2\pi)-\phi(0))
\end{aligned}
\tag{2.2.3}
$$

On the other hand, we have

$$\begin{aligned}\deg_{\partial\omega_j}\frac{z-a}{|z-a|} &= \frac{1}{2\pi i}\int_{\partial\omega_j}\frac{1}{\frac{z-a}{|z-a|}}\,d\frac{z-a}{|z-a|}\\ &= \frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{1}{e^{i\phi(\theta)}}\,d(e^{i\phi(\theta)})\\ &= \frac{1}{2\pi}(\phi(2\pi)-\phi(0)).\end{aligned}$$

Therefore we can see from (2.2.3) such that

$$\deg_{\partial\omega_j}\frac{z-a}{|z-a|} = 1, \quad \text{for } a\in\omega_j$$

and using the same reasoning we can easily derive that

$$\deg_{\partial\omega_j}\frac{z-a}{|z-a|} = 0,$$

when $a\notin\bar{\omega}_j$. □

2.3 Existence of a lifting and its properties

We can now establish lifting properties using the ideas introduced in section 2.2. We now define the space $\widetilde{H}^1(\Omega;\mathbb{S}^1)$ where $\Omega = \mathbb{R}^2\setminus\cup_{j=1}^N\bar{\omega}_j$, which we use frequently below.

Definition 2.3.1. *Let $\mathbb{B}_R\subset\mathbb{R}^2$ denote the disk centred at the origin with radius $R>0$. We say that $\mathbf{n}\in\widetilde{H}^1(\Omega;\mathbb{S}^1)$ if and only if $\mathbf{n}\in H^1(\Omega\cap\mathbb{B}_R;\mathbb{S}^1)$ for all sufficiently large R.*

Note that the space $\widetilde{H}^1(\Omega;\mathbb{S}^1)$ can be made into a complete metric space such that $\mathbf{n}^j\to\mathbf{n}$ in $\widetilde{H}^1(\Omega;\mathbb{S}^1)$ if and only if $\mathbf{n}^j\to\mathbf{n}$ in $H^1(\Omega\cap\mathbb{B}_R;\mathbb{S}^1)$ for all sufficiently large R and the complete proof can be found in [MV97, p.40].

We will also assume that the vector field $\mathbf{n}_j\in H^{1/2}(\partial\omega_j;\mathbb{S}^1)$ on each of the boundary $\partial\omega_j$ has the prescribed degree d_j, i.e. $\deg_{\partial\omega_j}\mathbf{n}_j = d_j$, $j=1,...,N$. Given a vector field $\mathbf{n}\in\widetilde{H}^1(\Omega;\mathbb{S}^1)$, we now define a new vector field $\mathbf{m}\in\widetilde{H}^1(\Omega;\mathbb{S}^1)$:

$$\mathbf{m} = \Pi_{j=1}^N h_j(z)^{-d_j}\mathbf{n}, \tag{2.3.1}$$

where $h_j(z) = \frac{z-a_j}{|z-a_j|}$, and $d_j = \deg_{\partial\omega_j}\mathbf{n}$. Note that $\{a_j\}_{j=1}^N$ are arbitrary points such that $a_j\in\omega_j$, $j=1,...,N$, respectively. Then we have the following proposition.

Proposition 2.3.2. *Given a vector field* $\mathbf{m} = \Pi_{j=1}^{N} h_j(z)^{-d_j}\mathbf{n}$, *for* $\mathbf{n}$ *in the admissible set*

$$A = \{\mathbf{n} \in \widetilde{H}^1(\Omega; \mathbb{S}^1) \,|\, \mathbf{n}|_{\partial\omega_j} = \mathbf{n}_j,\ j = 1, ..., N\},$$

we have that the degree of $\mathbf{m}$ *is zero on each boundary* $\partial\omega_j$, *and there is a lifting of* $\mathbf{m}$ *on each boundary* $\partial\omega_j$, *i.e.* $\mathbf{m} = \exp(i\phi_j)$ *for some* $\phi_j \in H^{1/2}(\partial\omega_j)$. *Further, we can extend* $\mathbf{m}$ *within each* ω_j, *and obtain an extended vector field* $\widetilde{\mathbf{m}} \in \widetilde{H}^1(\mathbb{R}^2; \mathbb{S}^1)$. *In each ball* $\mathbb{B}_R$, *there exists a lifting* ϕ_R *for the extended vector field* $\widetilde{\mathbf{m}}$, *i.e.* $\widetilde{\mathbf{m}} = \exp(i\phi_R)$ *for some* $\phi_R \in H^1(\mathbb{B}_R)$, *unique up to an integral multiple of* 2π.

Proof. Applying Proposition 2.2.4 we have that

$$\begin{aligned}
\deg_{\partial\omega_i}\mathbf{m} &= \deg_{\partial\omega_i}(\Pi_{j=1}^{N} h_j(z)^{-d_j}\mathbf{n}) \\
&= \Sigma_{j=1}^{N} \deg_{\partial\omega_i} h_j(z)^{-d_j} + \deg_{\partial\omega_i}\mathbf{n} \\
&= -\Sigma_{j\neq i}^{N} d_j \deg_{\partial\omega_i} h_j(z) - d_i \deg_{\partial\omega_i} h_i(z) + \deg_{\partial\omega_i}\mathbf{n} \qquad (2.3.2)\\
&= 0 - d_i + d_i \\
&= 0.
\end{aligned}$$

Therefore we can now apply Theorem 2.1.2 and write $\mathbf{m} = \Pi_{j=1}^{N} h_j(z)^{-d_j}\mathbf{n}$ on each boundary $\partial\omega_j$ as:

$$\mathbf{m} = \exp i\phi_j(z) \quad \text{for} \quad z \in \partial\omega_j, \tag{2.3.3}$$

where $j = 1, ..., N$ and $\phi_j(z) \in H^{1/2}(\partial\omega_j)$.

So next we can extend $\mathbf{m}$ inside each ω_j by solving the Dirichlet problems for $j = 1, ..., N$:

$$\begin{cases} \Delta\phi = 0 \\ \phi|_{\partial\omega_j} = \phi_j \end{cases} \tag{2.3.4}$$

Hence we obtain a new vector field $\widetilde{m}$ such that

$$\widetilde{\mathbf{m}} = \begin{cases} \Pi_{j=1}^{N} h_j(z)^{-d_j}\mathbf{n} & \text{for} \quad z \in \Omega \\ \exp(i\phi) & \text{for} \quad z \in \cup_{j=1}^{N} \bar{\omega}_j, \end{cases} \tag{2.3.5}$$

where ϕ are solved from the Dirichlet problem (2.3.4). From (2.3.5) we have that $\widetilde{\mathbf{m}} \in H^1(\mathbb{B}_R; \mathbb{S}^1)$ for any $R > 0$ sufficiently large such that $\cup_{j=1}^{N} \bar{\omega}_j \subset \mathbb{B}_R$. Since $\mathbb{B}_R$ is a simply-connected domain, we know from Theorem 3 in [BBM00] that there exists a lifting $\phi_R \in H^1(\mathbb{B}_R)$ such that $\widetilde{\mathbf{m}} = \exp(i\phi_R)$ in $\mathbb{B}_R$, and $\fint_{\mathbb{B}_1} \phi_R \in [0, 2\pi)$. □

We now extend the lifting of $\widetilde{\mathbf{m}}$ to the whole plane $\mathbb{R}^2$. We need the following modified version of the Poincaré inequality.

Lemma 2.3.3. *For each* $R > 1$ *there exists a a constant* C_R *such that*

$$\int_{\mathbb{B}_R} \Psi^2 \, dx \le C_R \Big(\int_{\mathbb{B}_R} |\nabla \Psi|^2 \, dx + \big(\fint_{\mathbb{B}_1} \Psi \, dx\big)^2 \Big) \tag{2.3.6}$$

for all $\Psi \in H^1(\mathbb{B}_R)$.

Proof. We prove it by contradiction. Assume (2.3.6) does not hold, then for any $k \in \mathbb{N}$ there exists a sequence Ψ_k, such that

$$\int_{\mathbb{B}_R} \Psi_k^2 \, dx > k \Big(\int_{\mathbb{B}_R} |\nabla \Psi_k|^2 \, dx + \big(\fint_{\mathbb{B}_1} \Psi_k \, dx\big)^2 \Big)$$

Without loss of generality, we can choose a sequence $\{\Psi_k\}_{k=1}^{\infty}$ such that $\int_{\mathbb{B}_R} |\Psi_k|^2 \, dx = 1$. Then we have

$$\int_{\mathbb{B}_R} |\nabla \Psi_k|^2 \, dx + \big(\fint_{\mathbb{B}_1} \Psi_k \, dx\big)^2 \to 0.$$

This implies that $\{\nabla \Psi_k\}_{k=1}^{\infty}$ is bounded in $L^2(\mathbb{B}_R)$. Recall also the fact that $\int_{\mathbb{B}_R} |\Psi_k|^2 \, dx = 1$, we have that $\{\Psi_k\}_{k=1}^{\infty}$ is a bounded sequence in $H^1(\mathbb{B}_R)$.

Hence there exists a subsequence of $\{\Psi_k\}_{k=1}^{\infty}$ (without loss of generality, we still write the subsequence as $\{\Psi_k\}_{k=1}^{\infty}$) and some $\Psi \in H^1(\mathbb{B}_R)$ such that

$$\Psi_k \rightharpoonup \Psi \quad \text{weakly in } H^1(\mathbb{B}_R),$$

which implies using standard weak lower semi-continuity results that

$$\int_{\mathbb{B}_R} |\nabla \Psi|^2 \, dx + \big(\fint_{\mathbb{B}_1} \Psi \, dx\big)^2 \le \lim_{k\to\infty} \Big(\int_{\mathbb{B}_R} |\nabla \Psi_k|^2 \, dx + \big(\fint_{\mathbb{B}_1} \Psi_k \, dx\big)^2 \Big) = 0,$$

and this is saying that $\Psi \equiv 0$.

On the other hand, by the compactness of the embedding of $H^1(\mathbb{B}_R)$ in $L^2(\mathbb{B}_R)$, we have

$$\Psi_k \to \Psi \quad \text{in } L^2(\mathbb{B}_R),$$

which implies $\int_{\mathbb{B}_R} |\Psi|^2 \, dx = 1$, and this is contradictory to the fact that $\Psi \equiv 0$. Therefore (2.3.6) must hold.

□

Theorem 2.3.4. *Suppose that a vector field* $\mathbf{m} \in A_m = \{\mathbf{m} : \mathbb{R}^2 \mapsto \mathbb{S}^1 \,|\, \nabla\mathbf{m} \in L^2(\mathbb{R}^2)\}$ *is such that in any ball* $\mathbb{B}_R \subset \mathbb{R}^2$ *there exists a lifting* $\phi_R \in H^1(\mathbb{B}_R)$, *unique up to an integral multiple of* 2π, *i.e.* $\mathbf{m} = \exp(i\phi_R)$. *Then there exists a* ϕ *satisfying* $\phi \in H^1(\mathbb{B}_R)$ *for any sufficiently large* R, *and* $\nabla\phi \in L^2(\mathbb{R}^2)$, *such that* $\mathbf{m} = \exp(i\phi)$, *i.e.* ϕ *is a global lifting for* $\mathbf{m} \in A_m$, *and is also unique up to an integral multiple of* 2π.

Proof. For each integer $k > 1$ there exists a lifting $\phi_k \in H^1(\mathbb{B}_k)$ such that in $\mathbb{B}_k$ we have $\mathbf{m} = \exp(i\phi_k)$, where ϕ_k satisfies $\frac{1}{2\pi}\int_{\mathbb{B}_1} \phi_k \in [0, 2\pi)$. Then

$$\int_{\mathbb{B}_R} |\nabla\phi_k|^2 \, dx = \int_{\mathbb{B}_R} |\nabla\mathbf{m}|^2 \, dx \leq \int_{\mathbb{R}^2} |\nabla\mathbf{m}|^2 \, dx \leq C < \infty.$$

It follows from a diagonal argument that there exists a subsequence $\{\phi_{k_l}\}_{l=1}^{\infty}$ of $\{\phi_k\}_{k=1}^{\infty}$ such that

$$\nabla\phi_{k_l} \rightharpoonup Y \quad \text{in} \quad L^2(\mathbb{B}_R), \tag{2.3.7}$$

for any R sufficiently large, and

$$\int_{\mathbb{B}_R} |Y|^2 \, dx \leq \liminf_{k\to\infty} \int_{\mathbb{B}_R} |\nabla\phi_k|^2 \, dx \leq C < \infty,$$

where C is some constant and by letting $R \to \infty$, we have $Y \in L^2(\mathbb{R}^2)$.

Also we can apply the Poincaré inequality (2.3.6) to the subsequence $\{\phi_{k_l}\}_{l=1}^{\infty}$ obtained above:

$$\int_{\mathbb{B}_R} \phi_{k_l}^2 \, dx \leq C_R \left(\int_{\mathbb{B}_R} |\nabla\phi_{k_l}|^2 \, dx + \Big(\fint_{\mathbb{B}_1} \phi_{k_l} \, dx\Big)^2 \right)$$

Therefore by using diagonal argument again, there exists some subsequence of $\{\phi_{k_l}\}_{l=1}^{\infty}$ (without loss of generality, we still write it as $\{\phi_{k_l}\}_{l=1}^{\infty}$), such that

$$\phi_{k_l} \rightharpoonup \phi \quad \text{in} \quad L^2(\mathbb{B}_R),$$

for any sufficiently large R and some $\phi \in \widetilde{L}^2(\mathbb{R}^2)$ (where $\phi \in \widetilde{L}^2(\mathbb{R}^2)$ means $\phi \in L^2(\mathbb{B}_R)$ for any sufficiently large ball $\mathbb{B}_R$), and it is easy to see that we have $Y = \nabla\phi$.

Writing the subsequence $\{\phi_{k_l}\}_{l=1}^{\infty}$ as $\{\phi_k\}_{k=1}^{\infty}$ it follows that

$$\phi_k \rightharpoonup \phi \quad \text{in} \quad H^1(\mathbb{B}_R),$$

for any R sufficiently large, and thus by the compactness of the embedding of $H^1(\mathbb{B}_R)$ in $L^2(\mathbb{B}_R)$

$$\phi_k \to \phi \quad \text{in} \quad L^2(\mathbb{B}_R).$$

Hence we may assume that

$$\phi_k \to \phi \quad \text{a.e.} \quad \text{in} \quad \mathbb{B}_R$$

for any R sufficiently large. Thus

$$\mathbf{m} = \exp(i\phi),$$

where $\nabla\phi \in L^2(\mathbb{R}^2)$ and $\phi \in H^1(\mathbb{B}_R)$, for any R sufficiently large, i.e. ϕ is a global lifting for $\mathbf{m}$.

□

The global lifting ϕ for the vector field $\mathbf{m} = \Pi_{j=1}^{N} h_j(z)^{-d_j}\mathbf{n} = e^{i\phi}$ satisfies the property stated in the following proposition.

Proposition 2.3.5. *For ϕ defined in (2.3.1) and $\mathbf{m}$ defined in Theorem 2.3.4 such that $\mathbf{m} = \Pi_{j=1}^{N} h_j(z)^{-d_j}\mathbf{n} = e^{i\phi}$, we have*

$$\int_{\mathbb{S}^1} (\phi_{,2}(r,\theta)\cos\theta - \phi_{,1}(r,\theta)\sin\theta)\, d\theta = 0 \quad \text{a.e. } r \in (R,\infty) \tag{2.3.8}$$

where R is sufficiently large such that $\cup_{j=1}^{N}\bar{\omega}_j \subset \mathbb{B}_R$.

Proof. First we need to show that

$$\nabla\phi = \begin{pmatrix} \frac{\partial\phi}{\partial x_1} \\ \frac{\partial\phi}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \phi_{,1} \\ \phi_{,2} \end{pmatrix}$$

is well defined on any large circle $\partial\mathbb{B}_r$ a.e. $r \in (R,\infty)$.

This is true by the result stated in [Zie12, Remark 2.1.5], saying that for any given function $\phi \in W^{1,p}(\Omega)$ it stays in the same Sobolev space on almost all lines in Ω that are parallel to the coordinate line without finding one. In our case this line is chosen to be any large circle using polar coordinate. For a proof, one can approximate $\phi \in W^{1,p}(\Omega)$ by smooth functions, and then by applying Fubini's theorem for almost every radius the functions converge in the $W^{1,p}$ norm.

Consequently, for a.e. $r \in (R, \infty)$ we have

$$\begin{aligned}\int_{\mathbb{S}^1} (\phi_{,2}(r,\theta)\cos\theta - \phi_{,1}(r,\theta)\sin\theta)\ d\theta &= \int_{\mathbb{S}^1} \nabla\phi \cdot \nu\, d\theta \\ &= \frac{1}{r}\int_{\partial\mathbb{B}_r} d\phi \\ &= \frac{1}{r}(\phi(r,\theta+2\pi) - \phi(r,\theta)) \\ &= 0.\end{aligned}$$

where in the first step of this integral, ν is the tangential direction on the given circle $\partial\mathbb{B}_r$. Hence proving our result. □

As a consequence, we have the following proposition summarizing Proposition 2.3.2, Theorem 2.3.4 and Proposition 2.3.5.

Proposition 2.3.6. *Given* **n** *in the admissible set*

$$A = \{\mathbf{n} \in \widetilde{H}^1(\Omega; \mathbb{S}^1) \,|\, \mathbf{n}|_{\partial\omega_j} = \mathbf{n}_j,\ j = 1, ..., N\},$$

we have:

$$\mathbf{n} = \Pi_{j=1}^N h_j(z)^{d_j} \exp(i\phi), \tag{2.3.9}$$

where ϕ *is in the admissible set*

$$A_\phi^m = \{\phi \in \widetilde{H}^1(\Omega) \,|\, \phi|_{\partial\omega_j} = \phi_j + 2m_j\pi,\ m_j \in \mathbb{Z},\ \text{for } j = 1, ..., N\}, \tag{2.3.10}$$

where on each of the boundary $\partial\omega_j$, ϕ_j *is given by*

$$\exp i\phi_j = \Pi_{i=1}^N h_i(z)^{-d_i}\, \mathbf{n}_j$$

as is shown by (2.3.3) in the proof of Proposition 2.3.2, *and*

$$h_j(z) = \frac{z - a_j}{|z - a_j|} \quad \text{for} \quad j = 1, ..., N.$$

Further more, we have

$$\int_{\mathbb{S}^1} (\phi_{,2}(r,\theta)\cos\theta - \phi_{,1}(r,\theta)\sin\theta)\ d\theta = 0 \quad \text{a.e. } r \in (R, \infty)$$

where R *is sufficiently large such that* $\cup_{j=1}^{N}\bar{\omega}_j \subset \mathbb{B}_R$.

One can see later in Theorem 4.3.2 that Proposition 2.3.5 plays a very important role in the proof. We also have the following result.

Proposition 2.3.7. *Given any large enough disk* $\mathbb{B}_R$ *such that* $\cup_{j=1}^{N}\bar{\omega}_j \subset \mathbb{B}_R$, *and for* $\mathbf{n}$ *in the admissible set*

$$A = \{\mathbf{n} \in \widetilde{H}^1(\Omega; \mathbb{S}^1) \,|\, \mathbf{n}|_{\partial\omega_j} = \mathbf{n}_j,\ j = 1, ..., N\},$$

we have

$$\oint_{\partial\mathbb{B}_R} \mathbf{n}^{-1}\, d\mathbf{n} = 2\pi\, k = 2\pi\, \Sigma_{j=1}^{N} d_j, \tag{2.3.11}$$

where the integral is taken in the anti-clockwise direction and k *is sum of the degrees of* $\mathbf{n}$ *on each boundary* $\partial\omega_j$. *Further, for* ϕ *defined in (2.3.1) and* $\mathbf{m}$ *defined in Theorem 2.3.4 such that* $\mathbf{m} = \Pi_{j=1}^{N} h_j(z)^{-d_j}\mathbf{n} = e^{i\phi}$, *we have*

$$\frac{1}{2\pi}\oint_{\partial\mathbb{B}_R} \mathbf{m}^{-1}\, d\mathbf{m} = 0, \tag{2.3.12}$$

where R *is sufficiently large such that* $\cup_{j=1}^{N}\bar{\omega}_j \subset \mathbb{B}_R$.

Proof. First we prove that

$$\oint_{\gamma} - \oint_{\cup_{j=1}^{N}\partial\omega_j} \mathbf{n}^{-1}\, d\mathbf{n} = 0.$$

Since $|\mathbf{n}| = (n_1^2 + n_2^2)^{1/2} = 1$ a.e. in Ω, we have $\mathbf{n}^{-1} = \bar{\mathbf{n}} = n_1 - in_2$. As a consequence we have

$$\begin{aligned}
&\oint_{\partial\mathbb{B}_R} - \oint_{\cup_{j=1}^{N}\partial\omega_j} \mathbf{n}^{-1}\, d\mathbf{n} \\
&= \oint_{\partial\mathbb{B}_R} - \oint_{\cup_{j=1}^{N}\partial\omega_j} (n_1 - in_2)\, d(n_1 + in_2) \\
&= \oint_{\partial\mathbb{B}_R} - \oint_{\cup_{j=1}^{N}\partial\omega_j} (n_1 - in_2)(n_{1,1}\, dx + n_{1,2}\, dy) + i(n_1 - in_2)(n_{2,1}\, dx + n_{2,2}\, dy) \qquad (2.3.13)\\
&= \oint_{\partial\mathbb{B}_R} - \oint_{\cup_{j=1}^{N}\partial\omega_j} (n_1 n_{1,1} + n_2 n_{2,1})\, dx + i(n_1 n_{2,1} - n_2 n_{1,1})\, dx \\
&\quad + (n_1 n_{1,2} + n_2 n_{2,2})\, dy + i(n_1 n_{2,2} - n_2 n_{1,2})\, dy.
\end{aligned}$$

Since $|\mathbf{n}| = (n_1^2 + n_2^2)^{1/2} = 1$ a.e. in Ω, we have

$$\begin{pmatrix} n_1 n_{1,1} + n_2 n_{2,1} \\ n_1 n_{1,2} + n_2 n_{2,2} \end{pmatrix} = \mathbf{0} \quad \text{a.e.} \tag{2.3.14}$$

As a result, we have

$$\begin{pmatrix} n_{1,1} & n_{2,1} \\ n_{1,2} & n_{2,2} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \mathbf{0} \quad a.e.$$

and therefore

$$\det \nabla \mathbf{n} = 0 \quad \text{a.e. in } \Omega. \tag{2.3.15}$$

By substituting (2.3.14) back into (2.3.13) and using Stokes Theorem, we have

$$\begin{aligned}
&\oint_{\partial \mathbb{B}_R} - \oint_{\cup_{j=1}^N \partial\omega_j} \mathbf{n}^{-1}\, d\mathbf{n} \\
&= i \oint_{\partial \mathbb{B}_R} - \oint_{\cup_{j=1}^N \partial\omega_j} (n_1 n_{2,1} - n_2 n_{1,1})\, dx + (n_1 n_{2,2} - n_2 n_{1,2})\, dy \\
&= i \iint_{\mathbb{B}_R \cap \Omega} \big(n_{1,1} n_{2,2} + n_1 n_{2,21} - n_{2,1} n_{1,2} - n_2 n_{1,21} - n_{1,2} n_{2,1} - n_1 n_{2,12} + n_{2,2} n_{1,1} + n_2 n_{1,12}\big)\, dx\, dy \\
&= 2i \int_{\mathbb{B}_R \cap \Omega} (n_{1,1} n_{2,2} - n_{1,2} n_{2,1})\, dx\, dy \\
&= 2i \iint_{\mathbb{B}_R \cap \Omega} \det \nabla \mathbf{n}\, dx\, dy \\
&= 0,
\end{aligned}$$

where the last step is due to (2.3.15). As a consequence:

$$\oint_{\gamma} \mathbf{n}^{-1}\, d\mathbf{n} = \Sigma_{j=1}^N \oint_{\partial\omega_j} \mathbf{n}^{-1}\, d\mathbf{n} = 2\pi\, \Sigma_{j=1}^N d_j = 2\pi k,$$

and hence we have proved (2.3.11).

In order to prove (2.3.12), we can apply the same reasoning as we did for $\mathbf{n}$ to

$$\mathbf{m} = \Pi_{j=1}^N h_j(z)^{-d_j}\, \mathbf{n},$$

and use the fact that $|\mathbf{m}| = 1$ a.e.. This gives:

$$\oint_{\gamma} \mathbf{m}^{-1}\, d\mathbf{m} = \Sigma_{j=1}^N \oint_{\partial\omega_j} \mathbf{m}^{-1}\, d\mathbf{m} = 0,$$

where we used (2.3.2) in the last step, hence completing our proof.

□

In the following chapters, we will study the minimization problem for a one-hole domain and an N-connected domain in the one-constant setting. The explicit form (2.3.9) for **n** and the associated admissible set (2.3.10) for ϕ will play an important role in solving the minimization problem. The role of the lifting ϕ is to reduce the nonlinear problem of solving a harmonic map with respect to **n** into a linear problem of solving Laplace's equation of ϕ. This technique to use the lifting appeared in various works, e.g. [BWW+03, Example 3.3.8]. One can also refer to [HW08] for more details on harmonic maps.

Chapter 3

Minimization problem for one-hole domain (one-constant case)

In this chapter, we study the minimization problem for the Oseen-Frank energy in the domain exterior to a circle given by $\Omega = \{x \,|\, a < |x| < \infty\} \subset \mathbb{R}^2$, where $a > 0$. We assume that the boundary vector field $\mathbf{n}_0 \in H^{1/2}(\partial\Omega; \mathbb{S}^1)$ have the prescribed degree k, i.e. $\deg_{\partial\Omega}\mathbf{n}_0 = k$, where $k \in \mathbb{Z}$. We also assume that the director $\mathbf{n}$ of the Oseen-Frank energy is in the admissible set:

$$A = \{\mathbf{n} \in \widetilde{H}^1(\Omega; \mathbb{S}^1) \,|\, \mathbf{n}|_{\partial\Omega} = \mathbf{n}_0\},$$

where and $\widetilde{H}^1(\Omega; \mathbb{S}^1)$ is defined in Definition 2.3.1. Some of the techniques we will use in this chapter can be applied directly to the proof of the N-connected (multi-hole) domain case in Chapter 4, except that we need a more delicate treatment for the N-connected domain case such as defining homotopy classes for the vector fields in the given admissible set.

We know by (1.2.6) that the Oseen-Frank free energy is:

$$\begin{aligned} I(\mathbf{n}) &= \int_\Omega F(x, \mathbf{n}, \nabla\mathbf{n})\, dx \\ &= \frac{1}{2}\int_\Omega (k_3|\nabla\mathbf{n}|^2 + (k_1 - k_3)(\operatorname{div}\mathbf{n})^2 + (k_2 - k_3)(\mathbf{n}\cdot\nabla\wedge\mathbf{n})^2 \\ &\quad + (k_2 + k_4 - k_3)\Big(\operatorname{tr}(\nabla\mathbf{n})^2 - (\operatorname{div}\mathbf{n})^2\Big)\, dx, \end{aligned} \tag{3.0.1}$$

for the given domain Ω. Since we have (1.2.4) and (1.2.5), we know that the last two terms containing $\operatorname{tr}(\nabla\mathbf{n})^2 - (\operatorname{div}\mathbf{n})^2$ and $(\mathbf{n}\cdot\nabla\wedge\mathbf{n})^2$ are zero for two-dimensional case. Thus we only need to consider the contribution of the first two terms.

Recall that by Proposition 2.3.6, $\mathbf{n} \in A$ can be written explicitly as:

$$\mathbf{n} = \Pi_{j=1}^{N} h_j(z)^{d_j} \exp(i\phi),$$

where for this one-hole circular domain, we have $N = 1$ and $d_1 = k$. Note also that in $h_1(z) = \frac{z-a_1}{|z-a_1|}$, a_1 can be chosen to be the origin and therefore in this one-hole domain $\boldsymbol{\Omega}$, $\mathbf{n}$ can be written as

$$\mathbf{n} = \left(\frac{z}{|z|}\right)^k \exp(i\phi) = \exp i(\phi + k\theta), \tag{3.0.2}$$

and ϕ is in the admissible set

$$A_\phi^m = \{\phi \in \widetilde{H}^1(\Omega) \,|\, \phi|_{\partial\Omega} = \phi_0 + 2m\pi\}, \tag{3.0.3}$$

where $m \in \mathbb{Z}$, and ϕ_0 on the boundary $\partial\Omega$ is given by (2.3.3) in Proposition 2.3.2, which is

$$e^{i\phi_0} = \mathbf{n}_0\, e^{-ik\theta}. \tag{3.0.4}$$

Without loss of generality, we assume that ϕ_0 satisfies

$$⨍_{\partial\Omega} \phi_0 = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \phi_0 \, dx \in [0, 2\pi). \tag{3.0.5}$$

Now by letting $\Phi = \phi + k\theta$, we have

$$\mathbf{n} = \exp(i\Phi). \tag{3.0.6}$$

As a result, by using polar coordinate, i.e. $x = (r\cos\theta, r\sin\theta)$, we can write the vector field $\mathbf{n}(x)$ as $\mathbf{n}(x) = (\cos\Phi(r,\theta), \sin\Phi(r,\theta))$. Hence, by denoting the partial derivatives of $\mathbf{n}$ with respect to x_1 and x_2 as $\mathbf{n}_{,1}$ and $\mathbf{n}_{,2}$ respectively, we have:

$$\begin{cases} \mathbf{n}_{,1} = (-\sin\Phi, \cos\Phi)\left(\Phi_r \cos\theta - \Phi_\theta \dfrac{\sin\theta}{r}\right) \\ \mathbf{n}_{,2} = (-\sin\Phi, \cos\Phi)\left(\Phi_r \sin\theta + \Phi_\theta \dfrac{\cos\theta}{r}\right). \end{cases} \tag{3.0.7}$$

Thus the free energy density becomes:

$$\begin{aligned} F(x, \mathbf{n}, \nabla\mathbf{n}) &= G(r, \theta, \Phi, \Phi_r, \Phi_\theta) \\ &= \frac{1}{2}k_3\Big(\Phi_r^2 + \frac{\Phi_\theta^2}{r^2}\Big) + \frac{1}{2}(k_1 - k_3)\Big(\Phi_r^2 \sin^2(\theta - \Phi) \\ &\quad + \frac{\Phi_\theta^2}{r^2}\cos^2(\theta - \Phi) + \frac{2\Phi_\theta\Phi_r}{r}\sin(\theta - \Phi)\cos(\theta - \Phi)\Big). \end{aligned} \tag{3.0.8}$$

Consequently, the free energy $\mathrm{I}(\mathbf{n})$ can be written in terms of Φ as:

$$
\begin{aligned}
\mathrm{I}(\mathbf{n}) = \mathrm{E}(\Phi) &= \int_a^\infty r \int_{\mathbb{S}^1} \mathrm{G}(r, \theta, \Phi, \Phi_r, \Phi_\theta)\, d\theta\, dr \\
&= \frac{1}{2}\int_a^\infty r \int_{\mathbb{S}^1} \Big(k_3 \Big(\Phi_r^2 + \frac{\Phi_\theta^2}{r^2}\Big) + (k_1 - k_3)\Big(\Phi_r^2 \sin^2(\Phi - \theta) \\
&\quad + \frac{\Phi_\theta^2}{r^2}\cos^2(\Phi - \theta) - \frac{2\Phi_\theta \Phi_r}{r}\sin(\Phi - \theta)\cos(\Phi - \theta)\Big)\Big)\, d\theta\, dr.
\end{aligned}
\tag{3.0.9}
$$

Writing $\Phi^\epsilon(x) = \Phi^\epsilon(r, \theta) = \Phi(r, \theta) + \epsilon\, \eta(r, \theta)$, where $\eta(r, \theta) \in C_0^\infty(\Omega)$, we have that:

$$
\Phi^\epsilon(r, \theta + 2\pi) = \Phi^\epsilon(r, \theta) + 2k\pi,
$$

if and only if η satisfies

$$
\eta(r, \theta + 2\pi) = \eta(r, \theta), \tag{3.0.10}
$$

since by Proposition 2.3.6, Φ satisfies

$$
\Phi(r, \theta + 2\pi) = \Phi(r, \theta) + 2k\pi.
$$

Consequently the first variation becomes:

$$
\delta \mathrm{E}(\Phi)\, \eta = \frac{d\, \mathrm{E}(\Phi^\epsilon)}{d\epsilon}\Big|_{\epsilon=0} = \int_a^\infty r \int_{\mathbb{S}^1} (\mathrm{G}_\Phi \eta + \mathrm{G}_{\Phi_r} \eta_r + \mathrm{G}_{\Phi_\theta} \eta_\theta)\, d\theta\, dr, \tag{3.0.11}
$$

and the second variation becomes:

$$
\begin{aligned}
\delta^2 \mathrm{E}(\Phi)(\eta, \eta) &= \frac{1}{2}\frac{d^2 \mathrm{E}(\Phi^\epsilon)}{d^2\epsilon}\Big|_{\epsilon=0} \\
&= \int_a^\infty r \int_{\mathbb{S}^1} \Big(\mathrm{G}_{\Phi\Phi}\eta^2 + \mathrm{G}_{\Phi_r\Phi_r}\eta_r^2 + \mathrm{G}_{\Phi_\theta\Phi_\theta}\eta_\theta^2 \\
&\quad + 2\mathrm{G}_{\Phi\Phi_r}\eta\eta_r + 2\mathrm{G}_{\Phi\Phi_\theta}\eta\eta_\theta + 2\mathrm{G}_{\Phi_r\Phi_\theta}\eta_r\eta_\theta \Big)\, d\theta\, dr.
\end{aligned}
\tag{3.0.12}
$$

In this chapter, we only study the one-constant case mentioned in (1.2.7), i.e.

$$
k = k_1 = k_2 = k_3; \quad k_4 = 0,
$$

and, without loss of generality, we can assume that $k = 1$.

From (3.0.8) we know that if $\mathrm{F}(x, \mathbf{n}, \nabla \mathbf{n}) = \frac{1}{2}|\nabla \mathbf{n}|^2$, we have :

$$
\mathrm{F}(x, \mathbf{n}, \nabla \mathbf{n}) = \frac{1}{2}|\nabla \mathbf{n}|^2 = \mathrm{G}(r, \theta, \Phi, \Phi_r, \Phi_\theta) = \frac{1}{2}\Big(\Phi_r^2 + \frac{\Phi_\theta^2}{r^2}\Big), \tag{3.0.13}
$$

and by substituting (3.0.13) into (3.0.9), we have:

$$E(\Phi) = \int_a^\infty \int_0^{2\pi} G(r, \theta, \Phi, \Phi_r, \Phi_\theta)\, d\theta\, dr = \frac{1}{2} \int_a^\infty r \int_0^{2\pi} \Big(\Phi_r^2 + \frac{\Phi_\theta^2}{r^2}\Big)\, d\theta\, dr, \tag{3.0.14}$$

and the first variation defined by (3.0.11) becomes:

$$\delta E(\Phi)\,\eta = \int_a^\infty r \int_{\mathbb{S}^1} (G_\Phi \eta + G_{\Phi_r} \eta_r + G_{\Phi_\theta} \eta_\theta)\, d\theta\, dr = \int_a^\infty r \int_0^{2\pi} \Big(\Phi_r\, \eta_r + \frac{\Phi_\theta}{r^2}\, \eta_\theta\Big)\, d\theta\, dr, \tag{3.0.15}$$

where $\eta \in C_0^\infty(\Omega)$ is smooth and has compact support in Ω and η satisfies (3.0.10), i.e.

$$\eta(r, \theta + 2\pi) = \eta(r, \theta).$$

In addition, the second variation in this one-constant case can therefore be derived by substituting (3.0.13) into (3.0.12):

$$\delta^2 E(\Phi)(\eta, \eta) = \frac{1}{2} \int_a^\infty r \int_0^{2\pi} \Big(\eta_r^2 + \frac{\eta_\theta^2}{r^2}\Big)\, d\theta\, dr \geq 0.$$

If further, we assume that $\Phi(x) = \Phi(r, \theta)$ is radius-independent:

$$\Phi(\theta, r) = \Phi(\theta) = \Phi(\frac{x}{|x|}),$$

then the first variation (3.0.15) becomes:

$$\delta E(\Phi)\,\eta = \int_a^\infty \int_0^{2\pi} \frac{\Phi'(\theta)}{r} \eta_\theta\, d\theta\, dr = -\int_a^\infty \frac{1}{r} \int_0^{2\pi} \Phi''(\theta)\, \eta(\theta, r)\, d\theta\, dr \tag{3.0.16}$$

where the last step comes from integration by parts and the constraint (3.0.10). And we can see that if $\delta E(\Phi)\,\eta = 0$, we must have

$$\Phi''(\theta) = 0,$$

which leads to:

$$\Phi(\theta) = k\theta + \beta, \tag{3.0.17}$$

where k is the given degree and β is an arbitrary constant depending on the boundary conditions of **n** in Ω which we will discuss in more detail later. Note that (3.0.17) agrees with $\Phi = \phi + k\theta$ (see p.33), only if ϕ_0 in (3.0.3) is constant.

We will introduce some other preliminary results in the following section before we state and prove our main result for the one-constant case on this one-hole circular domain.

3.1 Some preliminary results

Our aim is to prove the existence and uniqueness of a minimizer for the Oseen-Frank free energy which is given by (3.0.14):

$$I(\mathbf{n}) = \frac{1}{2}\int_\Omega |\nabla \mathbf{n}|^2\, dx = \frac{1}{2}\int_a^\infty r \int_0^{2\pi} \Big(\Phi_r^2 + \frac{\Phi_\theta^2}{r^2}\Big)\, d\theta\, dr = \int_a^\infty \int_0^{2\pi} G(r, \Phi_r, \Phi_\theta)\, d\theta\, dr = E(\Phi),$$

where $\mathbf{n}(x)$ is in the admissible set $A = \{\mathbf{n} \in \widetilde{H}^1(\Omega; \mathbb{S}^1) \,|\, \mathbf{n}|_{\partial\Omega} = \mathbf{n}_0; \deg_{\partial\Omega}\mathbf{n} = k\}$. Also, since we have (3.0.6), $\mathbf{n}$ can be characterized by $\mathbf{n}(x) = (\cos\Phi(x), \sin\Phi(x)) = (\cos\Phi(r,\theta), \sin\Phi(r,\theta))$, where Φ is given by

$$\Phi = \phi + k\theta,$$

for ϕ in the admissible set (3.0.3):

$$A_\phi^m = \{\phi \in \widetilde{H}^1(\Omega) \,|\, \phi|_{\partial\Omega} = \phi_0 + 2m\pi\},$$

where $m \in \mathbb{Z}$.

The Euler-Lagrange equation of $E(\Phi)$ is:

$$\frac{\partial}{\partial\theta}\frac{\partial G}{\partial \Phi_\theta} + \frac{\partial}{\partial r}\frac{\partial G}{\partial \Phi_r} = 0,$$

which leads to

$$(r\Phi_r)_r + \frac{1}{r}\Phi_{\theta\theta} = 0, \tag{3.1.1}$$

where $\Phi(r,\theta)$ satisfies the condition

$$\Phi(r, \theta + 2\pi) = \Phi(r,\theta) + 2k\pi,$$

and $k \in \mathbb{Z}$ is the given degree of $\mathbf{n}$.

We now apply a change of variable:

$$r = e^s$$

and we have

$$\begin{cases} \Phi_s = r\Phi_r \\ \Phi_{ss} = r\Phi_r + r^2\Phi_{rr}. \end{cases}$$

As a consequence, the Euler-Lagrange equation (3.1.1) becomes:

$$\Phi_{ss} + \Phi_{\theta\theta} = 0. \tag{3.1.2}$$

We now let

$$v(s, \theta) = \Phi(s, \theta) - k\theta - \beta, \tag{3.1.3}$$

where k is the given degree of $\mathbf{n}$, and β is some constant which will be determined by the boundary value of $v(s, \theta)$. Then

$$v(s, \theta + 2\pi) = v(s, \theta), \tag{3.1.4}$$

which implies that $v(s, \theta)$ is $2\pi - \text{periodic}$ with respect to θ, and $v(s, \theta)$ satisfies

$$v_{ss} + v_{\theta\theta} = 0. \tag{3.1.5}$$

In fact, we have $v(s, \theta) = \phi(s, \theta) - \beta$, where ϕ is in the admissible set (3.0.3). Before we prove our main result in the next section, we need the following lemma.

Lemma 3.1.1. *If* $\int_0^{2\pi} \Phi^2(\cdot, \theta)\, d\theta \in L^\infty(a, \infty)$, *then we have:*

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi(r, \theta)\, d\theta = \delta,$$

for some constant δ*, i.e.* $\frac{1}{2\pi} \int_0^{2\pi} \Phi(\cdot, \theta)\, d\theta$ *is independent of the radius* r.

Proof. As discussed in (3.1.1) above, by applying the change of variable $r = e^s$, we have (3.1.2):

$$\Phi_{ss} + \Phi_{\theta\theta} = 0,$$

and since

$$\frac{d^2}{ds^2} \frac{1}{2\pi} \int_0^{2\pi} \Phi(s, \theta)\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \Phi_{ss}(s, \theta)\, d\theta = -\frac{1}{2\pi} \int_0^{2\pi} \Phi_{\theta\theta}(s, \theta)\, d\theta = 0,$$

we have:

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi(s, \theta)\, d\theta = \alpha s + \delta,$$

for some constants α and δ.

Also, since

$$\int_0^{2\pi} \Phi^2(\cdot,\theta)\, d\theta \in L^\infty(a,\infty),$$

we have by Hölder's inequality:

$$\Big| \int_0^{2\pi} \Phi(s,\theta)\, d\theta \Big| \le c\, \|\Phi(s,\theta)\|_{L^2(0,2\pi)} \le M < \infty.$$

This implies $\alpha = 0$, and consequently we have:

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi(s,\theta)\, d\theta = \delta,$$

which is independent of s, and therefore independent of the radius r. □

With Lemma 3.1.1, we can now state and prove the following result:

Proposition 3.1.2. *If* $\Phi(r,\theta) \in \{\Phi(r,\theta) \in H^1((a,\infty)\times\mathbb{S}^1) \,|\, \Phi(a,\theta) = \phi_0(\theta) + k\theta; \Phi(r,\theta+2\pi) - \Phi(r,\theta) = 2k\pi,\ k \in \mathbb{Z}\}$ *is a weak solution to the Euler-Lagrange equation:*

$$\begin{cases} (r\Phi_r)_r + \dfrac{1}{r}\Phi_{\theta\theta} = 0 \\ \Phi(a,\theta) = \Phi_0(\theta) \end{cases} \tag{3.1.6}$$

where $\Phi_0(\theta) = \phi_0(\theta) + k\theta$ *with* $\phi_0 \in H^{1/2}(\mathbb{S}^1)$ *given by* (3.0.4) - (3.0.5)*, and* $\Phi(r,\theta)$ *satisfies*

$$\int_0^{2\pi} \Phi^2(\cdot,\theta)\, d\theta \in L^\infty(a,\infty), \tag{3.1.7}$$

then we have:

$$\lim_{r\to\infty} \Phi(r,\theta) = k\theta + \beta \quad \text{in} \quad L^2(\mathbb{S}^1),$$

$k \in \mathbb{Z}$ *is the given degree of the vector field* $\mathbf{n}(x) = (\cos\Phi(x), \sin\Phi(x))$, $\beta = \frac{1}{2\pi}\int_0^{2\pi} \phi_0(\theta)\, d\theta$.

Proof. Note that $(s,\theta) \in (s_0,\infty)\times\mathbb{S}^1$, where $s_0 = \log a$, and without loss of generality, we can let $a = 1$ (hence $s_0 = 0$).

Since by (3.1.5) we have $v(s,\theta) = \Phi(s,\theta) - k\theta - \beta$, and $v(s,\theta)$ satisfies:

$$\begin{cases} v_{ss} + v_{\theta\theta} = 0 \\ v(0,\theta) = \phi_0(\theta) - \beta. \end{cases} \tag{3.1.8}$$

As in step 3 of the proof of Proposition 2.1.3, $v(s,\theta)$ is smooth in $(0,\infty)\times\mathbb{S}^1$.

Now by letting

$$F(s) = \int_0^{2\pi} v^2(s,\theta)\, d\theta,$$

where $v(s,\theta) = \Phi(s,\theta) - k\theta - \beta$ as defined in (3.1.3), we have:

$$|F(s)| \in L^\infty(s_0,\infty), \tag{3.1.9}$$

and hence

$$|F(s)| \leq M < \infty. \tag{3.1.10}$$

Then we have:

$$F_s(s) = 2\int_0^{2\pi} v \cdot v_s\, d\theta,$$

and by the Cauchy-Schwarz inequality, we have :

$$|F_s| \leq 2|F(s)|^{\frac{1}{2}} \cdot \|v_s\|_{L^2(0,2\pi)} \leq c\,\|v(s)\|_{L^2(0,2\pi)},$$

where $c = 2M^{\frac{1}{2}}$ and the last step holds since we have (3.1.10).

Notice also

$$\begin{aligned} F_{ss}(s) &= 2\int_0^{2\pi} (v_s^2 + v \cdot v_{ss})\, d\theta = 2\int_0^{2\pi} (v_s^2 - v \cdot v_{\theta\theta})\, d\theta \\ &= 2\int_0^{2\pi} (v_s^2 - (v \cdot v_\theta)_\theta + v_\theta^2)\, d\theta \\ &= 2\int_0^{2\pi} (v_s^2 + v_\theta^2)\, d\theta \geq 0, \end{aligned} \tag{3.1.11}$$

where the last step holds since $v(s,\theta)$ is 2π-periodic in θ :

$$v(s,\theta + 2\pi) = v(s,\theta).$$

By (3.1.11) $F_s(s)$ is nondecreasing. Hence, since $|F_s|$ is bounded, $F_s(s) \to \alpha$ as $s \to \infty$ for some $\alpha \in [0,\infty)$. Since for any $M > 0$,

$$F(\tau + M) - F(\tau) = \int_\tau^{\tau+M} F_s(s)\, ds,$$

letting $\tau \to \infty$ we deduce from (3.1.10) that $\alpha = 0$. In particular $F_s(s) \leq 0$ for all $s \geq 0$, and so $\lim_{s\to\infty} F(s) = L$ for some constant $L \geq 0$.

We now prove that $L = 0$ by using Fourier series on $v(s,\theta)$. This technique will be used again in the proof of Theorem 4.3.2.

Since $v \in L^2((s_0,\infty)\times\mathbb{S}^1)$, v can be written as $v(s,\theta) = \sum_{j=0}^{\infty} v_j(s)\sin j\theta + u_j(s)\cos j\theta$, a.e.

Also, since $v(s,\theta)$ satisfies $v_{ss} + v_{\theta\theta} = 0$ in (3.1.8), we have:

$$\begin{cases} v_j''(s) - j^2 v_j(s) = 0 \\ u_j''(s) - j^2 u_j(s) = 0, \end{cases}$$

which has solutions of the following form:

$$\begin{cases} v_j(s) = a_j \cdot e^{js} + b_j \cdot e^{-js} \\ u_j(s) = c_j \cdot e^{js} + d_j \cdot e^{-js}, \end{cases}$$

where a_j, b_j, c_j, and d_j ($j = 1, 2, ...$) are some constants.

For $j = 0$, we have $u_0''(s) = 0$, hence $u_0(s) = \alpha_0 + \alpha_1 s$, and therefore we have:

$$v(s,\theta) = \alpha_0 + \alpha_1 s + \Sigma_{j=1}^{\infty}(a_j e^{js} + b_j e^{-js})\sin j\theta + (c_j e^{js} + d_j e^{-js})\cos j\theta, \tag{3.1.12}$$

and

$$\begin{cases} v_s = \alpha_1 + \Sigma_{j=1}^{\infty} j(a_j e^{js} - b_j e^{-js})\sin j\theta + j(c_j e^{js} - d_j e^{-js})\cos j\theta \\ v_\theta = \Sigma_{j=1}^{\infty} j(a_j e^{js} + b_j e^{-js})\cos j\theta - j(c_j e^{js} + d_j e^{-js})\sin j\theta. \end{cases}$$

Note also that by choosing $\beta = \frac{1}{2\pi}\int_0^{2\pi}\phi_0(\theta)\,d\theta$ we have $\int_0^{2\pi} v\,d\theta = \int_0^{2\pi}(\Phi(s,\theta) - k\theta - \beta)\,d\theta = 0$, hence $\alpha_0 = 0$.

Consequently

$$\int_0^{2\pi} v^2(s,\theta)\,d\theta = 2\pi\alpha_1^2 s^2 + \pi\,\Sigma_{j=1}^{\infty}\left((a_j^2 + c_j^2)\,e^{2js} + (b_j^2 + d_j^2)\,e^{-2js} + (a_j b_j + c_j d_j)\right). \tag{3.1.13}$$

Since we also know that

$$\lim_{s\to\infty} F(s) = \lim_{s\to\infty}\int_0^{2\pi} v^2(s,\theta)\,d\theta = L \geq 0,$$

we have

$$
\begin{aligned}
L &= \lim_{s\to\infty} \|v(s)\|^2_{L^2(0,2\pi)} \\
&= \lim_{s\to\infty} \pi\Sigma_{j=1}^{\infty} \left((a_j^2 + c_j^2)\, e^{2js} + (b_j^2 + d_j^2)\, e^{-2js} + (a_j b_j + c_j d_j)\right) + \lim_{s\to\infty} 2\pi\alpha_1^2 s^2 < \infty.
\end{aligned}
\tag{3.1.14}
$$

Therefore we must have $\alpha_1 = 0$, and

$$
a_j = c_j = 0, \quad j = 1, 2, ...
$$

As a consequence, $v(s, \theta)$ can be represented by Fourier series:

$$
v(s, \theta) = \sum_{j=1}^{\infty} b_j \cdot e^{-js} \sin j\theta + d_j \cdot e^{-js} \cos j\theta,
$$

and by (3.1.14), we have

$$
\sum_{j=1}^{\infty} b_j^2 + d_j^2 < \infty.
$$

Therefore we have

$$
L = \lim_{s\to\infty} F(s) = \lim_{s\to\infty} \int_0^{2\pi} v^2(s, \theta)\, d\theta = \lim_{s\to\infty} \pi \cdot \sum_{j=1}^{\infty} e^{-2js} (b_j^2 + d_j^2) = 0,
$$

hence

$$
\lim_{s\to\infty} \Phi(s, \theta) = k\theta + \beta, \quad \text{in} \quad L^2(0, 2\pi) \tag{3.1.15}
$$

Also from (3.1.13) we have

$$
F(s) = \int_0^{2\pi} v^2(s, \theta)\, d\theta = \pi\, \Sigma_{j=1}^{\infty} (b_j^2 + d_j^2) e^{-2js} = \pi\, e^{-2s} \Sigma_{j=1}^{\infty} (b_j^2 + d_j^2)\, e^{-2(j-1)s} \le \frac{C}{r^2}, \tag{3.1.16}
$$

where $C > 0$ is some constant and in the last step we used the change of variable $r = e^s$. (3.1.16) actually gives the asymptotic behavior of $\Phi(r, \theta)$, i.e.

$$
\|\Phi(x) - k\theta - \beta\|_{L^2(\mathbb{S}^1)} \le \frac{C_0}{r},
$$

as $r \to \infty$, completing the proof.

□

Remark 3.1.3. *In fact,* $L = 0$ *can also be proved directly by using Poincaré inequality, and we* *give this alternative proof in the following.*

Proof. We use Poincaré's inequality (see [Eva10]), that

$$||v_\theta||_{L^2(0,2\pi)} \geq \gamma \left\| v - \frac{1}{2\pi}\int_0^{2\pi} v \, d\theta \right\|_{L^2(0,2\pi)},$$

where $\gamma > 0$ is some constant. By Lemma 3.1.1, $\frac{1}{2\pi}\int_0^{2\pi} \phi(s,\theta)\, d\theta = \beta$ for all $s \geq 0$, and so $\frac{1}{2\pi}\int_0^{2\pi} v \, d\theta = 0$ and hence

$$||v_\theta||^2_{L^2(0,2\pi)} \geq \gamma ||v||^2_{L^2(0,2\pi)}.$$

Consequently the following inequality holds:

$$F_{ss} = 2\int_0^{2\pi} (v_s^2 + v_\theta^2)\, d\theta \geq 2\int_0^{2\pi} v_\theta^2 \, d\theta \geq 2\gamma\, ||v||^2_{L^2(0,2\pi)} = 2\gamma\, F \geq 2\gamma\, L. \tag{3.1.17}$$

Hence

$$F(s) \geq \gamma \cdot L \cdot s^2 + c \cdot s + d, \tag{3.1.18}$$

for constants c, d and by (3.1.10) we have $L = 0$.

□

Note that the limit in (3.1.15) is of the form (3.0.17), i.e. the solution to the Euler-Lagrange equation for the one-constant radius independent case. Proposition 3.1.2 implies that any weak solution to the Euler-Lagrange equation (3.1.6) tends to some radius independent solution at infinity if it satisfies the constraint (3.1.7).

We have proved in Proposition 3.1.2 that, by choosing $\beta = \frac{1}{2\pi}\int_0^\infty \phi_0 \, d\theta$ in $v(s,\theta) = \Phi(s,\theta) - k\theta - \beta$, any weak solution to the boundary value problem (3.1.8), i.e. the Euler-Lagrange equation of Oseen-Frank free energy after change of variable $r = e^s$, coupled with the boundary condition for $v(s,\theta)$:

$$\begin{cases} v_{ss} + v_{\theta\theta} = 0 \\ v(0,\theta) = \phi_0(\theta) - \beta \end{cases}$$

will tend to zero in $L^2(0,2\pi)$ when $s \to \infty$, where v is in the admissible set $\{v \in H^1((0,\infty)\times \mathbb{S}^1) \,|\, v(0,\theta) = \phi_0(\theta) - \beta;\ \int_0^{2\pi} v^2(s,\theta)\, d\theta \in L^\infty(0,\infty)\}$. Here the periodic condition $v(s,\theta+2\pi) = v(s,\theta)$ is automatically satisfied since $v \in H^1((0,\infty)\times\mathbb{S}^1)$. Also $v(s,\theta)$ and $\phi(s,\theta)$ only differs by a constant $\beta = \frac{1}{2\pi}\int_0^\infty \phi_0 \, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \Phi(1,\theta)\, d\theta - k\pi$, and without loss of generality, we can assume $\beta = 0$. Further, one can see that when $r \to \infty$ ($s \to \infty$), we have

$\lim_{r\to\infty}\Phi(r,\theta) = k\theta+\beta$, hence

$$\begin{cases} \lim\limits_{r\to\infty}\Phi_s = 0 \\ \lim\limits_{r\to\infty}\Phi_\theta = k, \end{cases}$$

and the free energy density becomes

$$G(r,\theta,\Phi_\theta,\Phi_r,\Phi) = \frac{1}{2}\Big(\Phi_r^2 + \frac{\Phi_\theta^2}{r^2}\Big) = \frac{k^2}{2r^2},$$

which after change of variable $r = e^s$ and integration over $s \in (0,\infty)$ becomes:

$$E(\Phi) = \frac{1}{2}\int_1^\infty r\int_0^{2\pi}(\Phi_r^2 + \frac{\Phi_\theta^2}{r^2})\,d\theta\,dr = \frac{1}{2}\int_0^\infty\int_0^{2\pi} k^2\,d\theta\,ds. \tag{3.1.19}$$

It is easy to see that the integral (3.1.19) is infinite, but it naturally lead us to conjecture that there exist minimizers for the modified functional

$$\hat{E}(\Phi) = \frac{1}{2}\int_1^\infty r\int_0^{2\pi}((\Phi_r^2 + \frac{\Phi_\theta^2}{r^2}) - \frac{k^2}{r^2})\,d\theta\,dr, \tag{3.1.20}$$

and we need to be careful when we define the admissible set A_Φ where $\Phi \in A_\Phi$, since $\Phi(r,\theta)$ has jumps when θ traverses around $\mathbb{S}^1$ due to the degree constraint, i.e.

$$\Phi(r,\theta+2\pi) - \Phi(r,\theta) = 2k\pi, \text{ a.e. } r \in (1,\infty).$$

We can define the admissible set A_Φ by using the change of variable $\phi(r,\theta) = \Phi(r,\theta) - k\theta$, which is 2π−periodic in $\theta \in \mathbb{S}^1$, and we have

$$A_\Phi = \{\Phi \,|\, \phi = \Phi - k\theta \in \widetilde{H}^1((1,\infty)\times\mathbb{S}^1);\ \phi(1,\theta) = \phi_0(\theta);\ \int_0^{2\pi}\phi(r,\theta)\,d\theta = 0 \text{ a.e. } r \in (1,\infty)\},$$

where $\widetilde{H}^1((1,\infty)\times\mathbb{S}^1)$ is defined in Definition 2.3.1. The condition $\int_0^{2\pi}\phi(r,\theta)\,d\theta = 0$ a.e. $r \in (1,\infty)$ implies that $\int_0^{2\pi}\Phi(r,\theta)\,d\theta$ is radius independent. We will prove the existence of minimizers to the functional (3.1.20) in the next section.

3.2 Main result for single-hole domain (one-constant case)

In this section we state and prove our main result for the one-constant radius-dependent minimization problem in a circular domain, which shows that there exists a unique minimizer to the functional:

$$E(\Phi) = \int_1^\infty r \int_0^{2\pi} ((\Phi_r^2 + \frac{\Phi_\theta^2}{r^2}) - \frac{k^2}{r^2})\, d\theta\, dr, \tag{3.2.1}$$

for Φ in the admissible set

$$A_\Phi = \{\Phi \,|\, \phi = \Phi - k\theta \in \widetilde{H}^1((1,\infty)\times\mathbb{S}^1);\ \phi(1,\theta) = \phi_0(\theta); \int_0^{2\pi} \phi(r,\theta)\, d\theta = 0 \ \text{ a.e. } r \in (1,\infty)\}.$$

Note that instead of using the notation $\hat{E}(\Phi)$ for the modified functional in (3.1.20), we used $E(\Phi)$ and omitted the coefficient $\frac{1}{2}$ as this will not change the conclusion. We will use the notation as in (3.2.1) throughout the rest of Chapter 3.

By change of variables

$$\begin{cases} r = e^s \\ \phi(r,\theta) = \Phi(r,\theta) - k\theta, \end{cases}$$

one can see that this is equivalent to saying that there exists a unique minimizer to the functional:

$$\widetilde{E}(\phi) = \int_0^\infty \int_0^{2\pi} ((\phi_s^2 + (\phi_\theta + k)^2) - k^2)\, d\theta\, ds, \tag{3.2.2}$$

for ϕ in the admissible set $A = \{\phi \in \widetilde{H}^1((0,\infty)\times\mathbb{S}^1) \,|\, \phi(0,\theta) = \phi_0(\theta); \int_0^{2\pi} \phi(s,\theta)\, d\theta = 0 \text{ a.e. } s \in (0,\infty)\}$, where $\phi_0(\theta) \in H^{1/2}(0,2\pi)$, and $\widetilde{H}^1((0,\infty)\times\mathbb{S}^1)$ is again defined in Definition 2.3.1.

In fact one can see that

$$\widetilde{E}(\phi) = \int_0^\infty \int_0^{2\pi} (\phi_s^2 + \phi_\theta^2)\, d\theta\, ds,$$

for $\phi \in A$, and we have the following theorem.

Theorem 3.2.1. *There exist a unique minimizer $\widetilde{\phi}$ to the functional*

$$\widetilde{E}(\phi) = \int_0^\infty \int_0^{2\pi} (\phi_s^2 + \phi_\theta^2)\, d\theta\, ds, \tag{3.2.3}$$

where ϕ is in the admissible set $A = \{\phi \in \widetilde{H}^1((0,\infty)\times\mathbb{S}^1) \,|\, \phi(0,\theta) = \phi_0(\theta); \int_0^{2\pi} \phi(s,\theta)\, d\theta = 0 \text{ a.e. } s \in (0,\infty)\}$. *Also, we have:*

$$\lim_{s\to\infty} \widetilde{\phi}(s,\theta) = 0 \ \text{ in } \ L^2(0,2\pi).$$

Proof. We prove this theorem in several steps.

Step 1

In this step we show that there exist a minimizer for (3.2.3). Select a minimizing sequence $\{\phi^{(l)}\}_{l=1}^{\infty} \subset \mathrm{A}$, such that

$$\lim_{l\to\infty} \widetilde{\mathrm{E}}(\phi^{(l)}) = \mathrm{m}.$$

We can assume $\mathrm{m} < \infty$ because for all $\phi \in \mathrm{A}$ one can multiply it by a suitable cut-off function so that it vanishes for large s, and it is still in the admissible set A.

Then we have

$$\widetilde{\mathrm{E}}(\phi^{(l)}) = \int_0^{\infty}\int_0^{2\pi} ((\phi_s^{(l)})^2 + (\phi_\theta^{(l)})^2)\, d\theta\, ds \le \mathrm{M} < \infty,$$

for some $\mathrm{M} > 0$.

Also, one can see that, by Poincaré's inequality (see [Nec11, p, 113]), we have:

$$\|\phi^{(l)} - \frac{1}{2\pi}\int_0^{2\pi} \phi^{(l)}\, d\theta\,\|_{L^2(0,2\pi)} \le c\,\|\phi_{\theta}^{(l)}\|_{L^2(0,2\pi)},$$

for some constant $c > 0$, and since $\phi^{(l)} \in \mathrm{A} = \{\phi \in \widetilde{\mathrm{H}}^1((0,\infty)\times\mathbb{S}^1)\,|\,\phi(0,\theta) = \phi_0(\theta); \int_0^{2\pi}\phi(s,\theta)\,d\theta = 0 \text{ a.e. } s \in (0,\infty)\}$, we have

$$\|\phi^{(l)}\|^2_{L^2(0,2\pi)} \le c^2\,\|\phi_\theta^{(l)}\|^2_{L^2(0,2\pi)}.$$

And as a consequence,

$$\int_0^{\infty}\int_0^{2\pi} (\phi^{(l)})^2\, d\theta\, ds \le c^2 \int_0^{\infty}\int_0^{2\pi} (\phi_\theta^{(l)})\, d\theta\, ds \le c^2 \int_0^{\infty}\int_0^{2\pi} ((\phi_\theta^{(l)})^2 + (\phi_s^{(l)})^2)\, d\theta\, ds < \infty, \tag{3.2.4}$$

which implies $\phi^{(l)} \in L^2((0,\infty)\times\mathbb{S}^1)$ for any $l \in (0, 1, ...)$.

Consequently, we know that $\{\phi^{(l)}\}_{l=1}^{\infty}$ is bounded in $H^1((0,j)\times\mathbb{S}^1)$, for any sufficiently large j. Then we can apply a diagonal argument: First we have that there exists a subsequence $\{\phi^{(l_{i_1})}\}_{i_1=1}^{\infty}$ and a function ϕ such that $\{\phi^{(l_{i_1})}\}_{i_1=1}^{\infty}$ converges weakly to ϕ in $H^1((0,j)\times\mathbb{S}^1)$. Then since $\{\phi^{(l_{i_1})}\}_{i_1=1}^{\infty}$ is also a abounded sequence in $H^1((0,j+1)\times\mathbb{S}^1)$, we can extract a subsequence $\{\phi^{(l_{i_2})}\}_{i_2=1}^{\infty}$ out of $\{\phi^{(l_{i_1})}\}_{i_1=1}^{\infty}$ and a function ϕ' such that $\{\phi^{(l_{i_2})}\}_{i_2=1}^{\infty}$ converges weakly to ϕ' in $H^1((0,j+1)\times\mathbb{S}^1)$. And we must have $\phi = \phi'$ in $H^1((0,j)\times\mathbb{S}^1)$. Then by repeating this process, we can therefore extract a subsequence $\{\phi^{(l_n)}\}_{n=1}^{\infty}$ of $\{\phi^{(l)}\}_{l=1}^{\infty}$ such that it converges weakly to some

$\widetilde{\phi}$ in $\mathrm{H}^1((0,\mathrm{j})\times\mathbb{S}^1)$ for any j sufficiently large, where $\widetilde{\phi}\in\widetilde{\mathrm{H}}^1((0,\infty)\times\mathbb{S}^1)$, i.e. $\widetilde{\phi}\in\mathrm{H}^1((0,\mathrm{j})\times\mathbb{S}^1)$ for any sufficiently large j.

Also, since $\mathrm{W}(\mathrm{s},\theta,\phi,\nabla\phi)=\frac{1}{2}(\phi_\mathrm{s}^2+\phi_\theta^2)$ is convex with respect to $\nabla\phi$, one can see that

$$\widetilde{\mathrm{E}}(\phi)=\int_0^\infty\int_0^{2\pi}(\phi_\mathrm{s}^2+\phi_\theta^2)\,\mathrm{d}\theta\,\mathrm{ds}$$

is lower semi-continuous, i.e. given a weakly convergent sequence $\phi^{(\mathrm{l_n})}\rightharpoonup\widetilde{\phi}$, we have

$$\liminf_{\mathrm{n}\to\infty}\widetilde{\mathrm{E}}(\phi^{(\mathrm{l_n})})\geq\widetilde{\mathrm{E}}(\widetilde{\phi}).$$

which implies that $\widetilde{\phi}$ is a minimizer for $\widetilde{\mathrm{E}}(\phi)$.

Step 2

In this step we show that $\widetilde{\phi}$ is in the admissible set $\mathrm{A}=\{\phi\in\widetilde{\mathrm{H}}^1((0,\infty)\times\mathbb{S}^1)\,|\,\phi(0,\theta)=\phi_0(\theta);\int_0^{2\pi}\phi(\mathrm{s},\theta)\,\mathrm{d}\theta=0\ \text{a.e. s}\in(0,\infty)\}$.

The periodicity of the weak limit $\widetilde{\phi}$ is automatic since we have a Sobolev space on a cylinder, hence we only need to show

$$\int_0^{2\pi}\phi(\mathrm{s},\theta)\,\mathrm{d}\theta=0\quad\text{a.e. s}\in(0,\infty).$$

Since we know that $\phi^{(\mathrm{l})}\rightharpoonup\widetilde{\phi}$ weakly in $\widetilde{\mathrm{H}}^1((0,\infty)\times\mathbb{S}^1)$, we have

$$\lim_{\mathrm{l}\to\infty}\int_0^\mathrm{R}\int_0^{2\pi}\phi^{(\mathrm{l})}\eta\,\mathrm{d}\theta\,\mathrm{ds}=\int_0^\mathrm{R}\int_0^{2\pi}\widetilde{\phi}\,\eta\,\mathrm{d}\theta\,\mathrm{ds}\tag{3.2.5}$$

for any $\eta(\mathrm{s},\theta)\in\mathrm{L}^2((0,\mathrm{R})\times\mathbb{S}^1)$ and any sufficiently large R.

We have by letting $\eta=\xi(\mathrm{s})\times\mathbb{1}_{(0,2\pi)}$, where $\forall\xi(\mathrm{s})\in\mathrm{L}^2(0,\mathrm{R})$, for any sufficiently large j, such that (3.2.5) becomes:

$$\lim_{\mathrm{l}\to\infty}\int_0^\mathrm{R}\Big(\int_0^{2\pi}\phi^{(\mathrm{l})}\,\mathrm{d}\theta\Big)\xi(\mathrm{s})\,\mathrm{ds}=\int_0^\mathrm{R}\Big(\int_0^{2\pi}\widetilde{\phi}\,\mathrm{d}\theta\Big)\xi(\mathrm{s})\,\mathrm{ds},$$

which by the fundamental lemma of the calculus of variations gives:

$$\int_0^{2\pi}\widetilde{\phi}\,\mathrm{d}\theta=\lim_{\mathrm{l}\to\infty}\int_0^{2\pi}\phi^{(\mathrm{l})}\,\mathrm{d}\theta=0\quad\text{a.e. s}\in(0,\infty).\tag{3.2.6}$$

Therefore we have $\widetilde{\phi} \in \mathrm{A} = \{\phi \in \widetilde{\mathrm{H}}^1((0,\infty)\times\mathbb{S}^1) \,|\, \phi(0,\theta) = \phi_0(\theta); \int_0^{2\pi} \phi(\mathrm{s},\theta)\,\mathrm{d}\theta = 0$ a.e. $\mathrm{s} \in (0,\infty)\}$.

Step 3

In this step we show that:

$$\lim_{\mathrm{s}\to\infty} \widetilde{\phi}(\mathrm{s},\theta) = 0 \quad \text{in} \quad \mathrm{L}^2(0,2\pi).$$

We first show that $\widetilde{\phi}$ is smooth in $(0,\infty)\times\mathbb{S}^1$ and satisfies:

$$\widetilde{\phi}_{\theta\theta} + \widetilde{\phi}_{\mathrm{ss}} = 0. \tag{3.2.7}$$

Recall that $\widetilde{\phi}$ is a minimizer to the functional

$$\widetilde{\mathrm{E}}(\phi) = \int_0^\infty \int_0^{2\pi} (\phi_{\mathrm{s}}^2 + \phi_\theta^2)\,\mathrm{d}\theta\,\mathrm{ds},$$

therefore we have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{\mathrm{E}}(\widetilde{\phi} + \tau\eta)\Big|_{\tau=0} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\infty \int_0^{2\pi} ((\widetilde{\phi}_{\mathrm{s}} + \tau\eta_{\mathrm{s}})^2 + (\widetilde{\phi}_\theta + \tau\eta_\theta)^2)\,\mathrm{d}\theta\,\mathrm{ds} \\
&= 2\int_0^\infty \int_0^{2\pi} (\widetilde{\phi}_{\mathrm{s}}\,\eta_{\mathrm{s}} + \widetilde{\phi}_\theta\,\eta_\theta)\,\mathrm{d}\theta\,\mathrm{ds} \\
&= -2\int_0^\infty \int_0^{2\pi} \widetilde{\phi}\cdot(\eta_{\theta\theta} + \eta_{\mathrm{ss}})\,\mathrm{d}\theta\,\mathrm{ds} \\
&= 0,
\end{aligned}$$

for any $\eta \in \mathrm{C}_0^\infty((0,\infty)\times\mathbb{S}^1)$. Hence we have that $\widetilde{\phi}$ is smooth in $(0,\infty)\times\mathbb{S}^1$, and satisfies (3.2.7).

Similar to (3.2.4), by applying Poincarés inequality on $\widetilde{\phi}$, we have

$$\int_0^\infty \int_0^{2\pi} \widetilde{\phi}^2\,\mathrm{d}\theta\,\mathrm{ds} \le \mathrm{c}\int_0^\infty \int_0^{2\pi} (\widetilde{\phi}_\theta^2 + \widetilde{\phi}_{\mathrm{s}}^2) = \mathrm{c}\,\mathrm{I}(\widetilde{\phi}) < \infty,$$

where c is some constant. Therefore, by letting $\widetilde{\mathrm{F}}(\mathrm{s}) = \int_0^{2\pi} \widetilde{\phi}^2\,\mathrm{d}\theta$, we have that

$$\int_0^\infty \widetilde{\mathrm{F}}(\mathrm{s})\,\mathrm{ds} \le \mathrm{M}'' < \infty.$$

Consequently, it follows from the same reasoning of how we proved (3.1.15) in Theorem 3.1.2

that

$$\lim_{s\to\infty} \widetilde{F}(s) = 0,$$

which implies

$$\lim_{s\to\infty} \widetilde{\phi}(s,\theta) = 0 \quad \text{in} \quad L^2(\mathbb{S}^1).$$

Step 4

At last, we show that the $\widetilde{\phi}(s,\theta)$ is the unique minimizer for the functional (3.2.3):

$$\widetilde{E}(\phi) = \int_0^{\infty} \int_0^{2\pi} (\phi_s^2 + \phi_\theta^2)\, d\theta\, ds,$$

for ϕ in the admissible set $A = \{\phi \in \widetilde{H}^1((0,\infty)\times\mathbb{S}^1)\,|\,\phi(0,\theta) = \phi_0(\theta); \int_0^{2\pi} \phi(s,\theta)\, d\theta = 0 \text{ a.e. } s \in (0,\infty)\}$.

Assume there is another minimizer $\widetilde{\phi}' \in A$ for the functional (3.2.3), then $\widetilde{\phi}'$ must also satisfies (3.2.7), i.e.

$$\widetilde{\phi}'_{\theta\theta} + \widetilde{\phi}'_{ss} = 0.$$

As a consequence, if we let $\phi^* = \widetilde{\phi} - \widetilde{\phi}'$, we will also have:

$$\phi^*_{\theta\theta} + \phi^*_{ss} = 0, \tag{3.2.8}$$

where ϕ^* is in the admissible set $A^* = \{\phi \in \widetilde{H}^1((0,\infty)\times\mathbb{S}^1)\,|\,\phi(0,\theta) = 0; \int_0^{2\pi} \phi\, d\theta = 0 \text{ a.e. } s \in (0,\infty)\}$. Multiply both sides of (3.2.8) with ϕ^*, we have:

$$\begin{aligned} 0 &= \int_0^j \int_0^{2\pi} (\phi^* \cdot \phi^*_{\theta\theta} + \phi^* \cdot \phi^*_{ss})\, d\theta\, ds \\ &= -\int_0^j \int_0^{2\pi} (\phi^{*2}_\theta + \phi^{*2}_s)\, d\theta\, ds + \int_0^{2\pi} \Big(\phi^*(j,\theta)\cdot\phi^*_s(j,\theta) - \phi^*(0,\theta)\cdot\phi^*_s(0,\theta)\Big)\, d\theta \\ &= -\int_0^j \int_0^{2\pi} (\phi^{*2}_\theta + \phi^{*2}_s)\, d\theta\, ds + \int_0^{2\pi} \phi^*(j,\theta)\cdot\phi^*_s(j,\theta)\, d\theta. \end{aligned} \tag{3.2.9}$$

One can easily see that we must also have

$$\lim_{s\to\infty} \phi^*(s,\theta) = 0 \quad \text{in} \quad L^2(\mathbb{S}^1), \tag{3.2.10}$$

which follows exactly from the same reasoning as we did for $\widetilde{\phi}(s,\theta)$.

As a consequence, we know from (3.2.9) and (3.2.10):

$$\begin{aligned}\int_0^\infty\int_0^{2\pi}(\phi_\theta^{*2}+\phi_s^{*2})\,d\theta\,ds &= \lim_{j\to\infty}\int_0^j\int_0^{2\pi}(\phi_\theta^{*2}+\phi_s^{*2})\,d\theta\,ds\\ &= \lim_{j\to\infty}\int_0^{2\pi}\phi^*(j,\theta)\cdot\phi_s^*(j,\theta)\,d\theta\\ &= \frac{1}{2}\lim_{j\to\infty}\frac{d}{ds}\int_0^{2\pi}\phi^{*2}(j,\theta)\,d\theta\\ &= 0.\end{aligned}$$

Therefore we have

$$\phi^*\equiv\text{constant}\quad\text{in}\quad(0,\infty)\times\mathbb{S}^1.$$

Since $\phi^*(0,\theta)=0$, we must have

$$\phi^*\equiv 0\quad\text{in}\quad(0,\infty)\times\mathbb{S}^1,$$

which implies the uniqueness of the minimizer for the functional (3.2.3), completing our proof.

□

3.3 Truncated minimization problem

In this section we first study the minimizer for a truncated version of the functional (3.2.2) in the domain $\Omega_j=(0,j)\times(0,2\pi)$ for all $j\geq 1$ with $\phi(s,\theta)$ satisfying the constraint $\phi(j,\theta)=0$, i.e. we prove that there exist a minimizer ϕ^j for the functional:

$$\widetilde{E}_j(\phi)=\int_0^j\int_0^{2\pi}(\phi_s^2+\phi_\theta^2)\,d\theta\,ds$$

where $\phi\in A^j=\{\phi\in H^1((0,j)\times\mathbb{S}^1)\,|\,\phi(0,\theta)=\phi_0(\theta)\,;\phi(j,\theta)=0;\int_0^{2\pi}\phi\,d\theta=0\text{ a.e. }s\in(0,j)\}$, and $\phi_0(\theta)\in H^{1/2}(0,2\pi)$.

Then we extend the result to $\Omega=(0,\infty)\times(0,2\pi)$. One can easily see that we can extend the minimizer $\phi^j(s,\theta)$ by zero for $s\in(j,\infty)$ and the extended function $\hat{\phi}^j$ will satisfy $\hat{\phi}^j\in\widetilde{H}^1((0,\infty)\times\mathbb{S}^1)$. Also, the limit of such extended functions $\hat{\phi}^j$ will tend to some $\hat{\phi}\in\widetilde{H}^1((0,\infty)\times\mathbb{S}^1)$, and we will show that $\hat{\phi}$ is in fact the unique minimizer of the functional (3.2.3) that we obtained in Theorem 3.2.1. We state the result in the following theorem.

Theorem 3.3.1. *There exists a minimizer* ϕ^{j} *to the functional*

$$\widetilde{\mathrm{E}}_{\mathrm{j}}(\phi) = \int_0^{\mathrm{j}} \int_0^{2\pi} (\phi_{\mathrm{s}}^2 + \phi_\theta^2)\, \mathrm{d}\theta\, \mathrm{ds} \tag{3.3.1}$$

for ϕ *in the admissible set* $\phi \in \mathrm{A}^{\mathrm{j}} = \{\phi \in \mathrm{H}^1((0,\mathrm{j}) \times \mathbb{S}^1) \,|\, \phi(0,\theta) = \phi_0(\theta)\,;\phi(\mathrm{j},\theta) = 0; \int_0^{2\pi} \phi\, \mathrm{d}\theta = 0 \text{ a.e. } \mathrm{s} \in (0,\mathrm{j})\}$ *for all* $\mathrm{j} \geq 1$. *Further, by extending the minimizer* ϕ^{j} *to* $(0,\infty) \times \mathbb{S}^1$ *such that the extended function* $\hat{\phi}^{\mathrm{j}} = 0$ *on* $(\mathrm{j},\infty) \times \mathbb{S}^1$, *we have that the sequence* $\{\hat{\phi}^{\mathrm{j}}\}$ *converges weakly to* $\widetilde{\phi}$, *where* $\widetilde{\phi}$ *is the unique minimizer to the functional*

$$\widetilde{\mathrm{E}}(\phi) = \int_0^{\infty} \int_0^{2\pi} (\phi_{\mathrm{s}}^2 + \phi_\theta^2)\, \mathrm{d}\theta\, \mathrm{ds} \tag{3.3.2}$$

in the admissible set $\mathrm{A} = \{\phi \in \widetilde{\mathrm{H}}^1((0,\infty) \times \mathbb{S}^1) \,|\, \phi(0,\theta) = \phi_0(\theta); \int_0^{2\pi} \phi(\mathrm{s},\theta)\, \mathrm{d}\theta = 0 \text{ a.e. } \mathrm{s} \in (0,\infty)\}$, *which we proved in Theorem 3.2.1 and it satisfies*

$$\lim_{\mathrm{j}\to\infty} \widetilde{\phi} \to 0 \quad \text{in} \quad \mathrm{L}^2(\mathbb{S}^1).$$

Proof. We prove this result in several steps.

Step 1

In this step we will show that there exist a minimizer for the functional (3.3.1) in the admissible set $\phi \in \mathrm{A}^{\mathrm{j}}$ for each $\mathrm{j} \geq 1$. By selecting a minimizing sequence $\{\phi^{(\mathrm{l})}\}_{\mathrm{l}=1}^{\infty}$ such that

$$\widetilde{\mathrm{E}}_{\mathrm{j}}(\phi^{(\mathrm{l})}) \to \mathrm{m},$$

and without loss of generality, we can assume that $-\infty < \mathrm{m} < \infty$. Since $\phi^{(\mathrm{l})} \in \mathrm{A}^{\mathrm{j}} = \{\phi \in \mathrm{H}^1((0,\mathrm{j}) \times \mathbb{S}^1) \,|\, \phi(0,\theta) = \phi_0(\theta)\,;\phi(\mathrm{j},\theta) = 0; \int_0^{2\pi} \phi\, \mathrm{d}\theta = 0 \text{ a.e. } \mathrm{s} \in (0,\mathrm{j})\}$, we have:

$$\widetilde{\mathrm{E}}_{\mathrm{j}}(\phi^{(\mathrm{l})}) = \int_0^{\mathrm{j}} \int_0^{2\pi} ((\phi_{\mathrm{s}}^{(\mathrm{l})})^2 + (\phi_\theta^{(\mathrm{l})})^2)\, \mathrm{d}\theta\, \mathrm{ds}.$$

Hence as a consequence we have:

$$\int_0^{\mathrm{j}} \int_0^{2\pi} ((\phi_{\mathrm{s}}^{(\mathrm{l})})^2 + (\phi_\theta^{(\mathrm{l})})^2)\, \mathrm{d}\theta\, \mathrm{ds} \leq \mathrm{M} < \infty \tag{3.3.3}$$

for some $M > 0$.

By Poincaré's inequality (see [MJ09, p. 69]), on $(0, 2\pi)$ we have:

$$\|\phi^{(l)} - \frac{1}{2\pi}\int_0^{2\pi} \phi^{(l)}\, d\theta\|_{L^2(0,2\pi)} \le c\, \|\phi^{(l)}_\theta\|_{L^2(0,2\pi)},$$

for some constant $c > 0$, and therefore:

$$C\, \|\phi^{(l)}\|^2_{L^2(0,2\pi)} \le \|\phi^{(l)}_\theta\|^2_{L^2(0,2\pi)}, \tag{3.3.4}$$

where $C = \frac{1}{c} > 0$ is some constant. Hence from (3.3.4) we have:

$$\widetilde{E}_j(\phi^{(l)}) = \int_0^j \int_0^{2\pi} ((\phi^{(l)}_s)^2 + (\phi^{(l)}_\theta)^2)\, d\theta\, ds \ge \int_0^j \int_0^{2\pi} (\phi^{(l)}_\theta)^2\, d\theta\, ds \ge C \int_0^j \int_0^{2\pi} (\phi^{(l)})^2\, d\theta\, ds \tag{3.3.5}$$

and this implies that $\phi^{(l)} \in L^2((0, j) \times \mathbb{S}^1)$.

From (3.3.3) and (3.3.5), we have

$$\|\phi^{(l)}\|_{H^1((0,j)\times(0,2\pi))} \le M' < \infty,$$

where M' is some positive constant. This implies that the minimizing sequence $\{\phi^{(l)}(s, \theta)\}_{l=1}^\infty$ is uniformly bounded in $H^1((0, j) \times \mathbb{S}^1)$. Therefore there exist a subsequence (without loss of generality, we still denote the subsequence by $\{\phi^{(l)}(s, \theta)\}_{l=1}^\infty$) and some $\phi^j(s, \theta) \in H^1((0, j) \times \mathbb{S}^1)$ such that

$$\phi^{(l)} \rightharpoonup \phi^j$$

weakly in $H^1((0, j) \times \mathbb{S}^1)$.

Also, by using the same reasoning in Step 2 of Theorem 3.2.1 which led to (3.2.6), we have:

$$\int_0^{2\pi} \phi^j\, d\theta = \lim_{l\to\infty} \int_0^{2\pi} \phi^{(l)}\, d\theta = 0 \quad \text{a.e. } s \in (0, j).$$

Consequently, ϕ^j is in the admissible set $A^j = \{\phi \in H^1((0, j) \times \mathbb{S}^1) \,|\, \phi(0, \theta) = \phi_0(\theta)\, ; \phi(j, \theta) = 0; \int_0^{2\pi} \phi\, d\theta = 0$ a.e. $s \in (0, j)\}$. Also since the functional

$$\widetilde{E}_j(\phi) = \int_0^j \int_0^{2\pi} (\phi_s^2 + \phi_\theta^2)\, d\theta\, ds$$

is convex in ϕ_θ and ϕ_s for each given ϕ, s and θ, we know that $\widetilde{E}_j(\phi)$ is weakly lower semicontinuous in $H^1((0,j)\times\mathbb{S}^1)$, we have from [Eva10]:

$$\widetilde{E}_j(\phi^j) \le \liminf_{l\to\infty} \widetilde{E}_j(\phi^{(l)}) = m.$$

Since $\phi^j \in A^j = \{\phi \in H^1((0,j)\times\mathbb{S}^1)\,|\,\phi(0,\theta) = \phi_0(\theta)\,;\phi(j,\theta) = 0; \int_0^{2\pi}\phi\, d\theta = 0 \text{ a.e. } s \in (0,j)\}$, we have

$$\widetilde{E}_j(\phi^j) \ge \inf_{\phi\in A} \widetilde{E}_j(\phi) = m.$$

Therefore we have shown that $\phi^j \in A^j = \{\phi \in H^1((0,j)\times\mathbb{S}^1)\,|\,\phi(0,\theta) = \phi_0(\theta)\,;\phi(j,\theta) = 0; \int_0^{2\pi}\phi(s,\theta)\, d\theta = 0 \text{ a.e. } s \in (0,j)\}$ is a minimizer for

$$\widetilde{E}_j(\phi) = \int_0^j \int_0^{2\pi} (\phi_s^2 + \phi_\theta^2)\, d\theta\, ds,$$

for all $j \ge 1$.

Step 2

In this step, we show that by extending ϕ^j to $(0,\infty)\times\mathbb{S}^1$, the extended function will converge weakly to some $\hat{\phi} \in \widetilde{H}^1((0,\infty)\times\mathbb{S}^1)$.

For each minimizer ϕ^j of functional (3.3.1), where $\phi^j \in A^j = \{\phi \in H^1((0,j)\times\mathbb{S}^1)\,|\,\phi(0,\theta) = \phi_0(\theta)\,;\phi(j,\theta) = 0; \int_0^{2\pi}\phi(s,\theta)\, d\theta = 0 \text{ a.e. } s \in (0,j)\}$, we extend the minimizer ϕ^j by zero on (j,∞) and denote the extended function by $\hat{\phi}^j(s,\theta)$. Therefore we have

$$\hat{\phi}^j \in A = \{\phi \in \widetilde{H}^1((0,\infty)\times\mathbb{S}^1)\,|\,\phi(0,\theta) = \phi_0(\theta); \int_0^{2\pi}\phi(s,\theta)\, d\theta = 0 \text{ a.e. } s \in (0,\infty)\}.$$

As a result, we have:

$$\begin{aligned}\widetilde{E}(\hat{\phi}^j) &= \int_0^\infty \int_0^{2\pi} ((\hat{\phi}^j_s)^2 + (\hat{\phi}^j_\theta)^2)\, d\theta\, ds \\ &= \int_0^j \int_0^{2\pi} ((\phi^j_s)^2 + (\phi^j_\theta)^2)\, d\theta\, ds \\ &= \widetilde{E}_j(\phi^j).\end{aligned}$$

By letting

$$\hat{A}^j = \{\phi \in \widetilde{H}^1((0,\infty)\times\mathbb{S}^1)\,|\,\phi(0,\theta) = \phi_0(\theta); \phi(s\ge j,\theta) = 0; \int_0^{2\pi}\phi(s,\theta)\, d\theta = 0 \text{ a.e. } s \in (0,\infty)\},$$

one can easily see that for $j_1 \le j_2$, we have $\hat{A}^{j_1} \subset \hat{A}^{j_2}$.

Also, since ϕ^{j} is a minimizer of

$$\widetilde{\mathrm{E}}_{\mathrm{j}}(\phi) = \int_0^{\mathrm{j}} \int_0^{2\pi} (\phi_{\mathrm{s}}^2 + \phi_\theta^2)\,\mathrm{d}\theta\,\mathrm{ds},$$

in the admissible set $\mathrm{A}^{\mathrm{j}} = \{\phi \in \mathrm{H}^1((0,\mathrm{j})\times\mathbb{S}^1)\,|\,\phi(0,\theta) = \phi_0(\theta)\,;\phi(\mathrm{j},\theta) = 0;\int_0^{2\pi}\phi\,\mathrm{d}\theta = 0$ a.e. $\mathrm{s} \in (0,\mathrm{j})\}$, we have:

$$\widetilde{\mathrm{E}}(\hat{\phi}^{\mathrm{j}_2}) \leq \widetilde{\mathrm{E}}(\hat{\phi}^{\mathrm{j}_1}), \tag{3.3.6}$$

which implies that $\{\widetilde{\mathrm{E}}(\hat{\phi}^{\mathrm{j}_\mathrm{i}})\}_{\mathrm{i}=1}^{\infty}$ is a decreasing sequence.

On the other hand, to find an upper bound for $\{\widetilde{\mathrm{E}}(\hat{\phi}^{\mathrm{j}})\}_{\mathrm{j}=1}^{\infty}$, we choose

$$\phi^* = \begin{cases} \phi_0(\theta)\,(1-\mathrm{s}), & \mathrm{s} \in [0,1] \\ 0, & \mathrm{s} \in (1,\infty), \end{cases}$$

and consequently we have:

$$\widetilde{\mathrm{E}}(\hat{\phi}^{\mathrm{j}}) \leq \widetilde{\mathrm{E}}(\hat{\phi}^{\mathrm{j}=1}) = \widetilde{\mathrm{E}}_{\mathrm{j}}(\phi^{\mathrm{j}=1}) \leq \widetilde{\mathrm{E}}_{\mathrm{j}}(\phi^*) = \mathrm{M} < \infty, \tag{3.3.7}$$

for all $\mathrm{j} \geq 1$, which implies:

$$\|\nabla\hat{\phi}^{\mathrm{j}}\|^2_{\mathrm{L}^2((0,\infty)\times\mathbb{S}^1)} \leq \mathrm{M} < \infty. \tag{3.3.8}$$

Now we prove that $\hat{\phi}^{\mathrm{j}}$ is uniformly bounded in $\mathrm{L}^2((0,\infty)\times\mathbb{S}^1)$. Again, by Poincaré's inequality on $(0,2\pi)$, we have:

$$\|\hat{\phi}^{\mathrm{j}} - \frac{1}{2\pi}\int_0^{2\pi}\hat{\phi}^{\mathrm{j}}\|_{\mathrm{L}^2(0,2\pi)} \leq \mathrm{c}\|\hat{\phi}^{\mathrm{j}}_\theta\|_{\mathrm{L}^2(0,2\pi)} \tag{3.3.9}$$

for some constant $\mathrm{c} > 0$, which is independent of s.

Also since

$$\int_0^{2\pi}\hat{\phi}^{\mathrm{j}} = \int_0^{2\pi}\phi^{\mathrm{j}} = 0,$$

we have from (3.3.9) such that:

$$\int_0^{2\pi}(\hat{\phi}^{\mathrm{j}})^2\,\mathrm{d}\theta \leq \mathrm{c}^2\int_0^{2\pi}(\hat{\phi}^{\mathrm{j}}_\theta)^2\,\mathrm{d}\theta,$$

therefore

$$\int_0^\infty \int_0^{2\pi} (\hat{\phi}^j)^2 \, d\theta \, ds \le c^2 \int_0^\infty \int_0^{2\pi} (\hat{\phi}^j_\theta)^2 \, d\theta \, ds,$$

where $c > 0$ is some constant, and

$$\int_0^\infty \int_0^{2\pi} (\hat{\phi}^j)^2 \, d\theta \, ds \le c^2 \int_0^\infty \int_0^{2\pi} (\hat{\phi}^j_\theta)^2 \, d\theta \, ds \le \widetilde{E}(\hat{\phi}^j) \le \widetilde{E}_j(\phi^*) = M < \infty.$$

As a consequence, we have

$$\|\hat{\phi}^j\|_{L^2((0,\infty)\times\mathbb{S}^1)} \le M' < \infty, \tag{3.3.10}$$

and from (3.3.8) to (3.3.10), we can see that:

$$\|\hat{\phi}^j\|_{H^1((0,\infty)\times\mathbb{S}^1)} \le M'' < \infty, \tag{3.3.11}$$

for all $j \ge 1$. Hence by (3.3.11), there exists a weakly convergent subsequence (without loss of generality, we can still denote it by the original sequence $\{\hat{\phi}^j\}_{j=1}^\infty$) and some $\hat{\phi} \in \widetilde{H}^1((0,\infty)\times\mathbb{S}^1)$ such that

$$\hat{\phi}^j \rightharpoonup \hat{\phi},$$

weakly in $H^1((0,\infty)\times\mathbb{S}^1)$.

Also, one can see that by (3.3.6), we have

$$\widetilde{E}(\hat{\phi}) = \inf_j \widetilde{E}(\hat{\phi}^j) = \lim_{j\to\infty} \widetilde{E}(\hat{\phi}^j). \tag{3.3.12}$$

Step 3

What left is to show that $\hat{\phi}$ is in fact the minimizer to (3.3.2). Assume $\widetilde{\phi} \in A$ is the unique minimizer to the functional (3.2.3) we proved in Theorem 3.2.1. Now choose a function $\rho^{(j)}(s)$:

$$\rho^{(j)}(s) = \begin{cases} 1, & s \in [0, j] \\ 2 - \frac{s}{j}, & s \in (j, 2j], \end{cases} \tag{3.3.13}$$

then we have

$$\begin{cases} (\widetilde{\phi}\rho^{(j)})_{,\theta} = \widetilde{\phi}_\theta \rho^{(j)} \\ (\phi\rho^{(j)})_{,s} = \widetilde{\phi}_s \rho^{(j)} + \widetilde{\phi}\rho^{(j)}_s, \end{cases} \tag{3.3.14}$$

and we now show that

$$\widetilde{E}(\widetilde{\phi}\rho^{(j)}) \to \widetilde{E}(\widetilde{\phi}) \quad \text{as} \quad j \to \infty.$$

To show this, we only need to prove

$$\int_j^\infty \int_0^{2\pi} \Big(((\widetilde{\phi}_s \rho^{(j)} + \widetilde{\phi}\rho_s^{(j)}))^2 + (\rho^{(j)}\widetilde{\phi}_\theta)^2 \Big) \, d\theta \, ds \to 0, \tag{3.3.15}$$

and since $\widetilde{\phi}$ is the minimizer for

$$\widetilde{E}(\phi) = \int_0^\infty \int_0^{2\pi} (\phi_s^2 + \phi_\theta^2) \, d\theta \, ds,$$

in the admissible set $A = \{\phi \in \widetilde{H}^1((0,\infty) \times \mathbb{S}^1) \,|\, \phi(0,\theta) = \phi_0(\theta); \int_0^{2\pi} \phi \, d\theta = 0 \text{ a.e. } s \in (0,\infty)\}$, we have

$$\int_j^\infty \int_0^{2\pi} (\widetilde{\phi}_s^2 + \widetilde{\phi}_\theta^2) \, d\theta \, ds \to 0 \quad \text{as} \quad j \to \infty. \tag{3.3.16}$$

By applying Poincaré's inequality on $\widetilde{\phi}$ over $(0, 2\pi)$, we have:

$$\|\widetilde{\phi} - \frac{1}{2\pi} \int_0^{2\pi} \widetilde{\phi}\|_{L^2(0,2\pi)} \le c \|\widetilde{\phi}_\theta\|_{L^2(0,2\pi)} \tag{3.3.17}$$

for some constant $c > 0$, which is independent of s. Recall the fact that

$$\int_0^{2\pi} \widetilde{\phi}(s,\theta) \, d\theta = 0 \quad \text{a.e. } s \in (0,\infty),$$

we have from (3.3.17) that

$$\int_0^{2\pi} \widetilde{\phi}^2 \, d\theta \le c^2 \int_0^{2\pi} \widetilde{\phi}_\theta^2 \, d\theta,$$

hence

$$\int_0^\infty \int_0^{2\pi} \widetilde{\phi}^2 \, d\theta \, ds \le c^2 \int_0^\infty \int_0^{2\pi} \widetilde{\phi}_\theta^2 \, d\theta \, ds,$$

and

$$\int_j^\infty \int_0^{2\pi} \widetilde{\phi}^2 \, d\theta \, ds \to 0 \quad \text{as} \quad j \to \infty, \tag{3.3.18}$$

will hold.

As a consequence, from (3.3.16) and (3.3.18), we have

$$
\begin{aligned}
&\int_{j}^{\infty}\int_{0}^{2\pi}\Big(((\widetilde{\phi}_s\rho^{(j)}+\widetilde{\phi}\rho_s^{(j)}))^2+(\rho^{(j)}\widetilde{\phi}_\theta)^2\Big)\,d\theta\,ds\\
&\leq \int_{j}^{\infty}\int_{0}^{2\pi}\Big(2((\widetilde{\phi}_s\rho^{(j)})^2+(\widetilde{\phi}\rho_s^{(j)})^2)+(\rho^{(j)}\widetilde{\phi}_\theta)^2\Big)\,d\theta\,ds\\
&\leq C\int_{j}^{\infty}\int_{0}^{2\pi}(\widetilde{\phi}_s^2+\widetilde{\phi}_\theta^2+\widetilde{\phi}^2)\,d\theta\,ds \to 0 \quad \text{as} \quad j\to\infty,
\end{aligned}
$$

and hence (3.3.15) holds. Notice that we used the fact that $0 \leq \rho^{(j)} \leq 1$, and $\rho_s^{(j)}$ is bounded for all $j \geq 1$ in the inequality.

On the other hand, one can also see that $\widetilde{\phi}\rho^{(j)} \in A^{2j}$, therefore

$$
\inf_j \widetilde{E}(\widetilde{\phi}\rho^{(j)}) \geq \inf_j \widetilde{E}(\hat{\phi}^{2j}) \geq \widetilde{E}(\widetilde{\phi}) = \min_{\phi\in A}\int_0^{\infty}\int_0^{2\pi}(\phi_s^2+\phi_\theta^2)\,d\theta\,ds,
$$

which implies

$$
\widetilde{E}(\hat{\phi}) = \lim_{j\to\infty}\widetilde{E}(\hat{\phi}^j) = \widetilde{E}(\widetilde{\phi}),
$$

and this shows that $\hat{\phi} \in A$ is a minimizer to the functional (3.3.2). The uniqueness of minimizer and asymptotic behaviour, i.e.

$$
\lim_{s\to\infty}\hat{\phi}(s,\theta) = 0 \quad \text{in} \quad L^2(\mathbb{S}^1),
$$

follows directly from Theorem 3.2.1, hence completing our proof.

□

With all above, we have proved in Theorem 3.2.1 such that there exists a unique minimizer $\widetilde{\Phi}$ to the functional

$$
E(\Phi) = \int_1^{\infty} r\int_0^{2\pi}((\Phi_r^2+\frac{\Phi_\theta^2}{r^2})-\frac{k^2}{r^2})\,d\theta\,dr,
$$

for Φ in the admissible set

$$
A_\Phi = \{\Phi\,|\,\phi=\Phi-k\theta\in\widetilde{H}^1((1,\infty)\times\mathbb{S}^1);\ \phi(1,\theta)=\phi_0(\theta);\int_0^{2\pi}\phi(r,\theta)\,d\theta=0 \ \text{ a.e. } r\in(1,\infty)\},
$$

and the minimizer $\widetilde{\Phi}$ satisfies

$$\lim_{r\to\infty} \widetilde{\Phi}(r,\theta) = k\theta + \beta \quad \text{in } L^2(\mathbb{S}^1).$$

Then in Theorem 3.3.1, we provided some insight to the minimizer obtained in Theorem 3.2.1 through truncating $E(\Phi)$ into finite intervals and proved that there exists a unique minimizer to the truncated $E(\Phi)$ in each interval, and then by extending these minimizers on each interval we showed that the extended minimizers will tend to the unique minimizer $\widetilde{\Phi}$.

In the next chapter, we will deal with a more general case where, instead of a one-hole circular domain, the domain Ω is a N-connected domain, i.e. with N 'holes' each having a boundary vector field from the boundary to $\mathbb{S}^1$ (in $H^{1/2}$) with prescribed degree on them respectively.

Chapter 4

Minimization problem for N-connected domain (one-constant case)

In this chapter, we consider the same minimization problem but in a more general domain. Consider an N-connected domain generated by N Jordan curves, $\Omega = \mathbb{R}^2 \setminus \cup_{j=1}^{N} \bar{\omega}_j$, where each ω_j is a bounded simply-connected domain with Lipschitz boundary $\partial\omega_j$ and $\bar{\omega}_i \cap \bar{\omega}_j = \emptyset$ for $i \neq j$. In particular, each $\partial\omega_j$ is a Jordan curve. We assume that on each boundary $\partial\omega_j$, the vector field $\mathbf{n}$ is given by $\mathbf{n}|_{\partial\omega_j} = \mathbf{n}_j \in H^{\frac{1}{2}}(\partial\omega_j; \mathbb{S}^1)$ and has prescribed degree d_j , i.e. $\deg_{\partial\omega_j}\mathbf{n} = d_j$. Further, we require the origin to be contained in one of the ω_j, and without loss of generality, we can assume $0 \in \omega_1$.

Now we consider the minimization problem:

$$I(\widetilde{\mathbf{n}}) = \min_{\mathbf{n}\in A} I(\mathbf{n}) = \min_{\mathbf{n}\in A} \int_\Omega |\nabla\mathbf{n}|^2 \, dx,$$

for $\mathbf{n}$ in the admissible set

$$A = \{\mathbf{n} \in \widetilde{H}^1(\Omega, \mathbb{S}^1) \,|\, \mathbf{n}|_{\partial\omega_j} = \mathbf{n}_j, \; j = 1, ..., N\},$$

where $\widetilde{H}^1(\Omega; \mathbb{S}^1)$ is defined in Definition 2.3.1.

As discussed at the end of Section 3.1, the infimum of $I(\mathbf{n}) = \frac{1}{2}\int_\Omega |\nabla\mathbf{n}|^2 \, dx$ is usually infinite, so that instead we consider the renormalized functional

$$\hat{I}(\mathbf{n}) = \frac{1}{2}\int_\Omega \left(|\nabla\mathbf{n}|^2 - \frac{k^2}{|x|^2}\right) dx. \tag{4.0.1}$$

where $k = \Sigma_{j=1}^{N} d_j$. Note that since we will use this renormalized functional (4.0.1) instead of the original one throughout Chapter 4, we will without loss of generality omit the coefficient $\frac{1}{2}$ and still use the notation $\mathrm{I}(\mathbf{n})$ instead of using $\hat{\mathrm{I}}(\mathbf{n})$.

Unlike one-hole circular domain studied in Chapter 3, we do not have uniqueness for the minimizer in the multi-hole setting, and this can be easily illustrated by the following counterexample.

4.1 Counterexample

The following counterexample illustrates the fact that the minimizer for the functional

$$\mathrm{I}(\mathbf{n}) = \int_\Omega \left(|\nabla \mathbf{n}|^2 - \frac{\mathrm{k}^2}{|\mathrm{x}|^2} \right) \mathrm{dx},$$

for $\mathbf{n} \in \mathrm{A} = \{\mathbf{n} \in \widetilde{\mathrm{H}}^1(\Omega; \mathbb{S}^1) \,|\, \mathbf{n}|_{\partial\omega_j} = \mathbf{n}_j,\ j = 1, 2\}$ is not unique.

Example 4.1.1. *Let* $\mathrm{N} = 2$ *and* ω_1, ω_2 *be the discs of radius* $\frac{1}{4}$ *with centres* $(-1, 0)$, $(1, 0)$ *respectively. To avoid confusion, note that we do not assume* $0 \in \omega_1$ *only in this counterexample. We assume that the boundary data* $\mathbf{n}_1$, $\mathbf{n}_2$ *are given by the exterior normals to* $\partial\omega_1$, $\partial\omega_2$, *namely* $\mathbf{n}_1 = \mathbf{n}(-1 + \frac{1}{4}\cos\theta, \frac{1}{4}\sin\theta) = (\cos\theta, \sin\theta)$, $\mathbf{n}_2 = \mathbf{n}(1 + \frac{1}{4}\cos\theta, \frac{1}{4}\sin\theta) = (\cos\theta, \sin\theta)$. *Let* $\mathbf{n}$ *be a minimizer of* $\mathrm{I}(\mathbf{n})$ *in* A, *and consider its value at the origin* $\mathbf{n}(0, 0) = (\mathrm{n}_1(0, 0), \mathrm{n}_2(0, 0))$. *We will see later that* $\mathbf{n}$ *is smooth, so that* $\mathbf{n}(0, 0)$ *is well-defined. Since* $\mathbf{n}(0, 0) \in \mathbb{S}^1$ *either* $\mathrm{n}_1(0, 0) \neq 0$ *or* $\mathrm{n}_2(0, 0) \neq 0$. *Assume first that* $\mathrm{n}_1(0, 0) \neq 0$. *Figure 4.1 illustrates this situation.*

Now we can define another vector field $\widetilde{\mathbf{n}}$ *from the given vector field* $\mathbf{n}$ *such that:*

$$\widetilde{\mathbf{n}}(\mathrm{x}_1, \mathrm{x}_2) = (-\mathrm{n}_1(-\mathrm{x}_1, \mathrm{x}_2), \mathrm{n}_2(-\mathrm{x}_1, \mathrm{x}_2))\,.$$

One can easily check that this $\widetilde{\mathbf{n}}$ *preserves the boundary value of* $\mathbf{n}$ *on* $\partial\omega_1$ *and* $\partial\omega_2$, *and we have*

$$\nabla\widetilde{\mathbf{n}}(\mathrm{x}_1, \mathrm{x}_2) = \begin{pmatrix} \mathrm{n}_{1,1}(-\mathrm{x}_1, \mathrm{x}_2) & -\mathrm{n}_{1,2}(-\mathrm{x}_1, \mathrm{x}_2) \\ -\mathrm{n}_{2,1}(-\mathrm{x}_1, \mathrm{x}_2) & \mathrm{n}_{2,2}(-\mathrm{x}_1, \mathrm{x}_2) \end{pmatrix}.$$

Consequently we have:

$$|\nabla\widetilde{\mathbf{n}}(\mathrm{x}_1, \mathrm{x}_2)|^2 = |\nabla\mathbf{n}(-\mathrm{x}_1, \mathrm{x}_2)|^2,$$

and also

$$\widetilde{\mathbf{n}}(0, 0) = (-\mathrm{n}_1(0, 0), \mathrm{n}_2(0, 0)) \neq \mathbf{n}(0, 0),$$

if $\mathrm{n}_1(0, 0) \neq 0$.

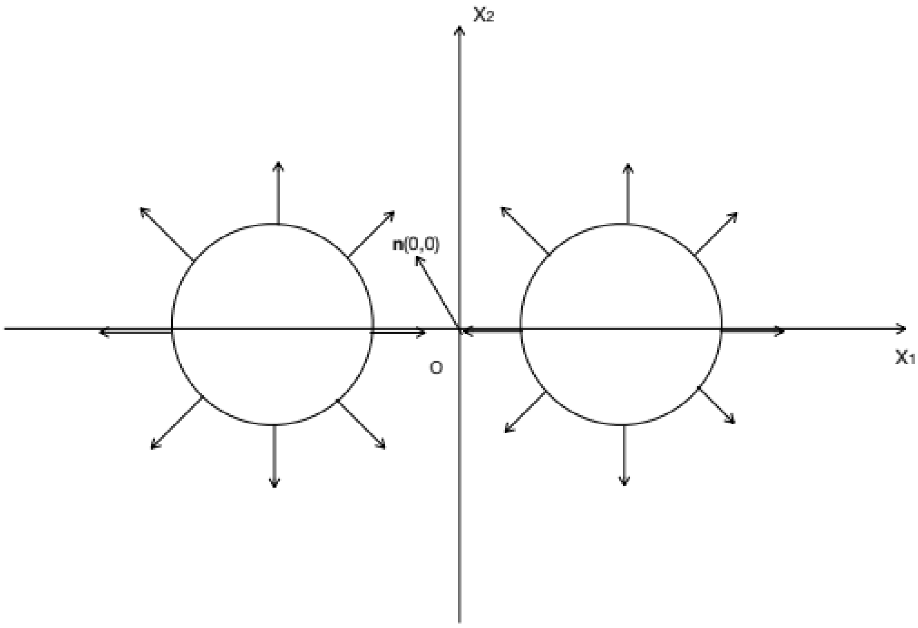

Figure 4.1: The vector field $\mathbf{n}(x_1, x_2)$ is normal on the boundary $\partial\omega_1$ and $\partial\omega_2$, which in our case are circles with radius $\frac{1}{4}$.

If, on the other hand, $\mathrm{n}_2(0,0) \neq 0$ *then we can define*

$$\widetilde{\mathbf{n}}(\mathrm{x}_1, \mathrm{x}_2) = (\mathrm{n}_1(\mathrm{x}_1, -\mathrm{x}_2), -\mathrm{n}_2(\mathrm{x}_1, -\mathrm{x}_2)) \, .$$

Again $\widetilde{\mathbf{n}}$ *preserves the boundary value of* $\mathbf{n}$ *on* $\partial\omega_1$ *and* $\partial\omega_2$*,*

$$|\nabla\widetilde{\mathbf{n}}(\mathrm{x}_1, \mathrm{x}_2)|^2 = |\nabla\mathbf{n}(\mathrm{x}_1, -\mathrm{x}_2)|^2,$$

and

$$\widetilde{\mathbf{n}} = (\mathrm{n}_1(0,0), -\mathrm{n}_2(0,0)) \neq \mathbf{n}(0,0).$$

Thus given a minimizer, we can always derive another minimizer from the given one for the same functional and this implies that there is no uniqueness in the minimization problem. Hence instead, we look for uniqueness in different homotopy classes.

Remark 4.1.2. *One may ask the question that whether this counterexample is implying the fact* *that the given minimizer has a defect in the origin, rather than implying the non-uniqueness. This is not true since one can see later in Theorem 4.3.2 that the minimizer* $\widetilde{\mathbf{n}}$ *is smooth in* Ω, *hence can not have any other defect or any pair of defects with total sum of degree being zero.*

To state our main result for this chapter, we need to define homotopy classes for our minimization problem in the next section.

4.2 Homotopy classes

With the properties established in Section 2.3, we can now define homotopy classes which play a significant role in our main result for this chapter. Then in Section 4.3, we will state and prove our main result for this chapter. Before we define homotopy classes, we need the following lemma.

Lemma 4.2.1. *Let* $\lambda \mapsto \mathbf{n}_\lambda$ *be a continuous map from* $[0,1] \to \mathrm{H}^1(\mathbb{B};\mathbb{S}^1)$, *where* $\mathbb{B} \subset \mathbb{R}^2$ *is any disc. Note that here we use the notation* $\mathbb{B}$ *to differentiate from the unit disc* $\mathbb{D}$. *Then there exists a continuous map* $\lambda \mapsto \phi_\lambda$ *from* $[0,1] \to \mathrm{H}^1(\mathbb{B})$ *such that* $\mathbf{n}_\lambda = \exp(\mathrm{i}\phi_\lambda)$.

Proof. Let $\mathbb{B}$ be any disc in $\mathbb{R}^2$ and let $\lambda \mapsto \mathbf{n}_\lambda$ be a continuous map from $[0,1]$ into $\mathrm{H}^1(\mathbb{B},\mathbb{S}^1)$.

For each λ there exists a lifting $\phi_\lambda \in \mathrm{H}^1(\mathbb{B})$ such that $\mathbf{n}_\lambda = \exp(\mathrm{i}\phi_\lambda)$, and if ϕ_λ and $\widetilde{\phi}_\lambda$ are two such liftings then we have $\phi_\lambda - \widetilde{\phi}_\lambda = 2\mathrm{m}\pi$ a.e. for some $\mathrm{m} \in \mathbb{Z}$. Hence, it follows that $\nabla\phi_\lambda$ is the same for any lifting of $\mathbf{n}_\lambda$. We prove the lemma in several steps.

Step 1

<u>Claim</u>: $\lambda \mapsto \nabla\phi_\lambda$ is continuous from $[0,1] \to \mathrm{L}^2(\mathbb{B};\mathbb{R}^2)$.

Suppose the claim is not true. Then there exists $\epsilon > 0$ and a sequence $\{\lambda_\mathrm{j}\}$ such that $\lambda_j \to \lambda$ in $[0,1]$ and

$$\int_{\mathbb{B}} |\nabla\phi_{\lambda_\mathrm{j}} - \nabla\phi_\lambda|^2 \mathrm{dx} > \epsilon, \quad \text{for all } \mathrm{j} \in \mathbb{N}.$$

Since $\int_{\mathbb{B}} |\nabla\phi_{\lambda_\mathrm{j}}|^2\,\mathrm{dx} = \int_{\mathbb{B}} |\nabla\mathbf{n}_{\lambda_\mathrm{j}}|^2\,\mathrm{dx}$, we have that $\{\nabla\phi_{\lambda_\mathrm{j}}\}_{\mathrm{j}=1}^\infty$ is bounded in $\mathrm{L}^2(\mathbb{B})$.

Without loss of generality by adding a suitable multiple of 2π to $\phi_{\lambda_\mathrm{j}}$ for each j, we can choose a lifting such that

$$⨍_{\mathbb{B}} \phi_{\lambda_\mathrm{j}}\,\mathrm{dx} := \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} \phi_{\lambda_\mathrm{j}}\,\mathrm{dx} \in [0,2\pi).$$

Thus by applying (2.3.6), we have that $\{\phi_{\lambda_\mathrm{j}}\}_{\mathrm{j}=1}^\infty$ is bounded in $\mathrm{H}^1(\mathbb{B})$ and hence there exists a subsequence of $\{\phi_{\lambda_\mathrm{j}}\}_{\mathrm{j}=1}^\infty$ (without loss of generality, we still write it as $\{\phi_{\lambda_\mathrm{j}}\}_{\mathrm{j}=1}^\infty$) converges weakly to some $\phi \in H^1(\mathbb{B})$. By the compactness of the embedding $H^1(\mathbb{B}) \hookrightarrow L^2(\mathbb{B})$, we can suppose that

$$\begin{cases} \phi_{\lambda_j} \to \phi \\ \nabla\phi_{\lambda_j} \rightharpoonup \nabla\phi \ \text{ weakly} \end{cases} \tag{4.2.1}$$

in $L^2(\mathbb{B})$ and that

$$\phi_{\lambda_j} \to \phi \quad \text{a.e.}$$

On the other hand, since $\lambda \mapsto \mathbf{n}_\lambda$ is continuous from $[0,1] \to H^1(\mathbb{B};\mathbb{S}^1)$, we know that

$$e^{i\phi_{\lambda_j}} = \mathbf{n}_{\lambda_j} \to \mathbf{n}_\lambda = e^{i\phi_\lambda} \quad \text{in} \quad H^1(\mathbb{B};\mathbb{S}^1),$$

and therefore ϕ is a lifting for $\mathbf{n}_\lambda$ and $\phi = \phi_\lambda + 2m\pi$ for some integer m.

Recall the fact that

$$\int_{\mathbb{B}} |\nabla\phi_{\lambda_j}|^2\,dx = \int_{\mathbb{B}} |\nabla\mathbf{n}_{\lambda_j}|^2\,dx \to \int_{\mathbb{B}} |\nabla\mathbf{n}_\lambda|^2\,dx = \int_{\mathbb{B}} |\nabla\phi_\lambda|^2\,dx.$$

Hence, by the weak convergence of $\nabla\phi_j$ we have that $\lim_{j\to\infty}\int_{\mathbb{B}} |\nabla\phi_{\lambda_j} - \nabla\phi_\lambda|^2\,dx = 0$, which is a contradiction to $\int_{\mathbb{B}} |\nabla\phi_{\lambda_j} - \nabla\phi_\lambda|^2 dx > \epsilon$.

Step 2

Claim: $\lambda \mapsto \exp(i\fint_{\mathbb{B}} \phi_\lambda\,dx)$ is continuous.

Here we note the fact that $\exp(i\fint_{\mathbb{B}} \phi_\lambda\,dx)$ is independent of the choice of lifting since liftings only differ by a multiple of 2π.

Assume for contradiction that $\lambda_j \to \lambda$ but $|\exp(i\fint_{\mathbb{B}} \phi_{\lambda_j}\,dx) - \exp(i\fint_{\mathbb{B}} \phi_\lambda\,dx)| > \epsilon$. Arguing as in Step 1 we can assume that $\fint_{\mathbb{B}} \phi_{\lambda_j}\,dx \to \fint_{\mathbb{B}} \phi_\lambda\,dx$, giving a contradiction.

Step 3

Since by Step2, $\lambda \mapsto e^{i\fint_{\mathbb{B}} \phi_\lambda\,dx}$ is continuous and since $\mathbb{R}$ covers $\mathbb{S}^1$, there is a continuous map $\xi : [0,1] \to \mathbb{R}$ such that

$$\exp(i\xi(\lambda)) = \exp\left(i\fint_{\mathbb{B}} \phi_\lambda\,dx\right).$$

Now for each λ, $\fint_{\mathbb{B}} \phi_\lambda\,dx = \xi(\lambda) + 2m(\lambda)\pi$ for some integer $m(\lambda)$. Hence $\fint_{\mathbb{B}} (\phi_\lambda - 2m(\lambda)\pi)\,dx = \xi(\lambda)$, so that $\widetilde{\phi}_\lambda = \phi_\lambda - 2m(\lambda)\pi$ is a lifting of $\mathbf{n}_\lambda$.

Then if $\lambda_j \to \lambda$ by the Poincaré inequality

$$\int_\Omega |\widetilde{\phi}_{\lambda_j} - \widetilde{\phi}_\lambda|^2\,dx \le C\left(|\fint_{\mathbb{B}} \widetilde{\phi}_{\lambda_j}\,dx - \fint_{\mathbb{B}} \widetilde{\phi}_\lambda\,dx|^2 + \fint_{\mathbb{B}} |\nabla\phi_{\lambda_j} - \nabla\phi_\lambda|^2\,dx\right)$$

which tends to zero as the first term on the right-hand side of this inequality equals $|\xi(\lambda_j)-\xi(\lambda)|^2$.

The continuity of $\lambda \mapsto \widetilde{\phi}_\lambda$ from $[0,1] \to H^1(\mathbb{B})$ now follows from Step 1, completing the proof.

□

Note that this result not just apply to the disc but to any simply-connected domain in Ω with sufficiently smooth boundary.

Now with the help of Lemma 4.2.1, we can define the homotopy classes for the given admissible set $A = \{\mathbf{n} \in \widetilde{H}^1(\Omega;\mathbb{S}^1) \,|\, \mathbf{n}|_{\partial\omega_j} = \mathbf{n}_j \ j = 1,...,N\}$.

Definition 4.2.2. *Assume* $\mathbf{n}$ *and* $\mathbf{n}'$ *are two given vector fields in the admissible set* A. $\mathbf{n}$ *and* $\mathbf{n}'$ *are homotopic if there exists a mapping* $\mathbf{n}(\lambda, x)$ *such that* $\mathbf{n}(\lambda,\cdot) : [0,1] \times \Omega \mapsto A$ *is continuous with respect to the norm topology of* $H^1(\Omega \cap \mathbb{B}_R)$, *for any* R *sufficiently large, where* $\mathbf{n}(0,x) = \mathbf{n}(x)$ *and* $\mathbf{n}(1,x) = \mathbf{n}'(x)$. *The vector fields in* A *that are homotopic to each other are said to be in the same homotopy class.*

Thus the admissible set A is the union of all the homotopy classes, and each homotopy class consists of vector fields that are homotopic to each other. The following theorem describes which vector fields are contained in each homotopy class. Without loss of generality, we choose $a_1 \in \omega_1$ to be the origin and assume that for each point a_j, it satisfies $a_j \in \omega_j$ for all $j \in \{2,...,N\}$.

Proposition 4.2.3. *The admissible set* $A = \{\mathbf{n} \in \widetilde{H}^1(\Omega;\mathbb{S}^1) \,|\, \mathbf{n}|_{\partial\omega_j} = \mathbf{n}_j, \ j = 1,...,N\}$ *splits into the distinct homotopy classes* $\{C_\mathbf{p}\}$, *where* $\mathbf{p} = \{p_2, p_3, ..., p_N\} \in P \cong \underbrace{\mathbb{Z}\times\cdots\times\mathbb{Z}}_{\text{N-1}} = \mathbb{Z}^{N-1}$. *Each homotopy class* $C_\mathbf{p}$ *depends on the lifting of* $\mathbf{n}_j$ *on each of the boundaries* $\partial\omega_j$, *and each* $C_\mathbf{p}$ *consists of vector field* $\mathbf{n} = \Pi_{j=1}^N h_j(z)^{d_j}\exp(i\phi)$, *where for each* $j = 1,...,N$, *we have* $h_j(z) = \frac{z-a_j}{|z-a_j|}$ *and* ϕ *is in the set:*

$$A_\phi^\mathbf{p} = \{\phi \in \widetilde{H}^1(\Omega) \,|\, \phi|_{\partial\omega_1} = \phi_1; \phi|_{\partial\omega_j} = \phi_j + 2p_j\pi, \ j = 2,...,N\}, \tag{4.2.2}$$

and ϕ_j *satisfies* $\exp(i\phi_j) = \Pi_{i=1}^N h_i^{-d_i}(z)\,\mathbf{n}_j$ *(see (2.3.3) in Proposition 2.3.2), for all* $j \in \{1,2,...,N\}$, *where without loss of generality, we assume that each* ϕ_j *satisfies* $\fint_{\partial\omega_j}\phi_j\,dx \in [0,2\pi)$ *for* $j = 1,...,N$.

Proof. Given $A = \{\mathbf{n} \in \widetilde{H}^1(\Omega;\mathbb{S}^1) \,|\, \mathbf{n}|_{\partial\omega_j} = \mathbf{n}_j, \ j = 1,...,N\}$, the corresponding admissible set $A_\mathbf{m}$ for $\mathbf{m} = \Pi_{j=1}^N h(z)^{-d_j}\mathbf{n}$ by

$$A_\mathbf{m} = \{\mathbf{m} \in \widetilde{H}^1(\Omega;\mathbb{S}^1) \,|\, \mathbf{m}|_{\partial\omega_j} = \mathbf{m}_j, \ j = 1,...,N\}.$$

By extending each $\mathbf{m} \in \mathrm{A}_{\mathbf{m}}$ within each ω_j as described in Proposition 2.3.2 we can define a map $\sigma : \mathbf{m} \mapsto \widetilde{\mathbf{m}}$ and hence the corresponding admissible set

$$\widetilde{\mathrm{A}}_{\mathbf{m}} = \{\widetilde{\mathbf{m}} \,|\, \widetilde{\mathbf{m}} = \sigma\mathbf{m},\ \mathbf{m} \in \mathrm{A}_{\mathbf{m}}\}. \tag{4.2.3}$$

Also from (2.3.3) in Proposition 2.3.2, we can write $\mathbf{m}$ on each boundary ω_j in the form $\mathbf{m}|_{\partial\omega_j} = \exp i(\phi_j + 2p_j\pi)$, for $j = 1, ..., N$, where $\{p_j\}_{j=1}^N$ are arbitrary integers, and without loss of generality, we can assume that $p_1 = 0$, and $⨍_{\partial\omega_j} \phi_j \, dx \in [0, 2\pi)$ for $j = 1, ..., N$. Hence we can obtain infinitely many sets with respect to ϕ given by

$$\mathrm{A}_\phi^{\mathbf{p}} = \{\phi \in \widetilde{\mathrm{H}}^1(\Omega) \,|\, \phi|_{\partial\omega_1} = \phi_1; \phi|_{\partial\omega_j} = \phi_j + 2p_j\pi,\ \text{ for } j = 2, ..., N\}.$$

Similar to (4.2.3), we define another map $\varsigma : \phi \mapsto \widetilde{\phi}$ by extending $\phi \in \mathrm{A}_\phi^{\mathbf{p}}$ into each ω_j as in (2.3.4), hence obtaining a corresponding $\widetilde{\mathrm{A}}_\phi^{\mathbf{p}}$,

$$\widetilde{\mathrm{A}}_\phi^{\mathbf{p}} = \{\widetilde{\phi} \in \widetilde{\mathrm{H}}^1(\mathbb{R}^2) \,|\, \widetilde{\phi} = \varsigma\phi,\ \phi \in \mathrm{A}_\phi^{\mathbf{p}}\}. \tag{4.2.4}$$

Next we show that for each fixed $\mathbf{p} = \{p_2, p_3, ..., p_N\} \in \mathrm{P} \cong \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{\text{N-1}} = \mathbb{Z}^{N-1}$, the vector fields $\mathbf{n} = \Pi_{j=1}^N h_j^{d_j}(z) \exp i\phi$, where $\phi \in \mathrm{A}_\phi^{\mathbf{p}}$, form a homotopy class.

It is obvious that $\mathbf{n}$ and $\mathbf{n}'$ are homotopic if they belong to the same $\mathrm{C}_{\mathbf{p}}$. This is because if $\phi, \phi' \in \mathrm{A}_\phi^{\mathbf{p}}$, we can choose $\mathbf{n}(\lambda, x) := \Pi_{j=1}^N h_j^{d_j}(z) \exp i\phi_\lambda$, where $\phi_\lambda := (1 - \lambda)\phi + \lambda\phi' \in \mathrm{A}_\phi^{\mathbf{p}}$, then $\mathbf{n}(\lambda, x)$ is continuous from $[0, 1] \times \Omega$ to A and $\mathbf{n}(\lambda, x)$ satisfies $\mathbf{n}(0, x) = \mathbf{n}$, $\mathbf{n}(1, x) = \mathbf{n}'$.

On the other hand, we also need to show that, if $\mathbf{n} = \Pi_{j=1}^N h_j^{d_j}(z) \exp(i\phi)$, $\mathbf{n}' = \Pi_{j=1}^N h_j^{d_j}(z) \exp(i\phi')$ are homotopic in $\widetilde{\mathrm{H}}^1(\Omega)$, then if $\phi \in \mathrm{A}_\phi^{\mathbf{p}}$ and $\phi' \in \mathrm{A}_\phi^{\mathbf{p}'}$, we must have $\mathbf{p} = \mathbf{p}'$. Suppose $\mathbf{n}, \mathbf{n}' \in \mathrm{A}$ are homotopic in A then we must have that $\mathbf{m} = \Pi_{j=1}^N h(z)^{-d_j}\mathbf{n}$ and $\mathbf{m}' = \Pi_{j=1}^N h(z)^{-d_j}\mathbf{n}'$ are homotopic in $\mathrm{A}_{\mathbf{m}}$, since $\Pi_{j=1}^N h(z)^{-d_j}$ is bounded and analytic in $\Omega \cap \mathbb{B}_R$ for any R sufficiently large. By extending $\mathbf{m}$ and $\mathbf{m}'$ using the map σ, we have two extended vector fields $\widetilde{\mathbf{m}} = \sigma\mathbf{m}$, $\widetilde{\mathbf{m}'} = \sigma\mathbf{m}'$, which are homotopic in $\mathrm{H}^1(\mathbb{B}_R; \mathbb{S}^1)$ for any sufficiently large R. Recall from Definition 4.2.2, $\widetilde{\mathbf{m}}$ and $\widetilde{\mathbf{m}'}$ are homotopic in $\mathrm{H}^1(\mathbb{B}_R; \mathbb{S}^1)$ if there exists a continuous mapping $\lambda \mapsto \widetilde{\mathbf{m}}_\lambda$ which is continuous from $[0, 1]$ to $\widetilde{\mathrm{H}}^1(\Omega; \mathbb{S}^1)$ such that $\widetilde{\mathbf{m}}_{\lambda=0} = \widetilde{\mathbf{m}}$ and $\widetilde{\mathbf{m}}_{\lambda=1} = \widetilde{\mathbf{m}'}$. Therefore by Lemma 4.2.1 we have that, there exists a continuous map $\lambda \mapsto \widetilde{\phi}_\lambda$ from $[0, 1] \mapsto \mathbb{H}^1(\mathbb{B}_R)$ such that $\widetilde{\mathbf{m}}_\lambda = \exp(i\widetilde{\phi_\lambda})$.

As a consequence, on each boundary $\partial\omega_j$, $j = 1, ..., N$, we must have

$$||\widetilde{\phi}_\lambda||^2_{L^2(\partial\omega_j)} \leq C_j||\widetilde{\phi}_\lambda||^2_{H^1(\omega_j)} \leq C_j||\widetilde{\phi}_\lambda||^2_{H^1(\mathbb{B}_R)},$$

where C_j is some constant independent on $\widetilde{\phi}_\lambda$. Consequently $\lambda \mapsto \widetilde{\phi}_\lambda$ is continuous from $[0, 1] \mapsto L^2(\partial\omega_j)$.

Since we have assumed without loss of generality such that $\int_{\partial\omega_j} \phi_j \, dx \in [0, 2\pi)$ for each $j = 1, ..., N$, we can easily see that on each boundary $\partial\omega_j$, $\widetilde{\phi}$ in the same homotopy class can not have boundary value differs by integral multiple of 2π. Otherwise $||\widetilde{\phi}_\lambda||^2_{L^2(\partial\omega_j)}$ will have a jump, which contradicts with the fact that $\lambda \mapsto \widetilde{\phi}_\lambda$ is continuous from $[0, 1] \mapsto L^2(\partial\omega_j)$. This implies that if $\mathbf{p} \neq \mathbf{p}'$, the corresponding $\mathbf{n} = \Pi_{j=1}^N h(z)^{d_j} \exp(i\phi)$, where $\phi \in A_\phi^{\mathbf{p}}$ are not homotopic, hence belong to different homotopy classes, completing the proof.

□

With the definition of homotopy classes for a given admissible set $A = \{\mathbf{n} \in \widetilde{H}^1(\Omega; \mathbb{S}^1) \,|\, \mathbf{n}|_{\partial\omega_j} = \mathbf{n}_j,\ j = 1, ..., N\}$ given by Proposition 4.2.3, we will be able to state and prove our main result for the minimization problem on this N-hole domain Ω in the next section. We will derive some preliminary estimate before the main result.

By Proposition 2.3.6 we have

$$\mathbf{n} = \Pi_{j=1}^N h_j(z)^{d_j} \exp(i\phi),$$

where $h_j(z) = \frac{z-a_j}{|z-a_j|}$, and a_j is some arbitrary point within ω_j, i.e. $a_j \in \omega_j$, for $j = 1, ..., N$. Now we define $\alpha_j = \alpha_j(r, \theta)$ so that

$$e^{i\alpha_j(r,\theta)} = e^{-i\theta} h_j(z), \tag{4.2.5}$$

for all $j = 1, ..., N$.

Since $h_j(z) = \frac{z-a_j}{|z-a_j|}$, we have

$$h_j(z) = \frac{z - a_j}{|z - a_j|} = \frac{re^{i\theta} - a_j}{|re^{i\theta} - a_j|} = \frac{e^{i\theta} - \frac{a_j}{r}}{|e^{i\theta} - \frac{a_j}{r}|} := e^{i(\theta+\alpha_j)}. \tag{4.2.6}$$

On the other hand, if we write $a_j = \rho_j e^{i\gamma_j}$, we will have

$$
\begin{aligned}
|e^{i\theta} - \frac{a}{r}|^2 &= (1 - \frac{ae^{-i\theta}}{r})(1 - \frac{\bar{a}e^{i\theta}}{r}) = 1 + \frac{|a|^2}{r^2} - \frac{1}{r}\Big(ae^{-i\theta} + \bar{a}e^{i\theta}\Big) \\
&= 1 + \frac{\rho_j^2}{r^2} - \frac{\rho_j}{r}\Big(e^{i(\gamma_j-\theta)} + e^{i(\theta-\gamma_j)}\Big) \\
&= 1 + \frac{\rho_j^2}{r^2} - \frac{2\rho_j}{r}\cos(\theta - \gamma_j).
\end{aligned}
$$

By substituting this back into (4.2.6) we have

$$
e^{i\theta} - \frac{\rho_j e^{i\gamma_j}}{r} = \Big(1 + \frac{\rho_j^2}{r^2} - \frac{2\rho_j}{r}\cos(\theta - \gamma_j)\Big)^{1/2} e^{i(\theta+\alpha_j)},
$$

hence

$$
e^{i\alpha_j} = \frac{1 - \frac{\rho_j}{r}e^{i(\gamma_j-\theta)}}{\Big(1 + \frac{\rho_j^2}{r^2} - \frac{2\rho_j}{r}\cos(\theta - \gamma_j)\Big)^{1/2}}.
$$

Also since $z = r\,e^{i\theta}$, we have

$$
\begin{cases}
\nabla\theta = \dfrac{ie^{i\theta}}{r} \\
\nabla r = \dfrac{z}{r} = e^{i\theta},
\end{cases}
\tag{4.2.7}
$$

where $\nabla = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}$. As a result we have

$$
\begin{aligned}
\nabla e^{i\alpha_j} &= ie^{i\alpha_j}\nabla\alpha_j \\
&= \frac{1}{\Big(1 + \frac{\rho_j^2}{r^2} - \frac{2\rho_j}{r}\cos(\theta - \gamma_j)\Big)}\Bigg(- \nabla(\frac{\rho_j}{r}e^{i(\gamma_j-\theta))})\Big(1 + \frac{\rho_j^2}{r^2} - \frac{2\rho_j}{r}\cos(\theta - \gamma_j)\Big)^{1/2} \\
&\quad - (1 - \frac{\rho_j}{r}e^{i(\gamma_j-\theta)})\nabla\Big(1 + \frac{\rho_j^2}{r^2} - \frac{2\rho_j}{r}\cos(\theta - \gamma_j)\Big)^{1/2}\Bigg) \\
&= -\frac{1}{2}\frac{1 - \frac{\rho_j}{r}e^{i(\gamma_j-\theta)}}{\Big(1 + \frac{\rho_j^2}{r^2} - \frac{2\rho_j}{r}\cos(\theta - \gamma_j)\Big)^{3/2}}\Big(- 2\frac{\rho_j^2}{r^3}e^{i\theta} + 2\frac{\rho_j}{r^2}e^{i\theta}\cos(\theta - \gamma_j) + 2\frac{\rho_j}{r}\sin(\theta - \gamma_j)\frac{ie^{i\theta}}{r}\Big).
\end{aligned}
\tag{4.2.8}
$$

Therefore one can easily see that the order of $|\nabla\alpha_j|$ with respect to $1/r$ is:

$$
|\nabla\alpha_j| \sim O(1/r^2),
\tag{4.2.9}
$$

i.e. $r^2\,|\nabla\alpha_j|$ is bounded.

Therefore from all the above, $H(z) = \Pi_{j=1}^{N} h_j(z)^{d_j}$ can be written as:

$$
H(z) = \Pi_{j=1}^{N} h_j(z)^{d_j} = \exp i(k\theta + \alpha),
$$

where

$$\begin{cases} k = \Sigma_{j=1}^{N} d_j \\ \alpha = \Sigma_{j=1}^{N} d_j\, \alpha_j. \end{cases} \tag{4.2.10}$$

One can see that we have

$$e^{i\alpha} = e^{-ik\theta}\, \Pi_{j=1}^{N} \frac{(z-a_j)^{d_j}}{|z-a_j|^{d_j}}, \tag{4.2.11}$$

and

$$\mathbf{n} = \Pi_{j=1}^{N} h_j(z)^{d_j} \exp(i\phi) = H(z) \exp(i\phi) = \exp i(k\theta + \alpha + \phi). \tag{4.2.12}$$

Also, we have the following result for $\alpha(r,\theta)$.

Lemma 4.2.4. *For $\alpha(r,\theta) = \Sigma_{j=1}^{N} d_j\, \alpha_j(r,\theta)$ defined in (4.2.10), we have*

$$\int_{\mathbb{S}^1} \big(\alpha_{,2}(R,\theta)\cos\theta - \alpha_{,1}(R,\theta)\sin\theta\big)\, d\theta = 0, \tag{4.2.13}$$

where the path of integral is in fact taken on the circle $\partial\mathbb{B}_R$, and the radius R is sufficiently large so that $\cup_{j=1}^{N} \bar{\omega}_j \subset \mathbb{B}_R$.

Proof. Since

$$e^{i\alpha_j} = e^{-i\theta} h_j(z) = \frac{|z|}{z}\frac{(z-a_j)}{|z-a_j|} = \frac{(1-\frac{a_j}{z})}{|1-\frac{a_j}{z}|}, \quad j = 1, ..., N$$

we have by (4.2.11):

$$e^{i\alpha} = \Pi_{j=1}^{N} \frac{(1-\frac{a_j}{z})^{d_j}}{|1-\frac{a_j}{z}|^{d_j}} = e^{-ik\theta}\Pi_{j=1}^{N} \frac{(z-a_j)^{d_j}}{|z-a_j|^{d_j}}.$$

On the other hand, we have

$$\begin{aligned} \int_{\mathbb{S}^1} \big(\alpha_{,2}(R,\theta)\cos\theta - \alpha_{,1}(R,\theta)\sin\theta\big)\, d\theta &= \frac{1}{2}\int_{\mathbb{S}^1} \Big(\langle \nabla\alpha, de^{i\theta}\rangle + \langle de^{i\theta}, \nabla\alpha\rangle\Big) \\ &= \mathrm{Re} \int_{\mathbb{S}^1} \langle \nabla\alpha, ie^{i\theta}\rangle\, d\theta \\ &= -\frac{1}{i}\mathrm{Re}\int_{\mathbb{S}^1} e^{-i\alpha}\nabla e^{i\alpha}\, de^{-i\theta} \\ &= \mathrm{Re}\frac{1}{i}\int_{\mathbb{S}^1} \frac{1}{e^{i\alpha}}\frac{\partial e^{i\alpha}}{\partial\theta}(R,\theta)\, d\theta \\ &= 2\pi\, \deg_{\partial\mathbb{B}_R} e^{i\alpha}. \end{aligned} \tag{4.2.14}$$

By applying Proposition 2.2.4 and Proposition 2.2.5 to (4.2.11) on the circle $\partial\mathbb{B}_R$, we have

$$\frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{1}{e^{i\alpha}}\frac{\partial e^{i\alpha}}{\partial\theta}(R,\theta)\, d\theta = -k + \Sigma_{j=1}^{N} d_j = 0, \tag{4.2.15}$$

hence completing our proof. □

From (4.2.12) we have

$$
\begin{aligned}
|\nabla \mathbf{n}|^2 &= |\nabla(k\theta + \alpha + \phi)|^2 \\
&= |k\frac{ie^{i\theta}}{r} + \nabla\alpha + \nabla\phi|^2 \\
&= \frac{k^2}{r^2} + |\nabla\alpha + \nabla\phi|^2 + \frac{k}{r}\langle ie^{i\theta}, \nabla\alpha + \nabla\phi\rangle + \frac{k}{r}\langle \nabla\alpha + \nabla\phi, ie^{i\theta}\rangle \\
&= \frac{k^2}{r^2} + |\nabla\alpha + \nabla\phi|^2 + \frac{k}{r}\Big(ie^{i\theta}(\phi_{,1} - i\phi_{,2}) - ie^{-i\theta}(\phi_{,1} + i\phi_{,2})\Big) \\
&\quad + \frac{k}{r}\Big(ie^{i\theta}(\alpha_{,1} - i\alpha_{,2}) - ie^{-i\theta}(\alpha_{,1} + i\alpha_{,2})\Big) \\
&= \frac{k^2}{r^2} + |\nabla\alpha + \nabla\phi|^2 + \frac{2k}{r}(\phi_{,2}\cos\theta - \phi_{,1}\sin\theta) + \frac{2k}{r}(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta),
\end{aligned}
$$

therefore

$$
|\nabla \mathbf{n}|^2 - \frac{k^2}{r^2} = |\nabla\alpha + \nabla\phi|^2 + \frac{2k}{r}(\phi_{,2}\cos\theta - \phi_{,1}\sin\theta) + \frac{2k}{r}(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta), \tag{4.2.16}
$$

and for any fixed R, we have

$$
\begin{aligned}
I(\mathbf{n}) &= \int_\Omega \left(|\nabla \mathbf{n}|^2 - \frac{k^2}{r^2}\right) dx \\
&= \int_\Omega \left(|\nabla\phi + \nabla\alpha|^2 + \frac{2k}{r}(\phi_{,2}\cos\theta - \phi_{,1}\sin\theta) + \frac{2k}{r}(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta)\right) dx \\
&= \int_\Omega |\nabla\phi + \nabla\alpha|^2\, dx + \int_{\Omega\cap\mathbb{B}_R} \left(\frac{2k}{r}(\phi_{,2}\cos\theta - \phi_{,1}\sin\theta) + \frac{2k}{r}(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta)\right) dx \\
&\quad + \int_{\mathbb{B}_R^c} \left(\frac{2k}{r}(\phi_{,2}\cos\theta - \phi_{,1}\sin\theta) + \frac{2k}{r}(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta)\right) dx \\
&= \int_\Omega |\nabla\phi + \nabla\alpha|^2\, dx + \int_{\Omega\cap\mathbb{B}_R} (2k(\phi_{,2}\cos\theta - \phi_{,1}\sin\theta) + 2k(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta))\, dr\, d\theta \\
&\quad + \int_{\mathbb{B}_R^c} (2k(\phi_{,2}\cos\theta - \phi_{,1}\sin\theta) + 2k(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta))\, dr\, d\theta \\
&= \int_\Omega |\nabla\phi + \nabla\alpha|^2\, dx \\
&\quad + \int_{\Omega\cap\mathbb{B}_R} (2k(\phi_{,2}\cos\theta - \phi_{,1}\sin\theta) + 2k(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta))\, dr\, d\theta,
\end{aligned} \tag{4.2.17}
$$

where the last step holds since we have (4.2.13), and also by Proposition 2.3.5:

$$
\int_{\partial\mathbb{B}_r} (\phi_{,2}\cos\theta - \phi_{,1}\sin\theta)\, d\gamma = 0, \tag{4.2.18}
$$

where the integral (4.2.18) is taken over the boundary of $\mathbb{B}_r$ for $r \geq R$.

Therefore by (4.2.9) and (4.2.17) we have,

$$
\begin{aligned}
\int_\Omega \Big(|\nabla \mathbf{n}|^2 - \frac{k^2}{r^2}\Big)\,dx &= \int_\Omega \Big(|\nabla\phi|^2 + |\nabla\alpha|^2 + 2\phi_{,1}\alpha_{,1} + 2\phi_{,2}\alpha_{,2}\Big)\,dx \\
&\quad + \int_{\Omega\cap\mathbb{B}_R} (2k(\phi_{,2}\cos\theta - \phi_{,1}\sin\theta) + 2k(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta))\,dr\,d\theta \\
&\leq \int_\Omega \Big(|\nabla\phi|^2 + |\nabla\alpha|^2 + 2|\nabla\phi|\,|\nabla\alpha|\Big)\,dx + \int_{\Omega\cap\mathbb{B}_R} \frac{2k}{r}(|\nabla\phi| + |\nabla\alpha|)\,dx \\
&\leq 2\int_\Omega \Big(|\nabla\phi|^2 + |\nabla\alpha|^2\Big)\,dx + \int_{\Omega\cap\mathbb{B}_R} \frac{2k}{r}|\nabla\phi|\,dx + \int_{\Omega\cap\mathbb{B}_R} \frac{2k}{r}|\nabla\alpha|\,dx \\
&\leq \int_\Omega \Big(3|\nabla\phi|^2 + 2|\nabla\alpha|^2\Big)\,dx + \int_{\Omega\cap\mathbb{B}_R} \Big(\frac{k^2}{r^2} + \frac{2k}{r}|\nabla\alpha|\Big)\,dx \\
&= 3\int_\Omega \Big(|\nabla\phi|^2 + O(1/r^4)\Big)\,dx + C_R,
\end{aligned} \tag{4.2.19}
$$

where C_R is some fixed constant depending only on R, as the second integral is taken over a bounded domain $\Omega \cap \mathbb{B}_R$. On the other hand, we also have

$$
\begin{aligned}
\int_\Omega \Big(|\nabla \mathbf{n}|^2 - \frac{k^2}{r^2}\Big)\,dx &= \int_\Omega \Big(|\nabla\phi|^2 + |\nabla\alpha|^2 + 2\phi_{,1}\alpha_{,1} + 2\phi_{,2}\alpha_{,2}\Big)\,dx \\
&\quad + \int_{\Omega\cap\mathbb{B}_R} (2k(\phi_{,2}\cos\theta - \phi_{,1}\sin\theta) + 2k(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta))\,dr\,d\theta \\
&\geq \int_\Omega \Big(|\nabla\phi|^2 + |\nabla\alpha|^2 - 2|\nabla\phi|\,|\nabla\alpha|\Big)\,dx - \int_{\Omega\cap\mathbb{B}_R} \frac{2k}{r}(|\nabla\alpha| + |\nabla\phi|)\,dx \\
&\geq \int_\Omega \Big((1-\epsilon)|\nabla\phi|^2 + (1-\frac{1}{\epsilon})|\nabla\alpha|^2\Big)\,dx - \int_{\Omega\cap\mathbb{B}_R} \Big(\frac{2k}{r}|\nabla\alpha| + \epsilon|\nabla\phi|^2 + \frac{1}{\epsilon}\frac{k^2}{r^2}\Big)\,dx \\
&\geq \int_\Omega \Big((1-2\epsilon)|\nabla\phi|^2 - |O(1/r^4)|\Big)\,dx - C'_R
\end{aligned} \tag{4.2.20}
$$

where $\epsilon > 0$ is sufficiently small such that $1 - 2\epsilon > 0$, and C'_R is some fixed constant depending only on R. The inequality holds due to the fact that:

$$
2|\nabla\phi|\,|\nabla\alpha| = 2\sqrt{\epsilon}|\nabla\phi|\,\frac{1}{\sqrt{\epsilon}}|\nabla\alpha| \leq \epsilon|\nabla\phi|^2 + \frac{1}{\epsilon}|\nabla\alpha|^2,
$$

and

$$
\frac{2k}{r}|\nabla\phi| = 2\sqrt{\epsilon}\frac{k}{r}\,\frac{1}{\sqrt{\epsilon}}|\nabla\phi| \leq \epsilon|\nabla\phi|^2 + \frac{1}{\epsilon}\frac{k^2}{r^2}.
$$

We will use (4.2.19) and (4.2.20) in the proof of the main theorem in the next section.

4.3 Main result for N-connected domain (one-constant case)

In this section, we will state and prove our main result for the minimization problem on the N-hole domain $\Omega = \mathbb{R}^2 \setminus \cup_{j=1}^{N} \bar{\omega}_j$. Same as before, we assume that the vector field $\mathbf{n}$ is given by $\mathbf{n}|_{\partial\omega_j} = \mathbf{n}_j \in H^{\frac{1}{2}}(\partial\omega_j; \mathbb{S}^1)$ on each boundary $\partial\omega_j$ and has prescribed degree $\deg_{\partial\omega_j}\mathbf{n} = d_j$. We also assume $a_1 \in \omega_1$ is the origin and $a_j \in \omega_j$ for all $j \in \{2, ..., N\}$. Before we start, we show the relation between the weak equilibrium solutions (also called harmonic maps) to the Euler-Lagrange equation for the given functional $\mathbf{I}(\mathbf{n})$ and the corresponding solutions to the weak form of Euler-Lagrange equations (see (4.3.1) below) for the associated ϕ given by (2.3.9) in Proposition 2.3.6.

The weak form of the Euler-Lagrange equation for $\mathbf{I}(\mathbf{n}) = \int_\Omega \left(|\nabla\mathbf{n}|^2 - \frac{k^2}{|x|^2}\right) dx$ is

$$\int_\Omega \nabla\mathbf{n} \cdot \nabla\mathbf{m} = \int_\Omega |\nabla\mathbf{n}|^2 \, \mathbf{n} \cdot \mathbf{m} \, dx \quad \forall\mathbf{m} \in C_0^\infty(\Omega; \mathbb{R}^2). \tag{4.3.1}$$

We know that by (4.2.12) we can write

$$\mathbf{n} = n_1 + i\, n_2 = H(z)\, e^{i\phi},$$

where $H(z) = \Pi_{j=1}^{N} h_j(z)^{d_j} \in C^\infty(\bar{\Omega}; \mathbb{C})$, when restricted to any ball $\mathbb{B} \subset \Omega$ has the form $H(z) = e^{iw}$, with $\Delta w = 0$ ($w = k\theta + \alpha$). We have the following proposition.

Proposition 4.3.1. $\mathbf{n}$ *satisfies (4.3.1) if and only if* ϕ *satisfies*

$$\int_\Omega \nabla\phi \cdot \nabla\Psi \, dx = 0 \quad \forall\Psi \in C_0^\infty(\Omega). \tag{4.3.2}$$

Proof. Let $\mathbb{B} \subset \Omega$ be a ball and $\Psi \in C_0^\infty(\mathbb{B})$. If $\mathbf{n}$ satisfies (4.3.1) then by approximation we have that

$$\int_\Omega \nabla\mathbf{n} \cdot \nabla\mathbf{m} = \int_\Omega |\nabla\mathbf{n}|^2 \, \mathbf{n} \cdot \mathbf{m} \, dx \quad \forall\mathbf{m} \in H_0^1(\mathbb{B}; \mathbb{R}^2). \tag{4.3.3}$$

Choose $\mathbf{m} = \mathbf{n}^\perp\Psi$, where $\mathbf{n}^\perp = \begin{pmatrix} -n_2 \\ n_1 \end{pmatrix}$, then we have

$$\int_{\mathbb{B}} \nabla\mathbf{n} \cdot \nabla(\mathbf{n}^\perp\Psi) \, dx = 0. \tag{4.3.4}$$

Conversely, if (4.3.4) holds for all $\Psi \in H^1_0(\mathbb{B})$ then if $\mathbf{m} \in H^1_0(\mathbb{B})$ we have $\mathbf{m} - (\mathbf{m}\cdot\mathbf{n})\mathbf{n} \perp \mathbf{n}$ and so $\mathbf{m} - (\mathbf{m}\cdot\mathbf{n})\mathbf{n} = \mathbf{n}^{\perp}\Psi$, where $\Psi = \mathbf{m}\cdot\mathbf{n}^{\perp} \in H^1_0(\mathbb{B})$, and hence (4.3.3) holds.

But

$$\begin{aligned}\nabla\mathbf{n}\cdot\nabla(\mathbf{n}^{\perp}\Psi) &= (n_1\nabla n_2 - n_2\nabla n_1)\cdot\nabla\Psi\\ &= \left(\cos^2(w+\phi)\,\nabla(w+\phi) + \sin^2(w+\phi)\,\nabla(w+\phi)\right)\cdot\nabla\Psi\\ &= (\nabla w + \nabla\phi)\cdot\nabla\Psi.\end{aligned}$$

Hence

$$\int_{\mathbb{B}} (\nabla w + \nabla\phi)\cdot\nabla\Psi\,dx = 0.$$

But since $\Delta w = 0$ in $\mathbb{B}$, we have

$$\int_{\mathbb{B}} \nabla w\cdot\nabla\Psi\,dx = 0.$$

Consequently

$$\int_{\mathbb{B}} \nabla\phi\cdot\nabla\Psi\,dx = 0.$$

Now suppose $\Psi \in C^{\infty}_0(\Omega)$ and let $K = \operatorname{supp}\Psi$. There exists a partition of unity $\{\Psi_i\}_{i=1}^{M}$ satisfying $\operatorname{supp}\Psi_i \subset \mathbb{B}_i \subset \Omega$, and $1 \le i \le M$, $\Sigma_{i=1}^{M}\Psi_i = 1$ on K. Then we have

$$\int_{\Omega} \nabla\phi\cdot\nabla\Psi\,dx = \Sigma_{i=1}^{M}\int_{\mathbb{B}_i} \nabla\phi\cdot\nabla(\Psi_i\,\Psi)\,dx = 0,$$

so that ϕ satisfies (4.3.1).

Conversely, suppose ϕ satisfies (4.3.1), then in the ball $\mathbb{B} \subset \Omega$ we have $\Delta\Psi = 0$, where $\Psi = \phi + w$. Let $\mathbf{m} \in C^{\infty}_0(\mathbb{B})$. Then in the ball $\mathbb{B}$

$$\begin{aligned}\nabla\mathbf{n}\cdot\nabla\mathbf{m} - |\nabla\mathbf{n}|^2\mathbf{n}\cdot\mathbf{m} = &-(\sin\Psi)\,\nabla\Psi\cdot\nabla m_1 - |\nabla\Psi|^2(\cos\Psi)\,m_1\\ &+ (\cos\Psi)\,\nabla\Psi\cdot\nabla m_2 - |\nabla\Psi|^2(\sin\Psi)\,m_2\end{aligned}$$

and it follows easily that

$$\int_{\mathbb{B}} \nabla\mathbf{n}\cdot\mathbf{m}\,dx = \int_{\mathbb{B}} |\nabla\mathbf{n}|^2\mathbf{n}\cdot\mathbf{m}\,dx$$

and using a partition of unity again we get (4.3.1).

□

Now we can state and prove our main result for this chapter.

Theorem 4.3.2. *There exists a unique minimizer* $\mathbf{n_p}$ *for the functional*

$$I(\mathbf{n}) = \int_\Omega \left(|\nabla \mathbf{n}|^2 - \frac{k^2}{|x|^2} \right) dx$$

in each homotopy class $C_\mathbf{p}$ *of the admissible set*

$$A = \{\mathbf{n} \in \widetilde{H}^1(\Omega; \mathbb{S}^1) \,|\, \mathbf{n}|_{\partial\omega_j} = \mathbf{n}_j, \; j = 1, ..., N\}$$

where $C_\mathbf{p}$ *is defined in Proposition* 4.2.3, *and* k *is the sum of the degree for each vector field* $\mathbf{n}_j$ *on the boundary* $\partial\omega_j$*, i.e.* $k = \Sigma_{j=1}^N d_j$*. Also, in each homotopy class* $C_\mathbf{p}$*, the minimizer* $\mathbf{n_p}$ *is smooth, i.e.* $\mathbf{n_p} \in C^\infty(\Omega; \mathbb{S}^1)$ *and can be written as* $\mathbf{n_p}(z) = \Pi_{j=1}^N h_j^{d_j}(z) \exp(i\phi_\mathbf{p})$*, where* $\phi_\mathbf{p}$ *is in the set (*4.2.2*):*

$$A_\phi^\mathbf{p} = \{\phi \in \widetilde{H}^1(\Omega) \,|\, \phi|_{\partial\omega_1} = \phi_1; \phi|_{\partial\omega_j} = \phi_j + 2p_j\pi, \; p_j \in \mathbb{Z}, \; \text{for } j = 2, ..., N\}.$$

Moreover, $\mathbf{n_p}$ *is smooth and satisfies:*

$$|\mathbf{n_p}(x) - \mathbf{n_p}^\infty(x)| \leq \frac{C_0}{r},$$

for some constant C_0*, and some sufficiently large* r*, where*

$$\mathbf{n_p}^\infty(x) = (\cos(k\theta + \beta), \sin(k\theta + \beta))$$

and $\beta \in \mathbb{R}$*.*

Further, the minimizer $\mathbf{n_p}$ *is also the unique harmonic map in each homotopy class* $C_\mathbf{p}$ *such that the corresponding* $\phi_\mathbf{p}$ *in* $\mathbf{n_p}(z) = \Pi_{j=1}^N h_j^{d_j}(z) \exp(i\phi_\mathbf{p})$ *satisfies*

$$\phi_\mathbf{p}(x) \in L^\infty((R, \infty) : L^2(\mathbb{S}^1)),$$

and there exists a harmonic map $\mathbf{n}^*$ *in each homotopy class* $C_\mathbf{p}$*, such that*

$$I(\mathbf{n}^*) = \int_\Omega (|\nabla \mathbf{n}^*|^2 - \frac{k^2}{|x|^2})\, dx = +\infty.$$

In addition, there exists a minimizer $\widetilde{\mathbf{n}}$ *for* $I(\mathbf{n})$ *in the union* $A = \bigcup_{\mathbf{p}\in\mathbb{Z}^{N-1}} C_\mathbf{p}$ *of all the* *homotopy classes. However,* $\widetilde{\mathbf{n}}$ *may not be unique.*

Remark 4.3.3. *As mentioned in Remark* 2.1.1*,* $\mathbf{n_p}$ *is treated as a vector field with range in the complex plane* $\mathbb{C}$*. Also* $\phi_{\mathbf{p}}(z)$ *is real and it is equivalent to write* $\phi_{\mathbf{p}}(z)$ *as* $\phi_{\mathbf{p}}(x)$*. Without loss of generality, we can assume that* $\fint_{\partial\omega_j} \phi_{\mathbf{p}}\, d\gamma \in [0, 2\pi)$ *(see Proposition* 4.2.3*).*

Proof. Step 1

First we prove that in each homotopy class $C_{\mathbf{p}}$ defined in Proposition 4.2.3, there exists a minimizer for the functional

$$I(\mathbf{n}) = \int_\Omega \left(|\nabla \mathbf{n}|^2 - \frac{k^2}{|x|^2} \right) dx,$$

and that this minimizer is unique. Recall that from (4.2.12), we have

$$\mathbf{n} = \Pi_{j=1}^{N} h_j(z)^{d_j} \exp(i\phi);$$

and for $\phi \in A_\phi^m$, we have from (4.2.17)

$$\begin{aligned} I(\mathbf{n}) &= \int_\Omega (|\nabla \mathbf{n}|^2 - \frac{k^2}{|x|^2})\, dx \\ &= \int_\Omega \left(|\nabla\phi + \nabla\alpha|^2 + \frac{2k}{r}(\phi_{,2}\cos\theta - \phi_{,1}\sin\theta) + \frac{2k}{r}(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta) \right) dx \qquad (4.3.5) \\ &= J(\phi). \end{aligned}$$

Let L denote the infimum of $I(\mathbf{n})$ in $C_{\mathbf{p}}$. Then $L < \infty$ because there exists $\phi \in \widetilde{H}^1(\Omega)$ satisfying the boundary conditions and we can multiply it by a suitable cut-off function so that it vanishes for large r. Let $\{\mathbf{n}^i\}$ be a minimizing sequence, so that

$$I(\mathbf{n}^i) = \int_\Omega \left(|\nabla \mathbf{n}^i|^2 - \frac{k^2}{|x|^2} \right) dx \to L$$

Hence we have a corresponding minimizing sequence $\{\phi^i\}_{i=1}^\infty \subset A_\phi^{\mathbf{p}}$ for the functional $J(\phi)$, so that

$$I(\mathbf{n}^i) = J(\phi^i) = \int_\Omega \left(|\nabla\phi^i + \nabla\alpha|^2 + \frac{2k}{r}\left(\phi^i_{,2}\cos\theta - \phi^i_{,1}\sin\theta\right) + \frac{2k}{r}(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta) \right) dx \to L.$$

We can easily see from (4.2.20) that L is bounded below from $-\infty$, (i.e. $L > -\infty$) and

$$\|\nabla\phi^i\|^2_{L^2(\Omega)} \le C < \infty.$$

Also we have by [MJ09] such that for some constant C_R depending only on $\Omega \cap \mathbb{B}_R$, we have

$$\int_{\Omega\cap\mathbb{B}_R} |\phi^i|^2\, dx \le C_R \left(\int_{\Omega\cap\mathbb{B}_R} |\nabla\phi^i|^2\, dx + \left(\int_{\partial\omega_j} |\phi_j|\, d\sigma + \Sigma_{j=2}^N \int_{\partial\omega_j} |\phi_j + 2p_j\pi|\, d\sigma \right)^2 \right) \le M_R < \infty.$$

Therefore the minimizing sequence $\{\phi^i\}$ is bounded in $H^1(\Omega\cap\mathbb{B}_R)$ for every R sufficiently large. By using a diagonal argument as in the proof of Theorem 2.3.4, we can extract a subsequence $\{\phi^{j_\iota}\}_{\iota=1}^{\infty}$ of $\{\phi^i\}_{i=1}^{\infty}$ such that the subsequence converges weakly to some $\phi_{\mathbf{p}} \in A_\phi^{\mathbf{p}}$ in $H^1(\Omega \cap \mathbb{B}_R)$ for every R sufficiently large.

Now we show that $I(\mathbf{n})$ is weakly lower semi-continuous i.e. given a weakly convergent sequence $\phi^i \rightharpoonup \phi_{\mathbf{p}}$ in $H^1(\Omega \cap \mathbb{B}_R)$ for any R sufficiently large enough, we have

$$\liminf_{i\to\infty} J(\phi^i) \ge J(\phi_{\mathbf{p}}).$$

Since by (4.2.17) we know

$$\begin{aligned} I(\mathbf{n}^i) = J(\phi^i) &= \int_\Omega \left(|\nabla\phi^i + \nabla\alpha|^2 + \frac{2k}{r}\left(\phi^i_{,2}\cos\theta - \phi^i_{,1}\sin\theta\right) + \frac{2k}{r}\left(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta\right)\right) dx \\ &= \int_\Omega |\nabla\phi^i + \nabla\alpha|^2\, dx + \int_{\Omega\cap\mathbb{B}_R} \left(\frac{2k}{r}\left(\phi^i_{,2}\cos\theta - \phi^i_{,1}\sin\theta\right) + \frac{2k}{r}\left(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta\right)\right) dx, \end{aligned}$$

for any R sufficiently large, therefore we have the following

$$\begin{aligned} \liminf_{i\to\infty} I(\mathbf{n}^i) &= \liminf_{i\to\infty} J(\phi^i) \\ &\ge \liminf_{i\to\infty} \int_{\Omega\cap\mathbb{B}_R} \left(|\nabla\phi^i + \nabla\alpha|^2 + \frac{2k}{r}\left(\phi^i_{,2}\cos\theta - \phi^i_{,1}\sin\theta\right) + \frac{2k}{r}\left(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta\right)\right) dx \\ &\ge \int_{\Omega\cap\mathbb{B}_R} \left(|\nabla\phi_{\mathbf{p}} + \nabla\alpha|^2 + \frac{2k}{r}\left(\phi_{\mathbf{p},2}\cos\theta - \phi_{\mathbf{p},1}\sin\theta\right) + \frac{2k}{r}\left(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta\right)\right) dx \end{aligned} \tag{4.3.6}$$

The last step is due to the property of the norm for a weakly convergent sequence (see [Eva10, Appendix D]). This is because from (4.2.8) and (4.2.9) we know that for any R sufficiently large, α is differentiable and integrable in $\Omega \cap \mathbb{B}_R$, and by letting $\overline{\phi}^i = \phi^i + \alpha$ we automatically have that $\overline{\phi}^i \rightharpoonup \overline{\phi}_{\mathbf{p}}$ in $H^1(\Omega \cap \mathbb{B}_R)$, hence (4.3.6) holds. By letting $R \to \infty$ we have

$$\liminf_{i\to\infty} J(\phi^i) \ge J(\phi_{\mathbf{p}}),$$

which proves the lower semi-continuity of $I(\phi)$. This implies that for each homotopy class $C_{\mathbf{p}}$, there exists a minimizer $\mathbf{n_p} = \Pi_{j=1}^{N} h_j(z)^{d_j} \exp(i\phi_\mathbf{p})$, where $\phi_\mathbf{p} \in A_\phi^\mathbf{p}$.

The uniqueness of this minimizer in a given homotopy class $C_\mathbf{p}$ follows immediately from the strict convexity of the integrand. If there were two distinct minimizers then the gradient of the difference would be zero, so that by the boundary conditions they are the same.

Step 2

We prove in this step that $\mathbf{n_p} \in C^\infty(\Omega; \mathbb{S}^1)$ and $\|\phi_\mathbf{p}(x) - \beta\|_{L^2(\mathbb{S}^1)} \leq \frac{C_0}{r}$, where $\mathbf{n_p}(x) = \Pi_{j=1}^{N} h_j^{d_j} \exp(i\phi_\mathbf{p})$, i.e. we determine the asymptotic behaviour of $\phi_\mathbf{p}$ in each homotopy class $C_\mathbf{p}$.

Since again from (4.2.17) and (4.2.18), we have

$$\begin{aligned} I(\mathbf{n_p}) &= \int_\Omega \left(|\nabla \mathbf{n_p}|^2 - \frac{k^2}{|x|^2} \right) dx \\ &= \int_\Omega \left(|\nabla \phi_\mathbf{p} + \nabla \alpha|^2 + \frac{2k}{r} \left(\phi_{\mathbf{p},2} \cos\theta - \phi_{\mathbf{p},1} \sin\theta \right) + \frac{2k}{r} \left(\alpha_{,2} \cos\theta - \alpha_{,1} \sin\theta \right) \right) dx \qquad (4.3.7) \\ &= J(\phi_\mathbf{p}) \end{aligned}$$

and again by letting $\overline{\phi}_\mathbf{p} = \phi_\mathbf{p} + \alpha$,

$$\begin{aligned} J(\phi_\mathbf{p}) &= \int_\Omega \left(|\nabla \phi_\mathbf{p} + \nabla \alpha|^2 + \frac{2k}{r} \left(\phi_{\mathbf{p},2} \cos\theta - \phi_{\mathbf{p},1} \sin\theta \right) + \frac{2k}{r} \left(\alpha_{,2} \cos\theta - \alpha_{,1} \sin\theta \right) \right) dx \\ &= \int_\Omega \left(|\nabla \overline{\phi}_\mathbf{p}|^2 + \frac{2k}{r} \left(\overline{\phi}_{\mathbf{p},2} \cos\theta - \overline{\phi}_{\mathbf{p},1} \sin\theta \right) \right) dx. \end{aligned}$$

Since in each homotopy class $C_\mathbf{p}$, $\mathbf{n_p}$ minimizes $I(\mathbf{n})$, hence $\phi_\mathbf{p}$ minimizes $J(\phi)$. Therefore we have for $\psi \in C_0^\infty(\Omega)$

$$\begin{aligned} &\frac{d}{d\tau} J(\phi_\mathbf{p} + \tau\psi)\Big|_{\tau=0} \\ &= \frac{d}{d\tau} \int_\Omega \left(|\nabla \overline{\phi}_\mathbf{p} + \tau \nabla \psi|^2 + \frac{2k}{r} \left(\left(\overline{\phi}_{\mathbf{p},2} + \tau\psi_{,2} \right) \cos\theta - \left(\overline{\phi}_{\mathbf{p},1} + \tau\psi_{,1} \right) \sin\theta \right) \right) dx \Big|_{\tau=0} \\ &= 0 \end{aligned}$$

which leads to

$$\int_\Omega \nabla \overline{\phi}_\mathbf{p} \cdot \nabla \psi \, dx = 0. \qquad (4.3.8)$$

Consequently we know that $\overline{\phi}_\mathbf{p}$ is harmonic in $\Omega \cap \mathbb{B}_R$ for any sufficiently large R, i.e.

$$\Delta \overline{\phi}_\mathbf{p} = 0, \qquad (4.3.9)$$

and this also implies that $\overline{\phi}_{\mathbf{p}}$ is smooth in Ω.

Also, by applying change the of variable $r = e^s$ to $x = re^{i\theta}$, we have

$$\begin{aligned} I(\mathbf{n_p}) = J(\phi_{\mathbf{p}}) &= \int_\Omega \left(|\nabla \overline{\phi}_{\mathbf{p}}|^2 + \frac{2k}{r} \left(\overline{\phi}_{\mathbf{p},2} \cos\theta - \overline{\phi}_{\mathbf{p},1} \sin\theta \right) \right) dx \\ &= \int_\Omega \left(\overline{\phi}^2_{\mathbf{p},s} + \overline{\phi}^2_{\mathbf{p},\theta} + 2k \overline{\phi}_{\mathbf{p},\theta} \right) d\theta\, ds \\ &= \int_\Omega \left(|\nabla_s \overline{\phi}_{\mathbf{p}}|^2 + 2k \overline{\phi}_{\mathbf{p},\theta} \right) d\theta\, ds \end{aligned}$$

where

$$\nabla_s = \frac{\partial}{\partial s} + i \frac{\partial}{\partial \theta},$$

and (4.3.9) becomes

$$\overline{\phi}_{\mathbf{p},ss} + \overline{\phi}_{\mathbf{p},\theta\theta} = 0. \tag{4.3.10}$$

We now let

$$F(s) = \int_0^{2\pi} \overline{\phi}^2_{\mathbf{p}}\, d\theta, \tag{4.3.11}$$

where s is sufficiently large, so that the corresponding r is sufficiently large and $\cup_{j=1}^{N} \bar{\omega}_j \subset \mathbb{B}_R \subset \mathbb{B}_r$. By applying (4.3.10) we have

$$\frac{d^2}{ds^2} F(s) = 2 \int_0^{2\pi} (\overline{\phi}^2_{\mathbf{p},s} + \overline{\phi}^2_{\mathbf{p},\theta})\, d\theta \geq 0.$$

Since $\mathbf{n_p}$ is the minimizer for the functional (4.3.7), we have

$$J(\phi_{\mathbf{p}}) = \int_\Omega \left(|\nabla \overline{\phi}_{\mathbf{p}}|^2 + \frac{2k}{r} \left(\overline{\phi}_{\mathbf{p},2} \cos\theta - \overline{\phi}_{\mathbf{p},1} \sin\theta \right) \right) dx < \infty, \tag{4.3.12}$$

which implies that

$$\int_\Omega |\nabla_s \overline{\phi}_{\mathbf{p}}|^2\, d\theta\, ds < \infty. \tag{4.3.13}$$

Note that (4.3.13) holds, since by (4.2.19) the term $\int_\Omega \frac{2k}{r} \left(\overline{\phi}_{\mathbf{p},2} \cos\theta - \overline{\phi}_{\mathbf{p},1} \sin\theta \right) dx$ is bounded , and as a consequence we have

$$F_s(s) - F_s(s_0) = 2 \int_{s_0}^{s} \int_0^{2\pi} (\overline{\phi}^2_{\mathbf{p},s} + \overline{\phi}^2_{\mathbf{p},\theta})\, d\theta\, ds < \infty. \tag{4.3.14}$$

By (4.3.14) $F_s(s)$ is nondecreasing. Hence $F_s(s) \to \alpha$, for some constant $\alpha \geq 0$. Therefore as

$s \to \infty$ we have

$$F(s) \sim \alpha s + c$$

for some constant α and c.

We now prove by contradiction that $\alpha = 0$. For arbitrary small $\epsilon > 0$, we fix R_0 ($R_0 = e^{s_0}$) large enough such that

$$\int_{s_0}^{s} \int_0^{2\pi} \overline{\phi}^2_{\mathbf{p},s} \, d\theta \, ds < \epsilon,$$

and we have

$$\overline{\phi}_{\mathbf{p}}(s, \theta) - \overline{\phi}_{\mathbf{p}}(s_0, \theta) = \int_{s_0}^{s} \overline{\phi}_{\mathbf{p},\tau}(\tau, \theta) \, d\tau \leq \left(\int_{s_0}^{s} 1^2 \, d\tau \right)^{\frac{1}{2}} \cdot \left(\int_{s_0}^{s} \overline{\phi}^2_{\mathbf{p},\tau} \, d\tau \right)^{\frac{1}{2}},$$

which leads to

$$\int_0^{2\pi} \frac{|\overline{\phi}_{\mathbf{p}}(s, \theta) - \overline{\phi}_{\mathbf{p}}(s_0, \theta)|^2}{s - s_0} \leq \int_{s_0}^{s} \int_0^{2\pi} \overline{\phi}^2_{\mathbf{p},\tau} \, d\tau \, d\theta < \epsilon.$$

Therefore we have

$$\begin{aligned} \lim_{s \to s_0} F_s(s_0) &= \lim_{s \to s_0} \int_0^{2\pi} \frac{|\overline{\phi}_{\mathbf{p}}(s, \theta) - \overline{\phi}_{\mathbf{p}}(s_0, \theta)|^2}{s - s_0} \\ &= \lim_{s \to s_0} \int_0^{2\pi} \frac{\overline{\phi}^2_{\mathbf{p}}(s, \theta) + \overline{\phi}^2_{\mathbf{p}}(s_0, \theta) - 2\overline{\phi}_{\mathbf{p}}(s, \theta) \, \overline{\phi}_{\mathbf{p}}(s_0, \theta)}{s - s_0} \\ &= \lim_{s \to s_0} \int_0^{2\pi} \frac{\overline{\phi}^2_{\mathbf{p}}(s, \theta) - \overline{\phi}^2_{\mathbf{p}}(s_0, \theta)}{s - s_0} < \epsilon, \end{aligned}$$

for any $\epsilon > 0$.

Consequently, we have $\alpha = 0$, and

$$\lim_{s \to \infty} F(s) = \lim_{s \to \infty} \int_0^{2\pi} \overline{\phi}^2_{\mathbf{p}} \, d\theta = c < \infty. \tag{4.3.15}$$

This implies that $F(s) \in L^\infty(0, \infty)$, and we can follow exactly the same argument using Fourier series in the proof of Proposition 3.1.2, and obtain

$$\|\overline{\phi}_{\mathbf{p}} - \beta\|_{L^2(\mathbb{S}^1)} \leq \frac{C}{r}, \tag{4.3.16}$$

as $r \to \infty$, where $\beta \in \mathbb{R}$ is some constant, hence proving the asymptotic behaviour of $\overline{\phi}_{\mathbf{p}}$.

Step 3

In this step, we prove there is a minimum $\widetilde{\mathbf{n}}$ (which is not necessarily unique), in the union of all homotopy classes $A = \bigcup_{\mathbf{p}\in\mathbb{Z}^{N-1}} C_{\mathbf{p}}$. We claim that for $\phi \in A^{\mathbf{p}}_{\phi}$, we have

$$\int_{\cup_{j\neq 1}\partial\omega_j} |\phi|^2 \, d\sigma + \int_{\Omega\cap\mathbb{B}_R} |\phi|^2 \, dx \le C\Big(\int_{\Omega\cap\mathbb{B}_R} |\nabla\phi|^2 \, dx + \int_{\partial\omega_1} |\phi|^2 \, d\sigma\Big), \tag{4.3.17}$$

for some constant C depending only on $\Omega \cap \mathbb{B}_R$. We prove this claim by contradiction.

Assume the claim were not true. Then there would exist a sequence $\{\phi^j\}_{j=1}^{\infty}$ such that

$$\int_{\cup_{j\neq 1}\partial\omega_j} |\phi^j|^2 \, d\sigma + \int_{\Omega\cap\mathbb{B}_R} |\phi^j|^2 \, dx > j\Big(\int_{\Omega\cap\mathbb{B}_R} |\nabla\phi^j|^2 \, dx + \int_{\partial\omega_1} |\phi^j|^2 \, d\sigma\Big). \tag{4.3.18}$$

On account of homogeneity, we may assume

$$\int_{\cup_{j\neq 1}\partial\omega_j} |\phi^j|^2 \, d\sigma + \int_{\Omega\cap\mathbb{B}_R} |\phi^j|^2 \, dx = 1. \tag{4.3.19}$$

Then from (4.3.18)

$$\int_{\Omega\cap\mathbb{B}_R} |\nabla\phi^j|^2 \, dx + \int_{\partial\omega_1} |\phi^j|^2 \, d\sigma < \frac{1}{j} \to 0, \quad \text{as} \quad j \to \infty. \tag{4.3.20}$$

Consequently we have

$$\nabla\phi^j \to 0 \quad \text{in} \quad L^2(\Omega \cap \mathbb{B}_R),$$

and

$$\phi^j \ \text{ is bounded in } \ L^2(\Omega \cap \mathbb{B}_R).$$

Therefore the sequence $\{\phi^j\}$ is bounded in $H^1(\Omega\cap\mathbb{B}_R)$ and there exists a subsequence $\{\phi^{j_\iota}\}_{\iota=1}^{\infty}$ of $\{\phi^j\}_{j=1}^{\infty}$, and some $\phi \in H^1(\Omega \cap \mathbb{B}_R)$, such that

$$\phi^{j_\iota} \rightharpoonup \phi \ \text{ in } \ H^1(\Omega \cap \mathbb{B}_R) \ \text{ weakly.}$$

By lower semicontinuity of the norm and the compactness of the trace operator we have that

$$\begin{cases} \nabla\phi = 0 \ \text{ in } \ \Omega \cap \mathbb{B}_R \\ \phi = 0 \ \text{ on } \ \partial\omega_1. \end{cases}$$

This implies that $\phi \equiv C_0$ in $\Omega \cap \mathbb{B}_R$ for some constant C_0 , and $C_0 = 0$.

Passing to the limit in (4.3.19) gives the desired contradiction. From (4.3.17) it follows immediately that

$$\int_{\Omega\cap\mathbb{B}_R} |\nabla\phi_{\mathbf{p}}|^2\,dx \to \infty \text{ as } |\mathbf{p}| \to \infty$$

and therefore

$$\begin{aligned} I(\mathbf{n_p}) &= \int_\Omega (|\nabla\mathbf{n_p}|^2 - \frac{k^2}{|x|^2})\,dx \\ &= \int_\Omega \left(|\nabla\phi_{\mathbf{p}} + \nabla\alpha|^2 + \frac{2k}{r}\left(\phi_{\mathbf{p},2}\cos\theta - \phi_{\mathbf{p},1}\sin\theta\right) + \frac{2k}{r}\left(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta\right)\right) dx \to \infty \end{aligned}$$

as $|\mathbf{p}| \to \infty$. Hence the minimizer in some $C_{\mathbf{p}}$ must be a minimizer $\widetilde{\mathbf{n}}$ in the union $A = \bigcup_{\mathbf{p}\in\mathbb{Z}^{N-1}} C_{\mathbf{p}}$. That $\widetilde{\mathbf{n}}$ is not necessarily unique was shown in Example 4.1.1.

Step 4

Now we show that in each homotopy class $C_{\mathbf{p}}$, the minimizer $\mathbf{n_p}$ is the unique harmonic map satisfying the condition

$$\phi_{\mathbf{p}}(x) \in L^\infty((R,\infty); L^2(\mathbb{S}^1)),$$

where $\mathbf{n_p}(x) = \Pi_{j=1}^N h_j^{d_j}\exp(i\phi_{\mathbf{p}})$, and $\phi_{\mathbf{p}} \in A_\phi^{\mathbf{p}}$ as defined in (4.2.2).

Assume there exists another map $\mathbf{n}'_{\mathbf{p}} \in A$ in the same homotopy class $C_{\mathbf{p}}$, so that

$$\mathbf{n}'_{\mathbf{p}}(x) = \Pi_{j=1}^N h_j^{d_j}\exp(i\phi'_{\mathbf{p}}),$$

for $\phi'_{\mathbf{p}} \in A_\phi^{\mathbf{p}}$.

Since $\mathbf{n}'_{\mathbf{p}}$ is a harmonic map, it satisfies

$$\int_\Omega \nabla\mathbf{n}'_p \cdot \nabla\mathbf{m} = \int_\Omega |\nabla\mathbf{n}'_p|^2\mathbf{n}'_p \cdot \mathbf{m}\,dx \quad \forall\mathbf{m} \in C_0^\infty(\Omega;\mathbb{R}^2),$$

which is the weak form of the Euler-Lagrange equation for the functional $I(\mathbf{n}) = \int_\Omega \left(|\nabla\mathbf{n}|^2 - \frac{k^2}{|x|^2}\right) dx$, for $\mathbf{n} \in A = \{\mathbf{n} \in \widetilde{H}^1(\Omega;\mathbb{S}^1)\,|\,\mathbf{n}|_{\partial\omega_j} = \mathbf{n}_j,\ j = 1,...,N\}$. Hence by (4.3.8) in Step 3

$$(\overline{\phi}'_{\mathbf{p}})_{ss} + (\overline{\phi}'_{\mathbf{p}})_{\theta\theta} = 0,$$

where $\overline{\phi}'_{\mathbf{p}} = \phi'_{\mathbf{p}} + \alpha$.

Denoting $\widetilde{\phi}_\mathbf{p} = \overline{\phi}_\mathbf{p} - \overline{\phi}'_\mathbf{p}$, we have that

$$\begin{cases} (\widetilde{\phi}_\mathbf{p})_{ss} + (\widetilde{\phi}_\mathbf{p})_{\theta\theta} = 0 \\ \widetilde{\phi}_\mathbf{p}|_{\partial\omega_j} = 0, \ j = 1, ..., N. \end{cases}$$

Since we know that $\widetilde{\phi}_\mathbf{p}$ is a weak solution of the harmonic map, we have

$$\int_\Omega \nabla\widetilde{\phi}_\mathbf{p} \cdot \nabla v \, dx = 0, \quad \forall v \in H^1_0(\Omega).$$

Choose $v = \widetilde{\phi}_\mathbf{p}\Psi_\epsilon$, where Ψ_ϵ is a function or radius $r = |x|$,

$$\Psi_\epsilon := \begin{cases} 0 & |x| \geq R| \\ \frac{R-|x|}{\epsilon} & |x| \in (R-\epsilon, R) \\ 1 & |x| \leq R-\epsilon \end{cases} \tag{4.3.21}$$

we have

$$\int_\Omega \nabla\widetilde{\phi}_\mathbf{p} \cdot v \, dx = \int_{\Omega\cap\mathbb{B}_{R-\epsilon}} |\nabla\widetilde{\phi}_\mathbf{p}|^2 \, dx + \int_{\mathbb{B}_R\setminus\mathbb{B}_{R-\epsilon}} \left(\Psi_\epsilon|\nabla\widetilde{\phi}_\mathbf{p}|^2 + \widetilde{\phi}_\mathbf{p}\nabla\widetilde{\phi}_\mathbf{p} \cdot \nabla\Psi_\epsilon\right) dx = 0.$$

Since we also have

$$\int_{\mathbb{B}_R\setminus\mathbb{B}_{R-\epsilon}} \Psi_\epsilon|\nabla\widetilde{\phi}_\mathbf{p}|^2 \, dx \leq \int_{\mathbb{B}_R\setminus\mathbb{B}_{R-\epsilon}} |\nabla\widetilde{\phi}_\mathbf{p}|^2 \, dx \to 0, \quad \epsilon \to 0$$

and

$$\int_{\mathbb{B}_R\setminus\mathbb{B}_{R-\epsilon}} \widetilde{\phi}_\mathbf{p}\nabla\widetilde{\phi}_\mathbf{p} \cdot \nabla\Psi_\epsilon \, dx = -\int_{\mathbb{B}_R\setminus\mathbb{B}_{R-\epsilon}} \widetilde{\phi}_\mathbf{p} \frac{\partial\widetilde{\phi}_\mathbf{p}}{\partial\nu} \frac{1}{\epsilon} r \, dr \, dS \to -\int_{\partial\mathbb{B}_R} \widetilde{\phi}_\mathbf{p} \frac{\partial\widetilde{\phi}_\mathbf{p}}{\partial\nu} R \, dS, \quad \epsilon \to 0,$$

we have that

$$\begin{aligned} \int_{\Omega\cap\mathbb{B}_R} |\nabla\widetilde{\phi}_\mathbf{p}|^2 \, dx &= \int_{\partial\mathbb{B}_R} \widetilde{\phi}_\mathbf{p} \frac{\partial\widetilde{\phi}_\mathbf{p}}{\partial\nu} R \, dS \\ &= \int_{\mathbb{S}^1} \widetilde{\phi}_\mathbf{p}(R,\theta) \frac{\partial\widetilde{\phi}_\mathbf{p}}{\partial\nu} (R,\theta) \ R \, d\theta \\ &= \int_{\mathbb{S}^1} \widetilde{\phi}_\mathbf{p} \frac{\partial\widetilde{\phi}_\mathbf{p}}{\partial s} (s_R, \theta) \ d\theta. \end{aligned} \tag{4.3.22}$$

The rest is to prove that the right hand side of (4.3.22) goes to zero as $R \to \infty$ ($s_R \to \infty$).

Recall that since

$$\phi'_{\mathbf{p}}(x) \in L^\infty((R,\infty); L^2(\mathbb{S}^1)),$$

and also we have from (4.3.15) that

$$\phi_{\mathbf{p}}(x) \in L^\infty((R,\infty); L^2(\mathbb{S}^1)).$$

Consequently, we have

$$\begin{aligned}\widetilde{F}(s) = \int_0^{2\pi} \widetilde{\phi}_{\mathbf{p}}^2 \, d\theta \, dr &= \int_0^{2\pi} |\overline{\phi}_{\mathbf{p}} - \overline{\phi'_{\mathbf{p}}}|^2 \, d\theta \, dr \\ &\leq \left(\int_0^{2\pi} \overline{\phi}_{\mathbf{p}}^2 \, d\theta + \int_0^{2\pi} \overline{\phi'_{\mathbf{p}}}^2 \, d\theta \right)^2 < \infty.\end{aligned}$$

By applying the same argument using Fourier series in the proof of Proposition 3.1.2, we have

$$\int_{\Omega\cap\mathbb{B}_R} |\nabla\widetilde{\phi}_{\mathbf{p}}|^2 \, dx = \int_{\mathbb{S}^1} \widetilde{\phi}_{\mathbf{p}} \frac{\partial\widetilde{\phi}_{\mathbf{p}}}{\partial s}(s = s_R, \theta) \, d\theta < \frac{\widetilde{C}}{R^2},$$

hence

$$\int_\Omega |\nabla\widetilde{\phi}_{\mathbf{p}}|^2 \, dx = \lim_{R\to\infty} \int_{\Omega\cap\mathbb{B}_R} |\nabla\widetilde{\phi}_{\mathbf{p}}|^2 \, dx = 0.$$

Therefore we have $\widetilde{\phi}_{\mathbf{p}} = \overline{\phi}_{\mathbf{p}} - \overline{\phi}'_{\mathbf{p}} \equiv 0$ (hence $\phi_{\mathbf{p}}(x) \equiv \phi'_{\mathbf{p}}(x)$) in Ω. This proves that the minimizer $\mathbf{n}_{\mathbf{p}}$ of $I(\mathbf{n})$ in each homotopy class $C_{\mathbf{p}}$ is also unique of harmonic maps in $L^\infty((R,\infty); L^2(\mathbb{S}^1))$.

Step 5

Now we prove that there exists a harmonic map $\mathbf{n}^*$ in each homotopy class $C_{\mathbf{p}}$ such that

$$I(\mathbf{n}^*) = \int_\Omega \left(|\nabla\mathbf{n}^*|^2 - \frac{k^2}{|x|^2} \right) dx = +\infty.$$

First we show that there exists a minimizer for the energy functional

$$E(\phi) = \int_\Omega |\nabla\phi - \nabla \log|x||^2 \, dx, \qquad (4.3.23)$$

for ϕ in the admissible set $A_\phi^{\mathbf{p}} = \{\phi \in \widetilde{H}^1(\Omega) \,|\, \phi|_{\partial\omega_1} = \phi_1; \phi|_{\partial\omega_j} = \phi_j + 2p_j\pi, \;\; j = 2, ..., N\}$, where $\mathbf{p} = \{p_2, p_3, ..., p_N\} \in \mathbb{Z}^{N-1}$.

To prove there exists a minimizer for (4.3.23), we choose

$$u = \phi - \log |x|,$$

and it follows that

$$E(\phi) = \int_\Omega |\nabla\phi - \nabla \log r|^2 \, dx = \int_\Omega |\nabla u|^2 \, dx = I(u),$$

where u is in the admissible set

$$U := \{u \in \widetilde{H}^1(\Omega) \,|\, u|_{\partial\omega_1} = \phi_1 - \log |x|;\ u|_{\partial\omega_j} = \phi_j - \log |x| + 2p_j\pi,\ \ j = 2, ..., N\}.$$

It follows from the direct method that there is a minimizer $u^* \in U$ for the functional

$$I(u) = \int_\Omega |\nabla u|^2 \, dx,$$

and $\phi^* = u^* + \log |x|$ is therefore a minimizer for the functional

$$E(\phi) = \int_\Omega |\nabla\phi - \nabla \log r|^2 \, dx,$$

for $\phi \in A^{\mathbf{p}}_\phi$.

As a result, we have

$$\left.\frac{dE(\phi^*)}{d\tau}\right|_{\tau=0} = 2 \int_{\Omega\setminus L_1} (\nabla\phi^* - \nabla \log r) \cdot \nabla\eta = 0,$$

for $\eta \in C_0^\infty(\Omega)$, and it follows from Proposition 4.3.1 that the corresponding $\mathbf{n}^* = \Pi_{j=1}^N h_j(z)^{d_j} \exp(i\phi^*)$ satisfies (4.3.1), hence a harmonic map.

Therefore there exist such harmonic map $\mathbf{n}^*$ in each homotopy class $C_{\mathbf{p}}$, and now we show that

$$I(\mathbf{n}^*) = \int_\Omega \left(|\nabla \mathbf{n}^*|^2 - \frac{k^2}{|x|^2} \right) dx = +\infty.$$

Recall that since $\mathbf{n}^* = \Pi_{j=1}^N h_j(z)^{d_j} \exp(i\phi^*) = \exp i(k\theta + \alpha + \phi^*)$, where

$$\begin{cases} k = \Sigma_{j=1}^N d_j \\ \alpha = \Sigma_{j=1}^N d_j \, \alpha_j, \end{cases}$$

we have from (4.2.17), such that

$$
\begin{aligned}
I(\mathbf{n}) &= \int_\Omega \left(|\nabla \mathbf{n}|^2 - \frac{k^2}{r^2} \right) dx \\
&= \int_\Omega |\nabla\phi + \nabla\alpha|^2\, dx + \int_{\Omega\cap\mathbb{B}_R} \left(\frac{2k}{r}\left(\phi_{,2}\cos\theta - \phi_{,1}\sin\theta\right) + \frac{2k}{r}\left(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta\right) \right) dx \\
&= E(\phi).
\end{aligned}
$$

By $\phi^* = u^* + \log|x|$, we have

$$
\nabla\phi^* = \nabla \log|x| + \nabla u^* = \frac{x}{r^2} + \nabla u^*,
$$

consequently by (4.2.20), we have

$$
\begin{aligned}
E(\phi^*) &= \int_\Omega |\nabla\phi^* + \nabla\alpha|^2\, dx + \int_{\Omega\cap\mathbb{B}_R} \left(\frac{2k}{r}\left(\phi^*_{,2}\cos\theta - \phi^*_{,1}\sin\theta\right) + \frac{2k}{r}\left(\alpha_{,2}\cos\theta - \alpha_{,1}\sin\theta\right) \right) dx \\
&\geq \int_\Omega \left((1-2\epsilon)|\nabla\phi^*|^2 - |O(1/r^4)| \right) dx - C_R \\
&= \int_\Omega \left((1-2\epsilon)\left(|\nabla u^*|^2 + \frac{2}{r}(u^*_{,1}\cos\theta + u^*_{,2}\sin\theta) + \frac{1}{r^2} \right) - |O(1/r^4)| \right) dx - C_R \\
&\geq \int_\Omega \left((1-2\epsilon)|\nabla u^*|^2 - (1-2\epsilon)\frac{2}{r}|\nabla u^*| + (1-2\epsilon)\frac{1}{r^2} - |O(1/r^4)| \right) dx - C_R,
\end{aligned}
$$

where C_R is some finite constant.

By using the inequality

$$
\frac{2}{r}|\nabla u^*| \leq \frac{1}{\epsilon}|\nabla u^*|^2 + \epsilon\frac{1}{r^2},
$$

we have

$$
\begin{aligned}
E(\phi^*) &\geq \int_\Omega \left((1-2\epsilon)|\nabla u^*|^2 - (1-2\epsilon)\frac{2}{r}|\nabla u^*| + (1-2\epsilon)\frac{1}{r^2} - |O(1/r^4)| \right) dx - C_R \\
&\geq \int_\Omega \left((1-2\epsilon)(1-\frac{1}{\epsilon})|\nabla u^*|^2 + (1-\epsilon)(1-2\epsilon)\frac{1}{r^2} - |O(1/r^4)| \right) dx - C_R.
\end{aligned}
$$

Therefore for sufficiently small $\epsilon > 0$ such that $(1-\epsilon)(1-2\epsilon) > 0$, we have

$$
I(\mathbf{n}^*) = E(\phi^*) = +\infty
$$

which is due to the fact that

$$\int_{\mathbb{B}_R^c} \frac{1}{r^2}\,dx = \int_{\mathbb{B}_R^c} \frac{1}{r}\,dr\,d\theta = +\infty,$$

and

$$\int_\Omega \left((1-2\epsilon)(1-\frac{1}{\epsilon})|\nabla u^*|^2 - |O(1/r^4)|\right) dx - C_R < \infty,$$

since $I(u^*) = \inf_{u\in U} \int_\Omega |\nabla u|^2\,dx$. This completes our proof. □

Remark 4.3.4. *L. Nguyen pointed out to us that, instead of working on this exterior domain* $\Omega = \mathbb{R}^2 \setminus \cup_{j=1}^N \bar{\omega}_j$ *, one can actually use an inversion to work on a puncture interior domain. In this sense, you are working on a multiply-connected and bounded domain with one point removed.*

If we write $u(x) = \phi(x/|x|^2)$ *(the Kelvin transform of* $\phi(x)$*), and if* $\phi(x)$ *is harmonic in* Ω*, then* $u(x)$ *is harmonic in* $\Omega' = \{x \neq 0 \,|\, x/|x|^2 \in \Omega\}$. *One can easily see that* Ω' *is a bounded multiply-connected domain with the origin removed. In Theorem 4.3.2,* $\phi_{\mathbf{p}}(x)$*, which is associated to the minimizer* $\mathbf{n}_{\mathbf{p}}$ *in each homotopy class* $C_{\mathbf{p}}$*, is bounded at* $|x| \to \infty$*. Then one can easily see that* $u(x)$ *is bounded when* $|x| \to 0$*, which is saying that the origin in the inverted domain* Ω' *is a removable singularity of* $u(x)$*, hence* $u(x)$ *is harmonic in* $\Omega' \cup \{0\}$*. To see more on removable singularities of solutions of elliptic equations, one can refer to [Ser64].*

A related issue is whether the minimizer in $\widetilde{H}^1(\Omega;\mathbb{S}^1)$ *is in fact a minimizer in* $H^1(\Omega' \cup \{0\};\mathbb{S}^1)$*, where* Ω' *is the inverted domain. If this is true, it will also explain the existence of the harmonic map* $\mathbf{n}^*$ *with infinite energy on the exterior domain* Ω *: this is a solution which gives infinite* H^1 *energy at the removed point (in our case it is the origin) on the inverted domain* Ω'*. We will not elaborate further on this problem in this thesis, but will do more study on this in our future research.*

In the following chapter, we will address the minimization problem on non-orientable line field. We will introduce an auxiliary vector field for the given non-orientable line field and apply Theorem 4.3.2 to the auxiliary vector field to solve the minimization problem in a given N-connected domain Ω.

Chapter 5

Line field models for uniaxial nematic liquid crystals

5.1 Definition of line field and its degree

The idea of the Oseen-Frank model is to associate to each point in the macroscopic physical space a director describing the preferred direction of the molecules at the point. The Oseen-Frank model takes this director to be a unit vector $\mathbf{n}$. But this setting has the deficiency of ignoring a physical symmetry of the material, namely the statistical head-to-tail symmetry of the rod-like molecules. One of the advantages of the more complex Q-tensor theory of de Gennes, which uses a symmetric trace-free tensor Q as the order parameter, is that remedies this deficiency. In the simplest constrained case of uniaxial Q-tensors with a constant scalar order parameter, these Q-tensors admit the form:

$$\mathrm{Q} = \mathrm{s}\left(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathrm{I}\right), \tag{5.1.1}$$

from which it is easily seen that there exists a bijective correspondence between such Q and pairs of antipodal unit vectors $\{\mathbf{n}, -\mathbf{n}\}$. Thus we can think of Q as representing the line through the origin parallel to $\mathbf{n}$. A tensor $\mathrm{Q} = \mathrm{Q}(\mathrm{x})$ of the form (5.1.1) can be interpreted as a line field. We will refer to this as the constrained Landau - de Gennes theory. It is shown in [BZ08] that in the case of a non-zero constant scalar order parameter s, the Landau-de Gennes theory is equivalent to the Oseen-Frank theory when the domain Ω is simply-connected, but not in general otherwise.

As in Chapter 3 and Chapter 4, we restrict our discussion to a 2D domain with $\mathbf{m}$ a 'planar' unit vector, i.e. $\mathbf{m} = (\mathrm{m}_1, \mathrm{m}_2, 0)$. Thus we can write the 'planar' unit vector $\mathbf{m} = (\mathrm{m}_1, \mathrm{m}_2, 0) \in \mathbb{S}^2$ as $\mathbf{m} = (\mathrm{m}_1, \mathrm{m}_2) \in \mathbb{S}^1$, and the line field is identified with the pair

$\{\mathbf{m}, -\mathbf{m}\}$. One can also easily see that it is equivalent to write the line field as a Q-tensor of the form (5.1.1), or simply as

$$M = \mathbf{m} \otimes \mathbf{m} = \begin{pmatrix} m_1^2 & m_1 m_2 \\ m_2 m_1 & m_2^2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{5.1.2}$$

since there is a bijective correspondence between (5.1.1) and (5.1.2). In this chapter, we will use the form (5.1.2) to represent a given line field M. We denote the set of such matrices by

$$\mathbb{Q} = \{M \in M^{2\times 2}(\mathbb{R}) \,|\, M = \mathbf{m} \otimes \mathbf{m} \text{ for some } \mathbf{m} \in \mathbb{S}^1\},$$

where $M^{2\times 2}(\mathbb{R})$ is the set of 2×2 matrices with real values.

Note that $M = M^T$ and $\operatorname{Tr} M = 1$ for any $M \in \mathbb{Q}$, and that $\mathbb{Q}$ can be bijectively identified with the real projective space $\mathbb{RP}^1$, and thus can be given the structure of a one-dimensional manifold.

Given an N-connected domain $\Omega = \mathbb{R}^2 \setminus \cup_{j=1}^N \bar{\omega}_j$ satisfying our previous hypotheses, we let

$$\widetilde{X} = \{M \in \widetilde{H}^1(\Omega; \mathbb{Q}) \,|\, \operatorname{Tr} M|_{\partial\omega_j} = M_j,\ j = 1, ..., N\},$$

where $M_j \in \mathbb{Q}$ are given. The notation $\operatorname{Tr} M|_{\partial\omega_j}$ denotes the trace of the line field M on each boundary $\partial\omega_j$, $j = 1, ..., N$. Similar to Definition 2.3.1, we define $\widetilde{H}^1(\Omega; \mathbb{Q})$ as follows:

Definition 5.1.1. *We say that a line field* $M \in \widetilde{H}^1(\Omega; \mathbb{Q})$ *if and only if* $M \in H^1(\Omega \cap \mathbb{B}_R; \mathbb{Q})$ *for any* R *sufficiently large.*

Note that $\widetilde{H}^1(\Omega; \mathbb{Q})$ is also a completely metrizable space, such that a sequence of line fields $M^j \to M$ in $\widetilde{H}^1(\Omega; \mathbb{Q})$ if and only if $M^j \to M$ in $H^1(\Omega \cap \mathbb{B}_R; \mathbb{Q})$ for any R sufficiently large.

We call a line field $M \in H^1(\Omega; \mathbb{Q})$ orientable (see [BZ08] and [BZ11]) if and only if there exists a vector field in the same functional space, that is an $\mathbf{m} \in H^1(\Omega; \mathbb{S}^1)$ such that $\mathbf{m}(x) \in \mathbb{S}^1$ and $M = \mathbf{m} \otimes \mathbf{m}$ except for possibly a set of measure zero. It is shown in [BZ11, Proposition 2] that an orientable line field $M \in H^1(\Omega; \mathbb{Q})$ can have only two orientations. More precisely, if given a unit vector $\mathbf{m} \in H^1(\Omega; \mathbb{S}^1)$, corresponding to a line field $M \in H^1(\Omega; \mathbb{Q})$ such that $M = \mathbf{m} \otimes \mathbf{m}$, there can be only one other vector field, namely $-\mathbf{m}$, corresponding to the same M.

Before we state and prove the main theorem of this chapter, we need to define the degree for a given line field $M \in \widetilde{X}$, and the corresponding homotopy classes. In order to define the degree of a line field on the boundaries $\{\partial\omega_j\}_{j=1}^N$ of the domain Ω, we first define following [BZ11] an auxiliary vector field A(M) for each line field $M = \mathbf{m} \otimes \mathbf{m} \in \widetilde{X}$

by

$$\begin{aligned} A(M) &= \mathbf{m}^2 \\ &= (m_1 + i\, m_2)^2 \\ &= m_1^2 - m_2^2 + 2i\, m_1 m_2 \\ &= M_{11} - M_{22} + 2i\, M_{12} \\ &= 2M_{11} - 1 + 2i\, M_{12}, \end{aligned} \tag{5.1.3}$$

where $A(M)$, $\mathbf{m}$ are treated here as vector fields with range in complex plane $\mathbb{C}$. One can also notice from (5.1.3) that we have $|A(M)| = 1$, and thus the auxiliary map $A(M) : \widetilde{X} \mapsto \widetilde{H}^1(\Omega; \mathbb{S}^1)$ allows one to associate to a planar line field an auxiliary unit vector field.

Also, by letting $\mathbf{n} = A(M)$ where $\mathbf{n} = n_1 + i\, n_2$, we have from (5.1.3):

$$M = \mathbf{m} \otimes \mathbf{m} = \begin{pmatrix} m_1^2 & m_1 m_2 \\ m_2 m_1 & m_2^2 \end{pmatrix} = \begin{pmatrix} \frac{n_1+1}{2} & \frac{n_2}{2} \\ \frac{n_2}{2} & \frac{1-n_1}{2} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}. \tag{5.1.4}$$

In fact, one can see from (5.1.3) that $\mathbf{n} \in \widetilde{H}^1(\Omega; \mathbb{S}^1)$, and by (5.1.4) we have

$$\begin{cases} m_1 = \kappa \left(\frac{n_1+1}{2}\right)^{1/2} \\ m_2 = \widetilde{\kappa} \left(\frac{1-n_1}{2}\right)^{1/2} \end{cases}$$

where $\kappa = \pm 1, \widetilde{\kappa} = \pm 1$.

On the other hand, we have from (5.1.4) that

$$m_1 m_2 = \kappa \widetilde{\kappa} \left(\frac{1 - n_1^2}{4}\right)^{1/2} = \kappa \widetilde{\kappa} \frac{|n_2|}{2} = \frac{n_2}{2},$$

and this implies that $\kappa \widetilde{\kappa} = \operatorname{sgn} n_2$ for $n_2 \neq 0$, which gives

when $n_2 \geq 0$:

$$\mathbf{m} = \pm \left(\left(\frac{n_1 + 1}{2}\right)^{\frac{1}{2}}, \left(\frac{1 - n_1}{2}\right)^{\frac{1}{2}} \right);$$

and when $n_2 < 0$:

$$\mathbf{m} = \pm \left(\left(\frac{n_1 + 1}{2}\right)^{\frac{1}{2}}, - \left(\frac{1 - n_1}{2}\right)^{\frac{1}{2}} \right)$$

and in either case, we have a pair of antipodal unit vectors. In particular, as is obvious from (5.1.4), given an auxiliary vector field $\mathbf{n}$ there can be only one possible corresponding line field denoted by a pair of antipodal unit vectors $\{\mathbf{m}, -\mathbf{m}\}$.

Thus one can see that given any such auxiliary vector field $A(M)$, there exists a unique line field, such that the line field M and the auxiliary vector field $A(M)$ are in the same Sobolev class. Note that this is also true for the line field $\operatorname{Tr} M|_{\partial\omega_j} = M_j$ on each of the boundaries $\cup_{j=1}^{N}\partial\omega_j$ and its associated auxiliary vector field $A(M_j)$.

Next we define the degree for a give line field $M \in \widetilde{X}$. It is shown in [BZ11, Proposition 3] and the remark after it that orientability in a domain implies orientability at the boundary. Moreover, it is proved in [BZ11, Proposition 6] that for any $j \in \{1, ..., N\}$, the trace $\mathrm{Tr}\,M \in H^{1/2}(\partial\omega_j, \mathbb{Q})$ of the line field M is orientable (in the space $H^{1/2}$) if and only if the degree of the corresponding auxiliary vector field $A(\mathrm{Tr}\,M)$ on $\partial\omega_j$ is an even integer, i.e. $\deg_{\partial\omega_j} A(\mathrm{Tr}\,M) \in 2\mathbb{Z}$, where the degree (for a $H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ map) $\deg_{\partial\omega_j} A(\mathrm{Tr}\,M)$ is defined by (2.1.5) in Theorem 2.1.2. In other words, given a line field M on a loop, then to it corresponds a vector field $A(M)$ that has even degree if and only if M is orientable. It is also proved that, if there exists a unit vector field $\mathbf{m}$ such that $\mathrm{Tr}\,\mathbf{m} \in H^{1/2}(\partial\omega_j; \mathbb{S}^1)$, and $\mathrm{Tr}\,M = \mathbf{m} \otimes \mathbf{m}$ a.e. on $\partial\omega_j$, then $\deg_{\partial\omega_j}\mathbf{m} = \frac{1}{2}\deg_{\partial\omega_j} A(\mathrm{Tr}\,M)$.

Further, [BZ11, Proposition 7] also implies the fact that the orientability of the given vector field M in the whole domain Ω can be determined by the orientability on each of the boundaries $\cup_{j=1}^{N}\partial\omega_j$, i.e. $M \in \widetilde{X}$ is orientable if and only if $\deg_{\partial\omega_j} A(M) \in 2\mathbb{Z}$, $j = 1, ..., N$. Therefore the orientability of a given line field M can be determined at the level of the auxiliary vector field $A(M)$ on each of the boundaries $\cup_{j=1}^{N}\partial\omega_j$, and the line field M is non-orientable if and only if at least one of the degrees of the auxiliary vector field on the boundary, i.e. $\deg_{\partial\omega_j} A(M)$, $j = 1, ..., N$, is odd. Thus we can define the degree of the given line field $M \in \widetilde{X}$ on each of the boundaries $\cup_{j=1}^{N}\partial\omega_j$ as follows:

Definition 5.1.2. *In the admissible set*

$$\widetilde{X} = \{M \in \widetilde{H}^1(\Omega; \mathbb{Q}) \,|\, \mathrm{Tr}\,M|_{\partial\omega_j} = M_j,\ j = 1, ..., N\},$$

the degree of a given line field $M \in \widetilde{X}$ *on each boundary* $\partial\omega_j$, $j = 1, ..., N$ *is defined as*

$$\deg_{\partial\omega_j} M = \frac{1}{2}\deg_{\partial\omega_j} A(\mathrm{Tr}\,M) = \frac{1}{2}\deg_{\partial\omega_j} A(M_j) \tag{5.1.5}$$

where $A(M_j)$ *defined in (5.1.3) is the auxiliary vector field associated to the given line field* M_j *on the boundary* $\partial\omega_j$, $j = 1, ..., N$.

One can notice from Definition 5.1.2 that if the line field $M \in \widetilde{X}$ is orientable on the boundary $\partial\omega_j$, $j = 1, ..., N$, that is, there exists a unit vector field $\mathbf{m} \in H^{1/2}(\partial\omega_j; \mathbb{S}^1)$ such that $M = \mathbf{m} \otimes \mathbf{m}$, then the degree of the line field M on the boundary is in fact the degree of the vector field $\mathbf{m}$ on the boundary $\partial\omega_j$, i.e. $\deg_{\partial\omega_j} M = \deg_{\partial\omega_j}\mathbf{m}$, which follows from Proposition 2.2.4 that $\deg_{\partial\omega_j}\mathbf{m}^2 = 2\deg_{\partial\omega_j}\mathbf{m}$; and if the line field M is non-orientable on the boundary $\partial\omega_j$, $j = 1, ..., N$, then the degree of the line field on the boundary is half of some odd integer.

5.2 Homotopy classes of line fields

We assume that on each boundary $\partial\omega_j$, the line field M is given by $\mathrm{Tr}\,M|_{\partial\omega_j} = M_j \in H^{1/2}(\partial\omega_j; \mathbb{Q})$ and has prescribed degree $\deg_{\partial\omega_j} M_j = d_j$, which is defined in Definition 5.1.2.

Let

$$\widetilde{X} = \{M \in \widetilde{H}^1(\Omega; \mathbb{Q}) \,|\, \mathrm{Tr}\,M|_{\partial\omega_j} = M_j,\ j = 1, ..., N\}$$

be the collection of line fields in $\widetilde{X}$ satisfying the given boundary conditions.

Definition 5.2.1. *Assume* M, $\widetilde{M}$ *are two given line fields in the admissible set* $\widetilde{X}$. M *and* $\widetilde{M}$ *are homotopic if there exists a mapping* $M(\lambda, \cdot)$, *such that* $[0,1] \times \Omega \mapsto \widetilde{X}$ *is continuous in* $\widetilde{H}^1(\Omega; \mathbb{Q})$, *where* $M(\lambda, \cdot)$ *satisfies* $M(0, \cdot) = M$ *and* $M(1, \cdot) = \widetilde{M}$. *All the line fields in* $\widetilde{X}$ *that are homotopic to each other are said to be in the same homotopy class.*

We can characterize the homotopy classes as follows:

Proposition 5.2.2. *The admissible set*

$$\widetilde{X} = \{M \in \widetilde{H}^1(\Omega; \mathbb{Q}) \,|\, \mathrm{Tr}\,M|_{\partial\omega_j} = M_j,\ j = 1, ..., N\}$$

is the union of homotopy classes $C_{\mathbf{p}}$, *where* $\mathbf{p} \in \{p_2, p_3, ...p_N\} \in P \cong \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{N\text{-}1} = \mathbb{Z}^{N-1}$. *Each homotopy class* $C_{\mathbf{p}}$ *consists of the line fields*

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + n_1 & n_2 \\ n_2 & 1 - n_1 \end{pmatrix} \tag{5.2.1}$$

where $\mathbf{n} = n_1 + i\,n_2 = \Pi_{j=1}^N h_j^{2d_j}(z)\exp(i\phi)$ *with* ϕ *in the set:*

$$A_\phi^{\mathbf{p}} = \{\phi \in \widetilde{H}^1(\Omega) \,|\, \phi|_{\partial\omega_1} = \phi_1; \phi|_{\partial\omega_j} = \phi_j + 2\,p_j\pi,\ j = 2, ..., N\}, \tag{5.2.2}$$

where ϕ_j *satisfies* $\exp(i\phi_j) = \Pi_{i=1}^N h_i^{-2d_i}(z)\,A(M_j)$ *(see (2.3.3) in Proposition 2.3.2), for all* $j \in \{1, 2, ..., N\}$*).*

Remark 5.2.3. *Note that without loss of generality, we can assume that,* $⨍_{\partial\omega_j} \phi_j\, d\gamma \in [0, 2\pi)$, *for all* $j \in \{1, ..., N\}$, *i.e. the average of line integral for each* ϕ_j *on each of the corresponding boundary* $\partial\omega_j$ *is in* $[0, 2\pi)$.

Proof. For any given $M \in \widetilde{X}$, we have by (5.1.3) an auxiliary vector field A(M), and by Definition 5.1.2, we have

$$\deg_{\partial\omega_j} M = \frac{1}{2}\deg_{\partial\omega_j} A(M) = d_j.$$

Thus any auxiliary vector field A(M) associated to the given line field $M \in \widetilde{X}$ is in the admissible set:

$$A_M = \{\mathbf{n} \in \widetilde{H}^1(\Omega; \mathbb{S}^1) \,|\, \mathbf{n}|_{\partial\omega_j} = A(M_j),\ j = 1, ..., N\}, \tag{5.2.3}$$

where $A(M_j) \in H^{1/2}(\partial\omega_j, \mathbb{S}^1)$ and $\deg_{\partial\omega_j}\mathbf{n} = 2d_j$. By applying Proposition 4.2.3 we know that the admissible set A_M splits into homotopy classes $\{\widetilde{C}_\mathbf{p}\}$, where $\mathbf{p} \in \{p_2, p_3, ..., p_N\} \in P \cong \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{\text{N-1}} = \mathbb{Z}^{N-1}$, which is dependent on the lifting of $A(M_j)$ on each of the boundaries $\partial\omega_j$, for all $j \in \{1, ..., N\}$. More specifically, each homotopy class $\widetilde{C}_\mathbf{p}$ consists of vector fields $\mathbf{n} = n_1 + in_2 = \Pi_{j=1}^N h_j^{2d_j}(z)\exp(i\phi)$, where ϕ is in the set

$$A_\phi^\mathbf{p} = \{\phi \in \widetilde{H}^1(\Omega) \,|\, \phi|_{\partial\omega_1} = \phi_1; \phi|_{\partial\omega_j} = \phi_j + 2\,p_j\pi,\ j = 2, ..., N\}.$$

But from (5.2.1) we have that

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + n_1 & n_2 \\ n_2 & 1 - n_1 \end{pmatrix}$$

and

$$\widetilde{M} = \begin{pmatrix} \widetilde{M}_{11} & \widetilde{M}_{12} \\ \widetilde{M}_{21} & \widetilde{M}_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + \widetilde{n}_1 & \widetilde{n}_2 \\ \widetilde{n}_2 & 1 - \widetilde{n}_1 \end{pmatrix}$$

from which it follows that M and $\widetilde{M}$ are homotopic if and only if $\mathbf{n}$ and $\widetilde{\mathbf{n}}$, completing the proof.

□

5.3 Main theorem

With the definition of degree and homotopy class for line fields, we can consider the minimization problem for the functional

$$I(M) = \int_\Omega \left(|\nabla M|^2 - \frac{2k^2}{|x|^2}\right) dx, \tag{5.3.1}$$

where $\mathbf{M}$ is in the admissible set

$$\widetilde{X} = \{M \in \widetilde{H}^1(\Omega; \mathbb{Q}) \,|\, \mathrm{Tr}\, M|_{\partial\omega_j} = M_j,\ j = 1, ..., N\}$$

and k is the sum of the degree on the boundary, i.e. $k = \Sigma_{j=1}^N d_j$.

Theorem 5.3.1. *There exists a unique minimizer* $M_{\mathbf{p}}$ *for the energy functional*

$$I(M) = \int_\Omega \left(|\nabla M|^2 - \frac{2k^2}{|x|^2} \right) dx$$

in each homotopy class $C_{\mathbf{p}}$ *of the admissible set*

$$\widetilde{X} = \{M \in \widetilde{H}^1(\Omega; \mathbb{Q}) \,|\, \mathrm{Tr}\, M|_{\partial\omega_j} = M_j,\ j = 1, ..., N\},$$

where $C_{\mathbf{p}}$ *is defined in Proposition* 5.2.2, *and* k *is the sum of the degree for each line field* M_j *on the boundary* $\partial\omega_j$*, i.e.* $k = \Sigma_{j=1}^N d_j$*. Also, in each homotopy class* $C_{\mathbf{p}}$*, the minimizer* $M_{\mathbf{p}}$ *is smooth, i.e.* $M_{\mathbf{p}} \in C^\infty(\Omega; \mathbb{Q})$*, and can be written as*

$$M_{\mathbf{p}} = \begin{pmatrix} M_{11}^{(\mathbf{p})} & M_{12}^{(\mathbf{p})} \\ M_{21}^{(\mathbf{p})} & M_{22}^{(\mathbf{p})} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + n_1^{(\mathbf{p})} & n_2^{(\mathbf{p})} \\ n_2^{(\mathbf{p})} & 1 - n_1^{(\mathbf{p})} \end{pmatrix}$$

where $\mathbf{n_p} = n_1^{(\mathbf{p})} + i\, n_2^{(\mathbf{p})} = \Pi_{j=1}^N h_j^{2d_j}(z) \exp(i\phi_{\mathbf{p}})$*, and* $\phi_{\mathbf{p}}$ *is in the set*

$$A_\phi^{\mathbf{p}} = \{\phi \in \widetilde{H}^1(\Omega) \,|\, \phi|_{\partial\omega_1} = \phi_1; \phi|_{\partial\omega_j} = \phi_j + 2\, p_j \pi,\ j = 2, ..., N\}.$$

Moreover, $\phi_{\mathbf{p}}$ *is smooth and satisfies:*

$$\|\phi_{\mathbf{p}}(x) - \beta\|_{L^2(\mathbb{S}^1)} \leq C_0 \frac{1}{r}$$

for some constant β*,* C_0*. In addition, there exists a minimizing line field* $\widetilde{M}$ *for* $I(M)$ *in the collection of all the homotopy classes* $C_{\mathbf{p}}$, i.e. $\widetilde{X} = \bigcup_{\mathbf{p}\in\mathbb{Z}^{N-1}} C_{\mathbf{p}}$. *However* $\widetilde{M}$ *may not necessarily be unique. Further, the minimizer* $M_{\mathbf{p}}$ *is also the unique weak equilibrium solution of* $I(M)$ *in each homotopy class* $C_{\mathbf{p}}$ *such that the corresponding* $\phi_{\mathbf{p}}$ *in the expression of the associated auxiliary vector field* $\mathbf{n_p} = \Pi_{j=1}^N h_j^{2d_j}(z) \exp(i\phi_{\mathbf{p}})$ *satisfies*

$$\phi_{\mathbf{p}}(x) \in L^\infty((R, \infty) : L^2(\mathbb{S}^1)),$$

and there exists a weak equilibrium M^* *in each homotopy class* $C_{\mathbf{p}}$, *such that*

$$I(M^*) = \int_\Omega (|\nabla M^*|^2 - \frac{2k^2}{|x|^2})\, dx = +\infty.$$

Remark 5.3.2. *Note that* ϕ_j *satisfies* $\exp(i\phi_j) = \Pi_{i=1}^N h_i^{-2d_i}(z)\, A(M_j)$, *and without loss of generality, we can assume* $\fint_{\partial\omega_j} \phi_j\, d\gamma \in [0, 2\pi)$.

Proof. Let $\mathbf{n} = A(M)$. Since by (5.1.4)

$$\begin{cases} M_{11} + M_{22} = 1 \\ M_{12} = M_{21} \end{cases}$$

we have that

$$\begin{aligned} |\nabla \mathbf{n}|^2 &= n_{1,1}^2 + n_{2,1}^2 + n_{1,2}^2 + n_{2,2}^2 \\ &= (2M_{11,1})^2 + (2M_{12,1})^2 + (2M_{11,2})^2 + (2M_{12,2})^2 \qquad (5.3.2) \\ &= 2\,|\nabla M|^2. \end{aligned}$$

Therefore

$$I(M) = \widetilde{I}(\mathbf{n}) = \frac{1}{2}\int_\Omega \left(|\nabla \mathbf{n}|^2 - \frac{\widetilde{k}^2}{|x|^2}\right) dx,$$

where $\widetilde{k} = \Sigma_{j=1}^N 2\, d_j$, i.e. the sum of the degrees for the auxiliary vector field associated to the line field M on each of the boundaries $\partial\omega_j$.

Thus, if there exists a minimizer for

$$I(M) = \int_\Omega \left(|\nabla M|^2 - \frac{2k^2}{|x|^2}\right) dx,$$

in the admissible set $\widetilde{X}$, then the associated auxiliary vector field $\mathbf{n} = A(M)$, which is defined in (5.1.3), is also a minimizer for the functional

$$\widetilde{I}(\mathbf{n}) = \frac{1}{2}\int_\Omega \left(|\nabla \mathbf{n}|^2 - \frac{\widetilde{k}^2}{|x|^2}\right) dx,$$

where $\mathbf{n}$ is in the admissible set

$$A_M = \{\mathbf{n} \in \widetilde{H}^1(\Omega; \mathbb{S}^1)\,|\,\mathbf{n}|_{\partial\omega_j} = A(M_j),\ j = 1, ..., N\},$$

where $A(M_j)$ is the auxiliary vector field associated to the boundary line field M_j, and satisfies $A(M_j) \in H^{1/2}(\partial\omega_j, \mathbb{S}^1)$, $\deg_{\partial\omega_j}\mathbf{n} = 2d_j$. It is also easy to see that the converse is true, since (5.2.1) implies the bijective correspondence between the given line field $M \in \widetilde{X}$ and the associated auxiliary vector field $\mathbf{n} = A(M) \in A_M$, where A_M is defined in (5.2.3). Hence given the minimization problem in the admissible set $\widetilde{X}$, we can consider instead the minimization problem for the associated auxiliary vector field $A(M)$ in the admissible set A_M, as the two minimization problems are equivalent.

Recall that from the proof of Proposition 5.2.2 that the admissible set A_M splits into homotopy classes $\{\widetilde{C}_{\mathbf{p}}\}$, where $\mathbf{p} \in \{p_2, p_3, ..., p_N\} \in P \cong \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{N-1} = \mathbb{Z}^{N-1}$, which depends on the lifting of $A(M_j)$ on each of the boundaries $\partial\omega_j$, for all $j \in \{1, ..., N\}$. Therefore by applying Theorem 4.3.2, in each homotopy class $\widetilde{C}_{\mathbf{p}}$ of the admissible set A_M, there exists a unique minimizer $\mathbf{n}_p$ for the functional

$$\widetilde{I}(\mathbf{n}) = \frac{1}{2}\int_\Omega \left(|\nabla\mathbf{n}|^2 - \frac{\widetilde{k}^2}{|x|^2}\right) dx,$$

which is smooth and can be written as $\mathbf{n}_p(z) = \Pi_{j=1}^N h_j^{2d_j}(z)\exp(i\phi_p)$. Also $\phi_{\mathbf{p}}$ is smooth and satisfies

$$\|\phi_{\mathbf{p}}(x) - \beta\|_{L^2(\mathbb{S}^1)} \leq C_0\,\frac{1}{r}$$

for some constant β, C_0.

Moreover, we know from Proposition 5.2.2 that each homotopy class $\widetilde{C}_{\mathbf{p}}$ of the admissible set A_M corresponds uniquely to the homotopy class $C_{\mathbf{p}}$ of the admissible set $\widetilde{X}$. Thus by (5.2.1) we know that the corresponding line field $M_{\mathbf{p}}$ is in fact the minimizer for the functional $I(M)$ and it is smooth, i.e. $M_{\mathbf{p}} \in C^\infty(\Omega; \mathbb{Q})$.

The fact that there exists a minimizing line field $\widetilde{M}$ in the union $\widetilde{X}$ of all the homotopy classes $C_{\mathbf{p}}$, i.e. $\widetilde{X} = \cup_{p\in\mathbb{Z}^{N-1}} C_{\mathbf{p}}$ follows directly from the existence of a minimizer $\widetilde{\mathbf{n}}$ for the functional

$$\widetilde{I}(\mathbf{n}) = \int_\Omega \left(|\nabla\mathbf{n}|^2 - \frac{\widetilde{k}^2}{|x|^2}\right) dx$$

in the admissible set $A_M = \{\mathbf{n} \in \widetilde{H}^1(\Omega; \mathbb{S}^1) \,|\, \mathbf{n}|_{\partial\omega_j} = A(M_j),\ j = 1, ..., N\}$, which is also the union of all the homotopy classes $\widetilde{C}_{\mathbf{p}}$. The existence of such $\widetilde{\mathbf{n}}$ follows immediately from Theorem

4.3.2, and we have

$$\widetilde{M} = \frac{1}{2}\begin{pmatrix} 1+\widetilde{n}_1 & \widetilde{n}_2 \\ \widetilde{n}_2 & 1-\widetilde{n}_1 \end{pmatrix}$$

where $\widetilde{\mathbf{n}} = \widetilde{n}_1 + i\,\widetilde{n}_2$. Note that in general the minimum $\widetilde{M}$ is not unique, see Remark 5.3.3.

Now what left to prove are the existence of weak equilibrium solutions M^* in each homotopy class $C_{\mathbf{p}}$, such that

$$I(M^*) = \int_\Omega \left(|\nabla M^*|^2 - \frac{2k^2}{|x|^2} \right) dx = +\infty;$$

and that if the $\phi_{\mathbf{p}}$ in $\mathbf{n}_{\mathbf{p}} = \Pi_{j=1}^N h_j^{2d_j}(z)\exp(i\phi_{\mathbf{p}})$ satisfies $\phi_{\mathbf{p}}(x) \in L^\infty((R,\infty) : L^2(\mathbb{S}^1))$, then the minimizer $M_{\mathbf{p}}$ of each homotopy class $C_{\mathbf{p}}$ is the unique weak equilibrium solution of $I(M)$. We show that if $M \in \widetilde{X}$ is a weak equilibrium solution of $I(M)$, then its associated auxiliary vector field $\mathbf{n} \in A_M$ is a weak equilibrium solution (i.e. a harmonic map) to $\widetilde{I}(\mathbf{n})$, and vice versa.

Let $M = \mathbf{m} \otimes \mathbf{m}$ be a equilibrium solution of the functional

$$I(M) = \int_\Omega \left(|\nabla M|^2 - \frac{2k^2}{|x|^2} \right) dx = \int_\Omega \left(|\nabla(\mathbf{m}\otimes\mathbf{m})|^2 - \frac{2k^2}{|x|^2} \right) dx,$$

The analogue of (4.3.1) is obtained by setting

$$\frac{d}{d\epsilon} \int_\Omega \left(\left| \nabla \left[\frac{\mathbf{m} + \epsilon \mathbf{m}^\perp \Psi|}{|\mathbf{m} + \epsilon \mathbf{m}^\perp \Psi|} \otimes \frac{\mathbf{m} + \epsilon\, \mathbf{m}^\perp \Psi|}{|\mathbf{m} + \epsilon\, \mathbf{m}^\perp \Psi|} \right] \right|^2 - \frac{2k^2}{|x|^2} \right) dx \Bigg|_{\epsilon=0} = 0,$$

where $\Psi \in H_0^1(\Omega)$, and so we obtain

$$\int_\Omega \nabla M \cdot \nabla \left([\mathbf{m} \otimes \mathbf{m}^\perp + \mathbf{m}^\perp \otimes \mathbf{m}]\Psi \right) dx = 0.$$

Recall that the auxiliary vector field $\mathbf{n} = A(M)$ associated to the given line field M is given by (5.1.4), and hence we have

$$\mathbf{m} \otimes \mathbf{m}^\perp + \mathbf{m}^\perp \otimes \mathbf{m} = \begin{pmatrix} -2m_1m_2 & m_1^2 - m_2^2 \\ m_1^2 - m_2^2 & 2m_1m_2 \end{pmatrix} = \begin{pmatrix} -n_2 & n_1 \\ n_1 & n_2 \end{pmatrix}.$$

Hence

$$\int_\Omega (\nabla n_2 \cdot \nabla(n_1\Psi) - \nabla n_1 \cdot \nabla(n_2\Psi))\, dx = 0,$$

i.e.

$$\int_{\Omega} \nabla \mathbf{n} \cdot \nabla(\mathbf{n}^{\perp}\Psi)\, dx = 0,$$

which as we have seen from the equivalence between (4.3.3) and (4.3.4), is equivalent to

$$\int_{\Omega} \nabla \mathbf{n} \cdot \nabla \mathbf{v}\, dx = \int_{\Omega} |\mathbf{n}|^2\, \mathbf{n} \cdot \mathbf{v}\, dx = 0 \quad \forall \mathbf{v} \in H_0^1(\Omega; \mathbb{R}^2). \tag{5.3.3}$$

Thus we see from Proposition 4.3.1 that the weak form (5.3.3) is equivalent to

$$\int_{\Omega} \nabla \phi \cdot \nabla \Psi\, dx = 0 \quad \forall \Psi \in H_0^1(\Omega)$$

where ϕ corresponds to the auxiliary vector field $A(M) = \mathbf{m}^2 = \mathbf{n}(z) = \Pi_{j=1}^{N} h_j^{2d_j}(z)\, e^{i\phi}$.

Thus by applying Theorem 4.3.2, we know that in each homotopy class $C_{\mathbf{p}}$, the minimizing line field $M_{\mathbf{p}} \in \widetilde{X}$ is in fact the unique weak equilibrium solution to the functional $I(M)$ such that the associated $\phi_{\mathbf{p}}$ satisfies

$$\phi_{\mathbf{p}}(x) \in L^{\infty}((R, \infty) : L^2(\mathbb{S}^1)),$$

since the associated auxiliary vector field $\mathbf{n}_{\mathbf{p}}$ is the unique harmonic map in each homotopy class $\widetilde{C}_{\mathbf{p}}$ such that the corresponding $\phi_{\mathbf{p}}$ in $\mathbf{n}_{\mathbf{p}} = \Pi_{j=1}^{N} h_j^{2d_j}(z) \exp(i\phi_{\mathbf{p}})$ satisfies

$$\phi_{\mathbf{p}}(x) \in L^{\infty}((R, \infty) : L^2(\mathbb{S}^1)).$$

Finally, the existence of a weak equilibrium solution M^* in each homotopy class $C_{\mathbf{p}}$ such that $I(M^*) = +\infty$ follows exactly the same reasoning and this completes our proof.

□

Remark 5.3.3. *The minimizer* $\widetilde{M}$ *in the union* $\widetilde{X}$ *of all the homotopy classes is in general not unique, which is illustrated in Figure 5.1. The line field* $M = \mathbf{m} \otimes \mathbf{m}$ *given in Figure 5.1 has an auxiliary vector field which is identical to the counterexample given in Example 4.1.1. Hence by applying Example 4.1.1 to the auxiliary vector field corresponding to the given line field in Figure 5.1, we have the nonuniqueness in the minimization problem for the auxiliary vector fields, and therefore the line field.*

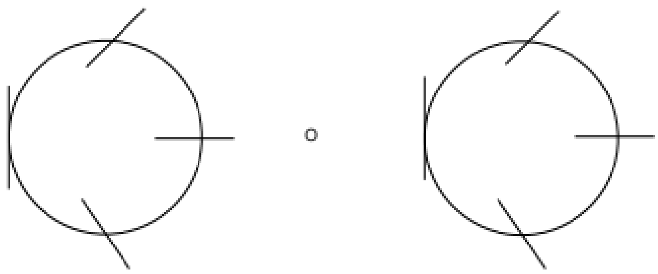

Figure 5.1: Let ω_1 and ω_2 be discs of radius $\frac{1}{4}$ with centres $(-1,0)$, $(1,0)$ respectively. The line field $M = \mathbf{m}\otimes\mathbf{m}$ is given by $\mathbf{m}|_{\partial\omega_1} = \mathbf{m}(-1+\frac{1}{4}\cos\theta, \frac{1}{4}\sin\theta) = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2})$, $\mathbf{m}|_{\partial\omega_2} = \mathbf{m}(1+\frac{1}{4}\cos\theta, \frac{1}{4}\sin\theta) = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2})$.

Theorem 5.3.1 applies to both orientable and non-orientable line fields. By introducing the auxiliary vector field $A(M)$ for any given line field $M \in \widetilde{X}$, the orientability on each boundary $\partial\omega_j$ depends on whether $\deg_{\partial\omega_j} M = d_j$ is an integer or not. As can be seen from Definition 5.1.2, the line field $M \in \widetilde{X}$ is orientable on the boundaries $\partial\omega_j$ if and only if d_j is an integer, and $M \in \widetilde{X}$ is non-orientable on the boundary $\partial\omega_j$ if and only if d_j is half of some odd integer. More specifically, if the given line field $M \in \widetilde{X}$ is orientable on each of the boundary $\partial\omega_j$, $\forall j \in \{1, ..., N\}$, i.e. there exists a vector field $\mathbf{m}_j \in H^{1/2}(\partial\omega_j; \mathbb{S}^1)$ such that $\mathrm{Tr}\, M|_{\partial\omega_j} = \mathbf{m}_j \otimes \mathbf{m}_j$, $\forall j \in \{1, ..., N\}$, then the degree of the line field M on the boundary is in fact the degree of the vector field $\mathbf{m}$ on the boundary $\partial\omega_j$, i.e. $\deg_{\partial\omega_j} M = \deg_{\partial\omega_j} \mathbf{m}$. Thus one can sees that if the line field $M \in \widetilde{X}$ is orientable on every $\partial\omega_j$ ($\forall j \in \{1, ..., N\}$), then Theorem 4.3.2 and Theorem 5.3.1 are in fact equivalent. On the other hand, when $M \in \widetilde{X}$ is non-orientable, i.e. there exist at least a $æ_0 \in \{1, ..., N\}$ such that d_{j_0} is not an integer, then we can apply Theorem 5.3.1 and recover the minimizing line field through the minimizing auxiliary vector field $\mathbf{n}_p$ in each homotopy class C_p ($p \in \mathbb{Z}^{N-1}$).

Chapter 6

Non-equal constant case

In this chapter, we study the minimization problem in the non-equal constant case for the energy functional, obtained by integrating the Oseen-Frank energy density (1.2.6) in a given domain Ω:

$$
\begin{aligned}
I(\mathbf{n}) &= \int_\Omega F(x, \mathbf{n}, \nabla\mathbf{n})\, dx \\
&= \frac{1}{2}\int_\Omega \Big(k_3|\nabla\mathbf{n}|^2 + (k_1 - k_3)(\operatorname{div}\mathbf{n})^2 + (k_2 - k_3)(\mathbf{n}\cdot\nabla\wedge\mathbf{n})^2 \\
&\quad + (k_2 + k_4 - k_3)(\operatorname{tr}(\nabla\mathbf{n})^2) - (\operatorname{div}\mathbf{n})^2)\Big)\, dx,
\end{aligned}
$$

where $\Omega = \{x \in \mathbb{R}^2 \,|\, 0 < a < |x| < b < \infty\}$ is an annulus. Note that the last term containing $\operatorname{tr}(\nabla\mathbf{n})^2 - (\operatorname{div}\mathbf{n})^2$ is zero since $\operatorname{tr}(\nabla\mathbf{n})^2 - (\operatorname{div}\mathbf{n})^2 = 0$ in some two-dimensional case as discussed previously in (1.2.4). Thus the last term neither contributes to the free energy nor the corresponding Euler-Lagrange equation. Also the term $(\mathbf{n}\cdot\nabla\wedge\mathbf{n})^2$ is zero for our two-dimensional case and, hence we only need to consider the contributions of the first two terms.

Recall that by (3.0.6), $\mathbf{n}$ can be identified with $\mathbf{n} = (\cos\Phi(r,\theta), \sin\Phi(r,\theta))$, and we have by (3.0.8):

$$
\begin{aligned}
F(x, \mathbf{n}, \nabla\mathbf{n}) &= \frac{1}{2}\Big(k_3|\nabla\mathbf{n}|^2 + (k_1 - k_3)(\operatorname{div}\mathbf{n})^2 + (k_2 - k_3)(\mathbf{n}\cdot\nabla\wedge\mathbf{n})^2\Big) \\
&= \frac{1}{2}k_3\left(\Phi_r^2 + \frac{\Phi_\theta^2}{r^2}\right) + \frac{1}{2}(k_1 - k_3)\Big(\Phi_r^2\sin^2(\Phi - \theta) \\
&\quad + \frac{\Phi_\theta^2}{r^2}\cos^2(\Phi - \theta) - \frac{2\Phi_\theta\Phi_r}{r}\sin(\Phi - \theta)\cos(\Phi - \theta)\Big).
\end{aligned}
$$

Again, by making the change of variable

$$r = e^s$$

we have $\Phi_s = r\,\Phi_r$, and as a result, the Oseen-Frank bulk energy (3.0.9) becomes:

$$\begin{aligned} I(\mathbf{n}) &= \int_a^b r \int_0^{2\pi} F(x, \mathbf{n}, \nabla\mathbf{n})\, d\theta\, dr \\ &= \frac{1}{2}\int_{\log a}^{\log b}\int_0^{2\pi} \Big(k_3(\Phi_s^2 + \Phi_\theta^2) + (k_1 - k_3)(\Phi_s^2 \sin^2(\Phi - \theta) \\ &\quad + \Phi_\theta^2 \cos^2(\Phi - \theta) - 2\Phi_s\Phi_\theta \sin(\Phi - \theta)\cos(\Phi - \theta))\Big)\, ds\, d\theta \\ &= \frac{1}{4}(k_1 + k_3)\, J(\Phi) \end{aligned} \tag{6.0.1}$$

where

$$J(\Phi) = \int_{\log a}^{\log b}\int_0^{2\pi} \Big((1 + \lambda\cos 2(\Phi - \theta))\,\Phi_s^2 + (1 - \lambda\cos 2(\Phi - \theta))\,\Phi_\theta^2 + 2\lambda\Phi_s\Phi_\theta \sin 2(\Phi - \theta)\Big)\, ds\, d\theta,$$

and

$$\lambda = \frac{k_3 - k_1}{k_1 + k_3}.$$

We now make another change of variable:

$$\phi(s, \theta) = \Phi(s, \theta) - \theta,$$

so that

$$\begin{aligned} \Phi_\theta &= \phi_\theta + 1, \\ \Phi_{\theta\theta} &= \phi_{\theta\theta}, \\ \Phi_s &= \phi_s, \\ \Phi_{ss} &= \phi_{ss}, \\ \Phi_{\theta s} &= \phi_{\theta s}. \end{aligned}$$

With this change of variable, $F(x, \mathbf{n}, \nabla\mathbf{n})$ can be written as

$$\begin{aligned} F(x, \mathbf{n}, \nabla\mathbf{n}) &= W(s, \theta, \phi, \phi_s, \phi_\theta) \\ &= \frac{1}{4}(k_1 + k_3)\,\Big((1 + \lambda\cos 2\phi)\phi_s^2 + (1 - \lambda\cos 2\phi)(1 + \phi_\theta)^2 + 2\lambda\phi_s(\phi_\theta + 1)\sin 2\phi\Big) \end{aligned} \tag{6.0.2}$$

and (6.0.1) becomes:

$$\begin{aligned} \mathrm{I}(\mathbf{n}) &= \frac{1}{4}(\mathrm{k}_1 + \mathrm{k}_3) \int_{\log a}^{\log b} \int_0^{2\pi} \Big((1 + \lambda \cos 2\phi)\phi_s^2 + (1 - \lambda \cos 2\phi)(1 + \phi_\theta)^2 \\ &\quad + 2\lambda \phi_s(\phi_\theta + 1) \sin 2\phi \Big) \, \mathrm{ds} \, \mathrm{d}\theta \\ &= \frac{1}{4}(\mathrm{k}_1 + \mathrm{k}_3) \widetilde{\mathrm{J}}(\phi) \end{aligned} \tag{6.0.3}$$

where

$$\lambda = \frac{\mathrm{k}_3 - \mathrm{k}_1}{\mathrm{k}_1 + \mathrm{k}_3}, \quad \mathrm{k}_1 \geq 0, \quad \mathrm{k}_3 \geq 0,$$

and

$$\widetilde{\mathrm{J}}(\phi) = \int_{\log a}^{\log b} \int_0^{2\pi} \Big((1 + \lambda \cos 2\phi)\phi_s^2 + (1 - \lambda \cos 2\phi)(1 + \phi_\theta)^2 + 2\lambda \phi_s(\phi_\theta + 1) \sin 2\phi \Big) \, \mathrm{ds} \, \mathrm{d}\theta.$$

By substituting (6.0.2) into the Euler-Lagrange equation for $\mathrm{W}(\mathrm{s}, \theta, \phi, \phi_s, \phi_\theta)$:

$$\frac{\partial \mathrm{W}}{\partial \phi} = \frac{\partial}{\partial \mathrm{s}} \frac{\partial \mathrm{W}}{\partial \phi_s} + \frac{\partial}{\partial \theta} \frac{\partial \mathrm{W}}{\partial \phi_\theta},$$

we have

$$\begin{aligned} &(1 - \lambda \cos 2\phi)\, \phi_{\theta\theta} + \lambda(\phi_\theta^2 - 1) \sin 2\phi + (1 + \lambda \cos 2\phi)\, \phi_{ss} + 2\lambda\, \phi_{\theta s} \sin 2\phi \\ &- \lambda \phi_s^2 \sin 2\phi + 2\lambda\, \phi_s\, \phi_\theta \cos 2\phi = 0. \end{aligned} \tag{6.0.4}$$

Furthermore, in one-constant case it is easy to see that any constant rotation, or rotation plus reflection, of an equilibrium solution is also an equilibrium as $\lambda = 0$. While this does not hold for non-equal constant case. Now we present an example showing that in 2-dimensional domain, a rigid rotation applied to the director field of an equilibrium for the Oseen-Frank energy need not be an equilibrium.

Again, we use polar coordinates $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2) = (\mathrm{r}, \theta)$ and consider the radial hedgehog

$$\hat{\mathbf{n}} = (\cos \theta, \sin \theta).$$

Rotating by angle α we have

$$\mathbf{n}^\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = \big(\cos(\theta + \alpha), \sin(\theta + \alpha) \big)^{\mathrm{T}}.$$

One can also see that

$$\operatorname{div} \mathbf{n}^\alpha = \frac{\cos \alpha}{\mathrm{r}}$$

and the Euler-Lagrange equation for the energy $\int_\Omega (\operatorname{div} \mathbf{n}^\alpha)^2\,\mathrm{dx}$ is

$$\nabla \operatorname{div} \mathbf{n}^\alpha = (\mathbf{n}^\alpha \cdot \nabla \operatorname{div} \mathbf{n}^\alpha)\, \mathbf{n}^\alpha. \tag{6.0.5}$$

Since we have

$$\nabla \operatorname{div} \mathbf{n}^\alpha = \begin{pmatrix} \frac{\partial(\operatorname{div} \mathbf{n}^\alpha)}{\partial x_1} \\ \frac{\partial(\operatorname{div} \mathbf{n}^\alpha)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -\frac{\cos\alpha}{r^2}\frac{x_1}{r} \\ -\frac{\cos\alpha}{r^2}\frac{x_2}{r} \end{pmatrix}$$

and

$$\mathbf{n}^\alpha \cdot \nabla \operatorname{div} \mathbf{n}^\alpha = -\frac{\cos^2\alpha}{r^2},$$

the Euler-Lagrange equation (6.0.5) becomes

$$\begin{cases} \sin\alpha\cos\alpha\,(x_1\sin\alpha + x_2\cos\alpha) = 0 \\ \sin\alpha\cos\alpha\,(x_2\sin\alpha - x_1\cos\alpha) = 0. \end{cases}$$

Therefore $\mathbf{n}^\alpha$ is equilibrium solution to the energy $\mathbf{I}(\mathbf{n}) = \frac{1}{2}\int_\Omega (\operatorname{div} \mathbf{n})^2\,\mathrm{dx}$ if and only if

$$\sin^2\alpha\cos\alpha = \cos^2\alpha\sin\alpha = 0,$$

i.e. $\sin 2\alpha = 0$ or $\alpha = \frac{n\pi}{2}$, $n \in \mathbb{Z}$.

As a consequence, given the hedgehog $\hat{\mathbf{n}} = (\cos\theta, \sin\theta)$, which is also the equilibrium solution to the energy $\mathbf{I}(\mathbf{n}) = \frac{1}{2}\int_\Omega (\operatorname{div} \mathbf{n})^2\,\mathrm{dx}$, the possible equilibrium solutions to $\mathbf{I}(\mathbf{n})$ derived by rotations are (i) negative hedgehog, i.e. a hedgehog with all the director pointing inwards (ii) concentric circular vector field with anticlockwise orientation (iii) concentric circular vector field with clockwise orientation.

6.1 Radius-independent minimizers for $\int_\Omega F(x, \mathbf{n}, \nabla\mathbf{n})\,\mathrm{dx}$

If we assume that the vector field $\mathbf{n}(x) = (\cos\Phi(r,\theta), \sin\Phi(r,\theta))$ is radius-independent, i.e. $\Phi(r,\theta) = \Phi(\theta) = \Phi(\frac{x}{|x|})$, we have:

$$\phi_s = \phi_{ss} = \phi_{\theta s} = 0.$$

Substituting this back into the Euler-Lagrange equation (6.0.4), we have:

$$(1 - \lambda\cos 2\phi)\,\phi'' + \lambda\,((\phi')^2 - 1)\sin 2\phi = 0, \tag{6.1.1}$$

where

$$\lambda = \frac{k_3 - k_1}{k_1 + k_3}, \quad k_1 \geq 0, \quad k_3 \geq 0.$$

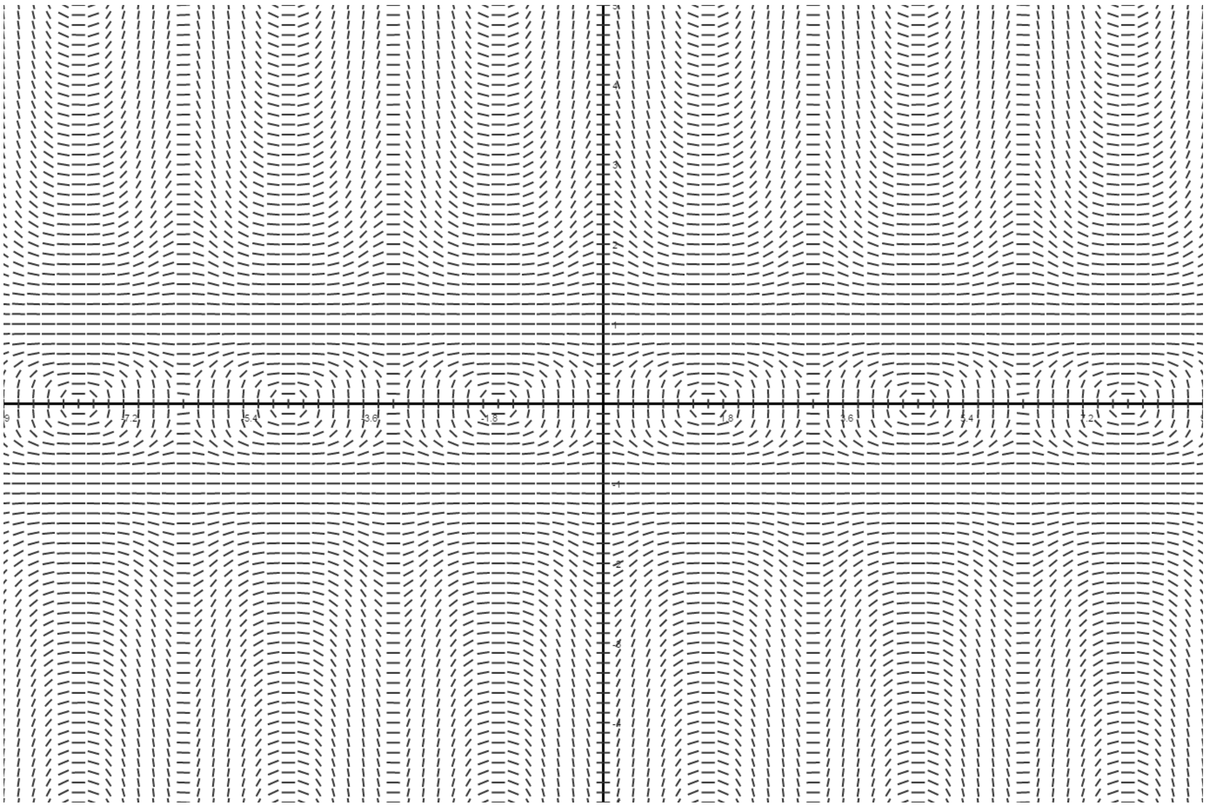

Figure 6.1: phase plane for equation (6.1.1) when $\lambda = 0.6$, $\phi \in (-10, 10)$ and $\phi_\theta \in (-5, 5)$.

In this section we consider the general case when $k_1 > 0$, $k_3 > 0$. One can easily see that $0 \leq |\lambda| < 1$ if $k_1 > 0$, $k_3 > 0$. Note that when $\mathbf{n}(x) = e^{i\Phi}$ is radius-independent, by letting $k_3 = 0$ and $k_1 = k_3$, one can easily recover the Euler-Lagrange equations for $\frac{1}{2}\int_D |\nabla \mathbf{n}|^2\,dx$ and $\frac{1}{2}\int_D (\operatorname{div}\mathbf{n})^2\,dx$ from (6.1.1), respectively. To help illustrate equation (6.1.1), we depict the phase plane for (6.1.1) in Figure 6.1. We can see that the stationary points on the phase plane are:

$$\begin{cases} \phi(\theta) = m\pi, \quad m \in \mathbb{Z} \\ \phi(\theta) = \dfrac{\pi}{2} + m\pi, \quad m \in \mathbb{Z} \end{cases} \tag{6.1.2}$$

Consider the annualus $\Omega = (a, b) \times \mathbb{S}^1$, $0 < a < b < \infty$, and let $k \in \mathbb{Z}$ be a given degree. We consider the problem of minimizing the functional

$$I(\mathbf{n}) = \frac{1}{2} \int_\Omega \left(k_3 |\nabla \mathbf{n}|^2 + (k_1 - k_3)(\operatorname{div} \mathbf{n})^2 \right) dx, \quad k_1 > 0,\ k_3 > 0$$

among $\mathbf{n}$ in the admissible set

$$A_k = \{\mathbf{n} \in H^1(\Omega; \mathbb{S}^1) \,|\, \deg \mathbf{n} = k\}$$

that depend only on $\theta \in \mathbb{S}^1$. Writing $\mathbf{n} = (\cos \Phi(\theta), \sin \Phi(\theta))$ this is equivalent to minimizing

$$J(\Phi) = \int_0^{2\pi} \Phi_\theta^2 \,(1 - \lambda \cos 2(\Phi - \theta))\, d\theta, \quad \lambda = \frac{k_3 - k_1}{k_1 + k_3}$$

in the set

$$A_k^\Phi = \{\Phi \in H^1(0, 2\pi) \,|\, \Phi(2\pi) - \Phi(0) = 2k\pi\},$$

or setting $\phi = \Phi - \theta$, to minimizing

$$\widetilde{J}(\phi) = \int_0^{2\pi} (\phi_\theta + 1)^2 (1 - \lambda \cos 2\phi)\, d\theta$$

in the set

$$A_k^\phi = \{\phi \in H^1(0, 2\pi) \,|\, \phi(2\pi) - \phi(0) = 2(k-1)\pi\}.$$

We have the following result.

Theorem 6.1.1. $I(\mathbf{n})$ *attains a minimum among* $\mathbf{n} \in A_k$ *that depend only on* $\theta \in \mathbb{S}^1$. *The minimizers* $\widetilde{\mathbf{n}}$ *are given by:*

(a) For $k = 0$, $\widetilde{\mathbf{n}} = (\cos \theta_0, \sin \theta_0)$ *for arbitrary* $\theta_0 \in [0, 2\pi)$.

(b) For $k = 1$,

if $k_1 > k_3$, $\widetilde{\mathbf{n}} = \pm \left(\cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2}) \right)$,

if $k_1 = k_3$, $\widetilde{\mathbf{n}} = (\cos(\theta + \theta_0), \sin(\theta + \theta_0))$ *for arbitrary* $\theta_0 \in [0, 2\pi)$,

if $k_1 < k_3$, $\widetilde{\mathbf{n}} = \pm (\cos \theta, \sin \theta)$.

(c) For $k = 2$, $\widetilde{\mathbf{n}} = (\cos 2(\theta + \theta_0), \sin 2(\theta + \theta_0))$ *for arbitrary* $\theta_0 \in [0, 2\pi)$.

(d) For $k > 2$, $\widetilde{\mathbf{n}} = (\cos(\phi_\alpha(\theta) + \theta), \sin(\phi_\alpha(\theta) + \theta))$, $\alpha \in \mathbb{R}^1$, *where* ϕ_α *is the unique solution of*

$$\phi'(\theta) = \sqrt{1 + \frac{c}{1 - \lambda \cos 2\phi}}$$

$$\phi(0) = \alpha$$

and c *is the unique constant such that*

$$\mathrm{F(c)} = \frac{\pi}{\mathrm{k}-1},$$
$$\mathrm{F(c)} = \int_0^{\pi} \frac{d\zeta}{\sqrt{1+\frac{c}{1-\lambda\cos 2\zeta}}}.$$

(e) For $\mathrm{k} < 1$, $\widetilde{\mathbf{n}} = (\cos(\phi_\alpha(\theta)+\theta), \sin(\phi_\alpha(\theta)+\theta))$, $\alpha \in \mathbb{R}^1$ *where* ϕ_α *is the unique solution of*

$$\phi'(\theta) = -\sqrt{1+\frac{c}{1-\lambda\cos 2\phi}}$$
$$\phi(0) = \alpha$$

and c *is the unique constant such that* $\mathrm{F(c)} = \frac{\pi}{1-\mathrm{k}}$.

Proof. Step 1

Let

$$\mathrm{G}(\theta, \Phi, \Phi_\theta) = \Phi_\theta^2 \left(1 - \lambda \cos 2(\Phi - \theta)\right).$$

Then since $|\lambda| < 1$, we have

$$\mathrm{G}(\theta, \Phi, \Phi_\theta) \geq (1 - |\lambda|)\Phi_\theta^2, \tag{6.1.3}$$

and

$$\mathrm{G}_{\Phi_\theta \Phi_\theta} \geq 2(1 - |\lambda|) > 0, \tag{6.1.4}$$

so that G is strictly convex in Φ_θ. Equation (6.1.4) holds since we have $0 \leq |\lambda| < 1$ for $\mathrm{k}_1 > 0$, $\mathrm{k}_3 > 0$.

Clearly $\mathrm{A}_\mathrm{k}^\Phi$ is nonempty. Let $\Phi^{(j)} \in \mathrm{A}_\mathrm{k}^\Phi$ be a minimizing sequence for $\mathrm{J}(\Phi)$. Then we assume that $\Phi^{(j)}(0) \in [0, 2\pi)$ for all j. By (6.1.3) $\Phi_\theta^{(j)}$ is bounded in $\mathrm{L}^2(0, 2\pi)$, and since

$$\Phi^{(j)}(\theta) - \Phi^{(j)}(0) = \int_0^\theta \Phi^{(j)}(s)\, ds$$

it follows easily that $\Phi^{(j)}$ is bounded in $\mathrm{H}^1(0, 2\pi)$. Hence we may assume that $\Phi^{(j)} \rightharpoonup \Phi$ in $\mathrm{H}^1(0, 2\pi)$, and by the compactness of the embedding $\mathrm{H}^1(0, 2\pi) \subset \mathrm{C}^0([0, 2\pi])$ we have that $\Phi \in \mathrm{A}_\mathrm{k}^\Phi$. By standard lower semicontinuity results we have

$$\mathrm{J}(\Phi) \leq \liminf_{j\to\infty} \mathrm{J}(\Phi^{(j)}),$$

so that $\widetilde{\mathbf{n}} = (\cos\Phi, \sin\Phi)$ is a minimizer.

Step 2 (Regularity)

Since for a minimizer Φ of J we have

$$|\mathrm{G}_\Phi| = |2\lambda \sin 2(\Phi - \theta)\Phi_\theta^2| \le 2\Phi_\theta^2,$$

it follows that $\mathrm{G}_\Phi \in \mathrm{L}^1(0, 2\pi)$ and hence (e.g. by Ball & Mizel [BM85, Theorem 2.10]) Φ is a smooth solution of the Euler-Lagrange equation on $[0, 2\pi]$, which in terms of ϕ has the form (6.1.1).

Step 3 (the case k = 1)

In this case, for any minimizer ϕ of $\widetilde{\mathrm{J}}$ we have by Jensen's inequality

$$\begin{aligned}\widetilde{\mathrm{J}}(\phi) &\ge (1 - |\lambda|) \int_0^{2\pi} (\phi_\theta + 1)^2 \,\mathrm{d}\theta \\ &\ge 2\pi(1 - |\lambda|) \left(\fint_{(0,2\pi)} (\phi_\theta + 1)^2 \,\mathrm{d}\theta \right)^2 \\ &= 2\pi(1 - |\lambda|),\end{aligned}$$

where we have used $\phi(2\pi) = \phi(0)$, with equality only if ϕ is constant and if $\lambda \ne 0$ satisfies $\cos 2\phi = \operatorname{sgn}\lambda$, i.e.

$$\phi(\theta) = \mathrm{c}_0 \quad \text{in} \quad [0, 2\pi]$$

$$\cos\, 2\mathrm{c}_0 = \operatorname{sgn}\lambda \quad (\text{no contradiction if } \lambda = 0)$$

where c_0 is some constant.

Therefore we have (i) if $\mathrm{k}_1 > \mathrm{k}_3$ ($\lambda < 0$, the nematic texture is easier to bend than to splay), we have

$$\phi = \frac{\pi}{2} + \mathrm{m}\pi, \quad \mathrm{m} \in \mathbb{Z}$$

which corresponds to

$$\Phi(\theta) = \theta + \frac{\pi}{2} + \mathrm{m}\pi, \quad \mathrm{m} \in \mathbb{Z}$$

and hence the minimizers are

$$\widetilde{\mathbf{n}} = (\cos\Phi(\theta), \sin\Phi(\theta)) = \left(\cos(\theta + \frac{\pi}{2} + \mathrm{m}\pi), \sin(\theta + \frac{\pi}{2} + \mathrm{m}\pi)\right) = \pm\left(\cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2})\right)$$

which is a 'pure bend' vector field, see [Ste04a].

(ii) if $k_1 = k_3$ ($\lambda = 0$) then ϕ is constant and so the minimizers are

$$\widetilde{\mathbf{n}} = (\cos(\theta + \theta_0), \sin(\theta + \theta_0)),$$

where $\theta_0 \in [0, 2\pi)$ is arbitrary.

(iii) if $k_1 < k_3$ ($\lambda > 0$, the nematic texture is easier to splay than to bend), we have

$$\phi(\theta) = m\pi, \quad m \in \mathbb{Z}$$

which corresponds to

$$\Phi(\theta) = \theta + m\pi, \quad m \in \mathbb{Z}$$

and hence

$$\widetilde{\mathbf{n}} = \pm(\cos\theta, \sin\theta),$$

which is a 'pure-splay' vector field.

Step 4 (the case $k \neq 1$)

If ϕ is a minimizer of $\widetilde{J}$ then since ϕ is smooth it satisfies the first integral (Du Bois-Reymond condition)

$$((\phi')^2 - 1)(1 - \lambda\cos 2\phi) = c,$$

for some constant c, and hence for each $\phi \in [0, 2\pi]$ we have

$$\frac{d\phi}{d\theta} = \pm\sqrt{1 + \frac{c}{1 - \lambda\cos 2\phi}}.$$

Suppose that for some θ_0 we have $\frac{d\phi(\theta_0)}{d\theta} = 0$. Then $c < 0$ and hence $\phi'(\theta)^2 < 1$ for all θ. But then

$$|2(k-1)\pi| = |\int_0^{2\pi} \phi'(\theta)\, d\theta| < 2\pi$$

so that $k = 1$, a contradiction. Therefore we have either that

$$\phi'(\theta) = +\sqrt{1 + \frac{c}{1 - \lambda\cos 2\phi}} \quad \text{for all } \theta \in [0, 2\pi], \tag{6.1.5}$$

or

$$\phi'(\theta) = -\sqrt{1 + \frac{c}{1 - \lambda \cos 2\phi}} \quad \text{for all } \theta \in [0, 2\pi], \tag{6.1.6}$$

the first case corresponding to k $>$ 1, and the second to k $<$ 1. In the first case $\phi'(\theta)$ is bounded above and below by positive constants, so that ϕ is invertible and given implicitly by the equation

$$\theta = \int_{\phi(0)}^{\phi(\theta)} \frac{\mathrm{d}\zeta}{\sqrt{1 + \frac{c}{1-\lambda \cos 2\zeta}}}.$$

Therefore c satisfies the equation

$$\begin{aligned} 2\pi &= \int_{\phi(0)}^{\phi(0)+2(\mathrm{k}-1)\pi} \frac{\mathrm{d}\zeta}{\sqrt{1 + \frac{c}{1-\lambda \cos 2\zeta}}} \\ &= \int_{0}^{2(\mathrm{k}-1)\pi} \frac{\mathrm{d}\zeta}{\sqrt{1 + \frac{c}{1-\lambda \cos 2\zeta}}} \\ &= 2(\mathrm{k}-1) \int_{0}^{\pi} \frac{\mathrm{d}\zeta}{\sqrt{1 + \frac{c}{1-\lambda \cos 2\zeta}}}, \end{aligned} \tag{6.1.7}$$

since the integrand is π periodic. Since the integrand is strictly monotone in c, c is unique, and in particular does not depend on $\phi(0)$. Therefore $\phi = \phi_\alpha$, where ϕ_α is the unique solution of (6.1.5) with initial data $\phi(0) = \alpha$. Now

$$\begin{aligned} \widetilde{\mathrm{J}}(\phi_\alpha) &= \int_0^{2\pi} ({\phi'_\alpha}^2 + 2\phi'_\alpha + 1)(1 - \lambda \cos 2\phi_\alpha)\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \left(2\phi'_\alpha + 2 + \frac{\mathrm{c}}{1 - \lambda \cos 2\phi_\alpha}\right)(1 - \lambda \cos 2\phi_\alpha)\,\mathrm{d}\theta \\ &= \int_0^{2\pi} (1 - \lambda \cos 2\phi_\alpha)\,\phi'_\alpha\,\mathrm{d}\theta + 2\pi\mathrm{c} + \int_0^{2\pi} 2(1 - \lambda \cos 2\phi_\alpha)\,\mathrm{d}\theta \\ &= \left(\phi_\alpha(\theta) - \frac{\lambda}{2} \sin 2\phi_\alpha(\theta)\right)\Bigg|_{\theta=0}^{\theta=2\pi} + 2\pi\mathrm{c} + 4\pi - 2\lambda \int_0^{2\pi} \cos 2\phi_\alpha\,\mathrm{d}\theta \\ &= 4(\mathrm{k}-1)\pi + 4\pi + 2\pi\mathrm{c} - 2\lambda \int_0^{2\pi} \cos 2\phi_\alpha\,\mathrm{d}\theta \\ &= 4\mathrm{k}\pi + 2\pi\mathrm{c} - 2\lambda \int_0^{2\pi} \cos 2\phi_\alpha\,\mathrm{d}\theta, \end{aligned}$$

and

$$\int_0^{2\pi} \cos 2\phi_\alpha\,\mathrm{d}\theta = \int_\alpha^{\alpha+2(\mathrm{k}-1)\pi} \frac{2\cos\phi}{\sqrt{1 + \frac{c}{1-\lambda \cos 2\phi}}}\,\mathrm{d}\phi,$$

which is independent of α. Hence the minimizers are

$$\widetilde{\mathbf{n}} = (\cos(\phi_\alpha(\theta) + \theta), \sin(\phi_\alpha(\theta) + \theta)).$$

In the case $k = 2$, we see from (6.1.7) that $c = 0$, and hence $\phi_\alpha = \theta + \alpha$.

For $k < 1$ we argue similarly using (6.1.6), and noting that if $k = 0$ then $c = 0$, and $\Phi(\theta)$ is an arbitrary constant.

□

Equilibrium solutions for $\widetilde{J}(\phi)$ are by definition weak solutions to the corresponding Euler-Lagrange equation, which are smooth solutions to (see (6.1.1))

$$(1 - \lambda \cos 2\phi)\phi'' + \lambda(\phi'^2 - 1) \sin 2\phi = 0, \quad \theta \in [0, 2\pi] \tag{6.1.8}$$

satisfying

$$\begin{cases} \phi(2\pi) - \phi(0) = 2(k-1)\pi, \\ \phi'(2\pi) = \phi'(0) \end{cases}$$

(note that weak solutions are these for which

$$\int_0^{2\pi} \Big((\phi_\theta + 1)(1 - \lambda \cos 2\phi)\, \Psi_\theta + \lambda(\phi_\theta + 1)^2 \sin 2\phi\, \Psi\Big)\, d\theta = 0$$

for all smooth Ψ satisfying $\Psi(2\pi) = \Psi(0)$, from which (6.1.8) and $\phi'(2\pi) = \phi'(0)$ follows).

The proof of Theorem 6.1.1 for $k \neq 1$ uses only the Euler-Lagrange equation and thus shows that the only equilibrium solution for $k \neq 1$ are the minimizers given in the theorem. However the proof does not show this for $k = 1$, and we remedy this in the next result.

Theorem 6.1.2. *For all* $k \in \mathbb{Z}$*, the only equilibrium solutions for* $\widetilde{J}(\phi)$ *are the minimizers given in Theorem 6.1.1.*

Proof. We have to show that the only equilibrium solutions for $\widetilde{J}(\phi)$ for $k = 1$ are the minimizers given in Theorem 6.1.1, i.e. there are no non-constant solutions ϕ to (6.1.8) satisfying

$$\phi(0) = \phi(2\pi)$$
$$\phi'(0) = \phi'(2\pi).$$

This is obvious if $\lambda = 0$, so we may assume $\lambda \neq 0$. We may also assume that $\lambda > 0$, since if ϕ is a non-constant function for $\lambda < 0$, then $\widetilde{\phi}(\theta) = \frac{\pi}{2} - \phi(\theta)$ is such a solution for $\lambda > 0$.

Let ϕ be a non-constant equilibrium solution, which thus extends by periodicity to a 2π−periodic solution of (6.1.8) on $\mathbb{R}$. It has a critical point at some $\theta_0 \in [0, 2\pi]$, so that $\phi'(\theta_0) = 0$, and without loss of generality we can assume $\theta_0 = 0$.

Since ϕ is not constant, $\phi(0)$ is not an integer multiple of $\frac{\pi}{2}$. Also, noting that $\pm\phi(\theta) + \mathrm{m}\pi$, where $\mathrm{m} \in \mathbb{Z}$ is also a solution, we can assume that $0 < \phi(0) < \frac{\pi}{2}$.

Since ϕ is smooth, there exists a constant c such that

$$\phi'(\theta)^2 = 1 + \frac{\mathrm{c}}{1 - \lambda \cos 2\phi(\theta)} \quad \text{for all } \theta. \tag{6.1.9}$$

In particular

$$1 - \lambda \cos 2\phi(0) + \mathrm{c} = 0$$

and so $\phi'(\theta) = 0$ if and only if $\phi(\theta) = \mathrm{m}\pi \pm \phi(0)$ for some $\mathrm{m} \in \mathbb{Z}$.

Let θ_1 be the first value of $\theta > 0$ where $\phi'(\theta) = 0$. Then since $\phi''(0) > 0$, $\phi(\theta_1) = \pi - \phi(0)$. Hence $\phi(\theta) = \phi(2\theta_1 - \theta)$ for $\theta_1 \le \theta \le 2\theta_1$ (because $\widetilde{\phi}(\theta) = \phi(2\theta_1 - \theta)$ satisfies $\widetilde{\phi}(\theta_1) = \phi(\theta_1)$, and $\widetilde{\phi}'(\theta_1) = \phi'(\theta_1) = 0$) and so $\phi(2\theta_1) = \phi(0)$ and $2\theta_1$ is the first positive zero of ϕ. Hence we have $2\theta_1 \le 2\pi$.

Integrating (6.1.9) we obtain

$$\phi_1 = \int_{\phi(0)}^{\pi-\phi(0)} \frac{\mathrm{d}\zeta}{\sqrt{1 + \frac{c}{1-\lambda\cos 2\zeta}}} = 2\int_{\phi(0)}^{\frac{\pi}{2}} \frac{\mathrm{d}\zeta}{\sqrt{1 + \frac{c}{1-\lambda\cos 2\zeta}}}.$$

Making the change of variables

$$\mathrm{v} = \frac{1 + \mathrm{c} - \lambda\cos 2\zeta}{\zeta}$$
$$\mathrm{s} = 1 + \mathrm{c} + \lambda,$$

noting that $\cos 2\zeta$ is monotone on $\left(\phi(0), \frac{\pi}{2}\right)$, we obtain after a calculation

$$\theta_1 = \int_0^1 \frac{\sqrt{1 + \lambda - s(1-v)}}{\sqrt{v(1-v)}\sqrt{2\lambda - s(1-v)}}\,\mathrm{dv}.$$

The integrand is monotone increasing in s, since $1 + \lambda > 2\lambda$, and so

$$\theta_1 > \int_0^1 \sqrt{\frac{1+\lambda}{2\lambda}} \frac{\mathrm{dv}}{\sqrt{v(1-v)}} = \pi\sqrt{\frac{1+\lambda}{2\lambda}} > \pi,$$

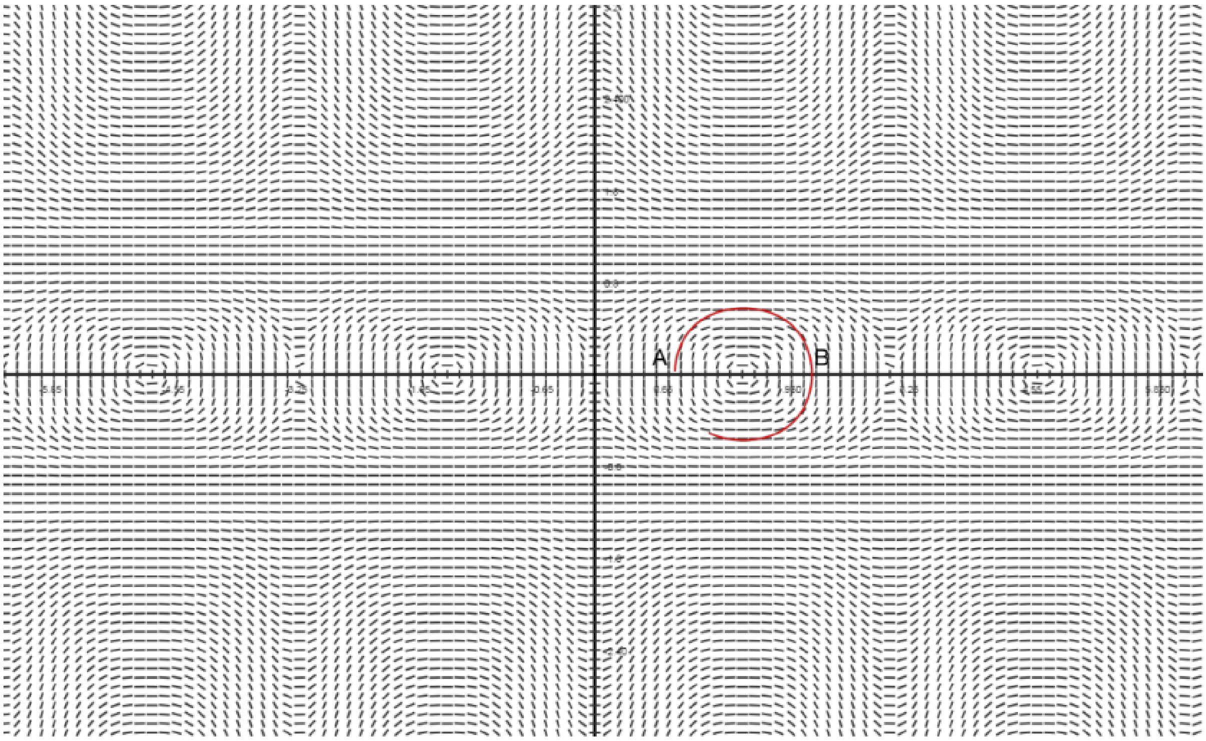

Figure 6.2: Red circle depicts solution on phase plane for equation (6.1.8) given $\lambda = 0.6$, starting from $(\phi(0), 0)$ (point A on the phase plane), as θ going from 0 to 2π. Point B represents $(\phi(\theta_1), 0)$ where $\phi(\theta_1)$ is given in the proof of Theorem 6.1.2. Since it is incomplete, we have $\phi(0) \neq \phi(2\pi)$, it is not an equilibrium solution.

which is a contradiction, completing the proof.

□

Remark 6.1.3. *One can refer to Figure 6.2 for an illustration of the proof for Theorem 6.1.2.*

6.2 The case when $k_1 = 0$ or $k_3 = 0$

Theorem 6.1.1 will not hold automatically for the case when $k_1 = 0$ or $k_3 = 0$. It is easily seen that (6.1.5) and (6.1.6) are not well defined everywhere when $\lambda = \pm 1$ hence the proof of Theorem 6.1.1 will not hold for the case when $\lambda = \pm 1$. Without loss of generality we can assume $k_1 = 1$ when $k_3 = 0$ and vice versa. The case $k_1 = 1$, $k_3 = 0$ corresponds to $\lambda = \frac{k_3 - k_1}{k_3 + k_1} = -1$ and

$$I(\mathbf{n}) = \frac{1}{2} \int_{\Omega} (\operatorname{div} \mathbf{n})^2 \, dx,$$

while $k_1 = 0$, $k_3 = 1$ corresponds to $\lambda = 1$ and

$$I(\mathbf{n}) = \frac{1}{2} \int_{\Omega} |\mathbf{n} \wedge \nabla \wedge \mathbf{n}|^2 \, dx.$$

Remark 6.2.1. *In fact the two degenerate cases are equivalent. If we write* $\mathbf{n} = (n_1, n_2)$ *and let*

$$Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so that $Q\,\mathbf{n} = (-n_2, n_1)$ *then a direct computation shows that*

$$(\operatorname{div} Q\,\mathbf{n})^2 = |\mathbf{n} \wedge \nabla \wedge \mathbf{n}|^2 \tag{6.2.1}$$

and

$$|Q\,\mathbf{n} \wedge \nabla \wedge Q\,\mathbf{n}|^2 = (\operatorname{div} \mathbf{n})^2. \tag{6.2.2}$$

Hence $\mathbf{n}$ *is a minimizer of* $\int_\Omega (\operatorname{div} \mathbf{n})^2\, dx$ *if and only if* $Q\,\mathbf{n}$ *is a minimizer of* $\int_\Omega |\mathbf{n} \wedge \nabla \wedge \mathbf{n}|^2\, dx$, *with appropriately changed boundary conditions.*

Hence with Remark 6.2.1 we only need to study when either $\lambda = 1$ or $\lambda = -1$. Before we study the minimization problem for the degenerate cases ($k_1 = 1$, $k_3 = 0$ or $k_1 = 0$, $k_3 = 1$), we first study the equilibria for the functional when $\lambda = -1$ ($k_1 = 1$, $k_3 = 0$):

$$I(\mathbf{n}) = \frac{1}{2} \int_\Omega (\operatorname{div} \mathbf{n})^2\, dx.$$

Further, as in Section 6.1 we assume that the vector field is radius independent, i.e. $\Phi(\theta, r) = \Phi(\theta) = \Phi(\frac{x}{|x|})$ (hence $\phi(s, \theta) = \phi(\theta)$), so that by (6.0.3) the bulk energy becomes:

$$\begin{aligned} I(\mathbf{n}) &= \frac{1}{2} \int_\Omega (\operatorname{div} \mathbf{n})^2\, dx \\ &= \frac{1}{4} \int_{\log a}^{\log b} \int_0^{2\pi} (1 + \cos 2\phi)\,(1 + \phi_\theta)^2\, ds\, d\theta \end{aligned}$$

and consequently the Euler-Lagrange equation (6.0.4) becomes:

$$(1 + \cos 2\phi)\,\phi_{\theta\theta} + (1 - \phi_\theta^2)\,\sin 2\phi = 0,$$

which can also be written as

$$\phi_{\theta\theta} \cos^2 \phi - \phi_\theta^2 \sin \phi \,\cos \phi = -\sin \phi\, \cos \phi. \tag{6.2.3}$$

Note that below, we will not distinguish between the notations $\phi'(\theta)$, $\phi_\theta(\theta)$ and $\frac{d\phi}{d\theta}$.

One can easily see that for equation (6.2.3) to hold, we must have either $\cos\phi = 0$, or

$$\phi'' \cos\phi - \phi'^2 \sin\phi = -\sin\phi. \tag{6.2.4}$$

(a) If $\cos\phi = 0$, then $\Phi(\theta) = \theta + \frac{\pi}{2} + \mathrm{m}\pi$, $\mathrm{m} \in \mathbb{Z}$.
(b) If (6.2.4) holds, we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\phi' \cos\phi) = -\sin\phi,$$

and consequently,

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}(\sin\phi) + \sin\phi = 0. \tag{6.2.5}$$

Since $\phi(\theta) = \Phi(\theta) - \theta$ satisfies $\phi(\theta + 2\pi) = \phi(\theta) + 2\mathrm{m}\pi$, $\mathrm{m} \in \mathbb{Z}$, (6.2.5) is equivalent to

$$\sin(\Phi(\theta) - \theta) = \mathrm{A}\sin(\theta + \theta_0),$$

where A and θ_0 are some constants depending on the given boundary conditions. It is not immediately obvious to see how $\Phi(\theta)$ behaves in general, but some special cases worth discussion.

(i) For $A = 0$, we have $\Phi(\theta) - \theta = \mathrm{m}\pi$, therefore $\Phi(\theta) = \theta + \mathrm{m}\pi$, $\mathrm{m} \in \mathbb{Z}$.

(ii) For $A = -1$ we have $\sin(\Phi - \theta) + \sin(\theta + \theta_0) = 0$, hence

$$\sin\left(\frac{\Phi + \theta_0}{2}\right)\cos\left(\frac{\Phi - 2\theta - \theta_0}{2}\right) = 0. \tag{6.2.6}$$

We can see from equation (6.2.6) that if $\sin(\frac{\Phi+\theta_0}{2}) = 0$, then $\Phi = -\theta_0 + 2\mathrm{m}\pi$; and if $\cos(\frac{\Phi-2\theta-\theta_0}{2}) = 0$, then $\Phi(\theta) = 2\theta + \pi + \theta_0 + 2\mathrm{m}\pi$, where θ_0 is some constant.

Therefore when $\mathrm{A} = -1$, we have

$$\begin{cases} \Phi(\theta) = -\theta_0 + 2\mathrm{m}\pi, \text{ or} \\ \Phi(\theta) = 2\theta + \theta_0 + (2\mathrm{m}+1)\pi, \end{cases}$$

where $\mathrm{m} \in \mathbb{Z}$ and $\theta_0 \in \mathbb{R}$ is some constant.

(iii) When $A = 1$, by a similar calculation we have

$$\begin{cases} \Phi(\theta) = -\theta_0 + (2\mathrm{m}+1)\pi, \text{ or} \\ \Phi(\theta) = 2\theta + \theta_0 + 2\mathrm{m}\pi, \end{cases}$$

where $\mathrm{m} \in \mathbb{Z}$ and $\theta_0 \in \mathbb{R}$ is some constant.

Therefore, one can see that $\Phi(\theta)$ of the following forms

$$\Phi(\theta) = \theta_0 + m\pi, \tag{6.2.7}$$

$$\Phi(\theta) = \theta + \frac{\pi}{2} + m\pi, \tag{6.2.8}$$

and

$$\Phi(\theta) = 2\theta + \theta_0 + m\pi, \tag{6.2.9}$$

where $\theta \in [0, 2\pi)$, and $m \in \mathbb{Z}$, $\theta_0 \in [0, 2\pi)$ are constants, form a group of special equilibrium solutions for $\frac{1}{2}\int_\Omega (\operatorname{div}\mathbf{n})^2\,dx$ for Φ in the admissible set $\{\Phi \in H^1((a,b)\times(0,2\pi)) \,|\, \Phi(\theta + 2\pi) - \Phi(\theta) = 2k\pi, k \in \mathbb{Z}\}$, where $k \in \mathbb{Z}$ is the given degree. One can easily see that the degree k of $\mathbf{n} = e^{i\Phi}$, where Φ takes the form (6.2.7) is zero; for (6.2.8) the degree is $k = 1$ and for (6.2.9), the degree is $k = 2$. Also, all the equilibrium solutions (6.2.7) to (6.2.9) (corresponding to $A = 0, -1, 1$) are also equilibria for the functional $I(\mathbf{n}) = \frac{1}{2}\int_\Omega |\nabla\mathbf{n}|^2\,dx$ as they have the form (3.0.17).

We now compute the second variation and bulk free energy for the above equilibria (6.2.7) to (6.2.9) under the assumption that $\Phi(r,\theta)$ is radius-independent.

Since by (3.0.12), the second variation for a given vector field $\mathbf{n} = e^{i\Phi}$ is:

$$\delta^2 E(\Phi)(\eta,\eta) = \frac{1}{2}\int_a^b r\int_{\mathbb{S}^1}\Big(G_{\Phi\Phi}\eta^2 + G_{\Phi_r\Phi_r}\eta_r^2 + G_{\Phi_\theta\Phi_\theta}\eta_\theta^2 + 2G_{\Phi\Phi_r}\eta\eta_r + 2G_{\Phi\Phi_\theta}\eta\eta_\theta + 2G_{\Phi_r\Phi_\theta}\eta_r\eta_\theta\Big)\,d\theta\,dr,$$

where $\eta \in C_0^\infty(\Omega)$ is smooth and has compact support in Ω and η satisfies

$$\eta(r, \theta + 2\pi) = \eta(r,\theta).$$

Also, when $k_3 = 0$ and $k_1 = 1$, $G(r,\theta,\Phi,\Phi_r,\Phi_\theta)$ is given by (3.0.8) as:

$$G(r,\theta,\Phi,\Phi_r,\Phi_\theta) = \frac{1}{2}\left(\Phi_r^2\sin^2(\Phi-\theta) - \frac{2\Phi_\theta\Phi_r}{r}\sin(\Phi-\theta)\cos(\Phi-\theta) + \frac{\Phi_\theta^2}{r^2}\cos^2(\Phi-\theta)\right).$$

As a consequence, the second variation $\delta^2 I(\Phi)$ becomes:

$$\begin{aligned}\delta^2 E(\Phi)(\eta,\eta) = \frac{1}{2}\int_a^b r\int_{\mathbb{S}^1}\Big(&\sin^2(\Phi-\theta)\eta_r^2 + \frac{1}{r^2}\cos^2(\Phi-\theta)\eta_\theta^2 + \Big(\Phi_r^2\cos 2(\Phi-\theta) - \frac{\Phi_\theta^2}{r^2}\cos 2(\Phi-\theta)\\ &+ \frac{2\Phi_\theta\Phi_r}{r}\sin 2(\Phi-\theta)\Big)\eta^2 + 2\Big(\Phi_r\sin 2(\Phi-\theta) - \frac{\Phi_\theta}{r}\cos 2(\Phi-\theta)\Big)\eta\eta_r\\ &+ 2\Big(-\frac{\Phi_r}{r}\cos 2(\Phi-\theta) - \frac{\Phi_\theta}{r^2}\sin 2(\Phi-\theta)\Big)\eta\eta_\theta - 2\Big(\frac{1}{r}\sin(\Phi-\theta)\cos(\Phi-\theta)\Big)\eta_\theta\eta_r\Big)\,d\theta\,dr.\end{aligned}$$

Note that if we assume Φ is radius-independent, i.e. $\Phi(\theta, r) = \Phi(\theta)$, we obtain:

$$\delta^2 E(\Phi)(\eta,\eta) = \frac{1}{2}\int_a^b r \int_{\mathbb{S}^1} \Big(\Big(\frac{1}{r}\cos(\Phi-\theta)\eta_\theta - \sin(\Phi-\theta)\eta_r\Big)^2 - \frac{(\Phi'(\theta))^2}{r^2}\cos 2(\Phi-\theta)\eta^2 - 2\frac{\Phi'(\theta)}{r^2}\sin 2(\Phi-\theta)\eta\eta_\theta - 2\frac{\Phi'(\theta)}{r}\cos 2(\Phi-\theta)\eta\eta_r \Big)\, d\theta\, dr.$$

As a consequence,

(i) For

$$\Phi(\theta) = \theta_0 + m\pi,$$

the energy is:

$$E(\Phi) = \frac{1}{4}\int_a^b \int_{\mathbb{S}^1} \frac{(\Phi')^2}{r}\cos^2(\Phi-\theta)\, d\theta\, dr = 0,$$

and the corresponding second variation is:

$$\delta^2 E(\Phi)(\eta,\eta) = \frac{1}{2}\int_a^b r \int_{\mathbb{S}^1} (\frac{1}{r}\cos(\Phi-\theta)\eta_\theta - \sin(\Phi-\theta)\eta_r)^2 d\theta\, dr \geq 0. \tag{6.2.10}$$

(ii) For

$$\Phi(\theta) = \theta + \frac{\pi}{2} + m\pi,$$

the free energy is:

$$E(\Phi) = \frac{1}{4}\int_a^b \int_{\mathbb{S}^1} \frac{(\Phi')^2}{r}\cos^2(\Phi-\theta)\, d\theta\, dr = \frac{1}{4}\log\frac{b}{a}\int_{\mathbb{S}^1}\cos^2(\frac{\pi}{2}+m\pi)\, d\theta = 0,$$

and the second variation is:

$$\begin{aligned}\delta^2 E(\Phi)(\eta,\eta) &= \frac{1}{2}\int_a^b r \int_{\mathbb{S}^1} \Big(\Big(\frac{1}{r}\cos(\frac{\pi}{2}+m\pi)\eta_\theta - \sin(\frac{\pi}{2}+m\pi)\eta_r\Big)^2 - \frac{1}{r^2}\cos(\pi+2m\pi)\eta^2 \\ &\quad - 2\frac{1}{r^2}\sin(\pi+2m\pi)\eta\,\eta_\theta - 2\frac{1}{r}\cos(\pi+2m\pi)\eta\,\eta_r\Big)\, d\theta\, dr \\ &= \frac{1}{2}\int_a^b r \int_{\mathbb{S}^1} \Big(\eta_r^2 + \frac{1}{r^2}\eta^2 + 2\frac{1}{r}\eta\,\eta_r\Big)\, d\theta\, dr \\ &= \frac{1}{2}\int_a^b r \int_{\mathbb{S}^1} \Big(\eta_r + \frac{1}{r}\eta\Big)^2\, d\theta\, dr \geq 0.\end{aligned} \tag{6.2.11}$$

(iii) For

$$\Phi(\theta) = 2\theta + \theta_0 + m\pi,$$

the free energy becomes:

$$E(\Phi) = \frac{1}{4}\int_a^b \int_{\mathbb{S}^1} \frac{(\Phi')^2}{r} \cos^2(\theta - \Phi)\, d\theta\, dr = \frac{1}{4}\log\frac{b}{a}\int_{\mathbb{S}^1} \cos^2(\theta + \theta_0)\, d\theta = \frac{\pi}{4}\log\frac{b}{a},$$

and the second variation is:

$$\begin{aligned}
\delta^2 E(\Phi)(\eta,\eta) &= \frac{1}{2}\int_a^b r \int_{\mathbb{S}^1} \Big(\Big(\sin(\theta+\theta_0)\eta_r - \frac{1}{r}\cos(\theta+\theta_0)\eta_\theta\Big)^2 \\
&\quad - \frac{4}{r}\cos 2(\theta+\theta_0)\eta\eta_r - \frac{2}{r^2}(\sin 2(\theta+\theta_0)\eta^2)_{,\theta}\Big)\, d\theta\, dr \\
&= \frac{1}{2}\int_a^b r \int_{\mathbb{S}^1} \Big(\sin(\theta+\theta_0)\eta_r - \frac{1}{r}\cos(\theta+\theta_0)\eta_\theta\Big)^2\, d\theta\, dr \\
&\quad - \int_{\mathbb{S}^1} \Big(\eta^2(b,\theta) - \eta^2(a,\theta)\Big)\cos 2(\theta+\theta_0)\, d\theta - \int_a^b \frac{1}{r}\Big(\sin 2(\theta+\theta_0)\, \eta(r,\theta)\Big)\Big|_{\theta=0}^{\theta=2\pi} dr \\
&= \frac{1}{2}\int_a^b r \int_{\mathbb{S}^1} \Big((\sin(\theta+\theta_0)\eta_r - \frac{1}{r}\cos(\theta+\theta_0)\eta_\theta\Big)^2\, d\theta\, dr \geq 0.
\end{aligned} \tag{6.2.12}$$

Since the second variation (6.2.10) to (6.2.12) of the equilibrium solutions (6.2.7) to (6.2.9) are non-negative, this leads us to ask the question whether these equilibria are also minimizers of the functional

$$I(\mathbf{n}) = \frac{1}{2}\int_\Omega (\operatorname{div}\mathbf{n})^2\, dx,$$

for $\mathbf{n}$ in the admissible set $\in A_k = \{\mathbf{n} \in H^1(\Omega;\mathbb{S}^1) \,|\, \deg\mathbf{n} = k\}$, where $\mathbf{deg}\,\mathbf{n}$ denotes the degree of the vector field $\mathbf{n}$ in view of Theorem 2.1.2 and Proposition 2.1.3.

In fact for these two cases:

$$I(\mathbf{n}) = \frac{1}{2}\int_\Omega (\operatorname{div}\mathbf{n})^2\, dx,$$

and

$$I(\mathbf{n}) = \frac{1}{2}\int_\Omega |\mathbf{n}\wedge\nabla\wedge\mathbf{n}|^2\, dx,$$

in order to have a similar result to Theorem 6.1.1, we need to study the minimization problem in the space $W^{1,1}(\Omega;\mathbb{S}^1)$ instead of $H^1(\Omega;\mathbb{S}^1)$ and replace the admissible set A_k by

$$\hat{A}_k = \{\mathbf{n} \in W^{1,1}(\Omega;\mathbb{S}^1) \,|\, \deg\mathbf{n}|_{\mathbb{S}_r} = k \text{ for a.e. } r \in (a,b)\}, \tag{6.2.13}$$

where $\mathbb{S}_r = \{x \,|\, |x| = r\}$. Note that $\mathbf{n}|_{\mathbb{S}_r} \in W^{1,1}(\mathbb{S}_r;\mathbb{S}^1)$ a.e. and so the degree $\mathbf{deg}\,\mathbf{n}|_{\mathbb{S}_r}$ is well defined. We claim that Theorem 6.1.1 still holds (but in a different admissible set) and prove it using an adaptation of the proof in [Ces83, Chapter 12].

Theorem 6.2.2. $\mathrm{I}(\mathbf{n})$ *attains a minimum among* $\mathbf{n} \in \hat{\mathrm{A}}_{\mathrm{k}}$ *that depend only on* $\theta \in \mathbb{S}^1$. *The minimizers* $\widetilde{\mathbf{n}}$ *are given by:*

(a) For $\mathrm{k} = 0$, $\widetilde{\mathbf{n}} = (\cos\theta_0, \sin\theta_0)$ *for arbitrary* $\theta_0 \in [0, 2\pi)$.

(b) For $\mathrm{k} = 1$,

if $\mathrm{k}_1 = 1, \mathrm{k}_3 = 0$, $\widetilde{\mathbf{n}} = \pm\left(\cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2})\right)$,

if $\mathrm{k}_1 = 0, \mathrm{k}_3 = 1$, $\widetilde{\mathbf{n}} = \pm(\cos\theta, \sin\theta)$.

(c) For $\mathrm{k} = 2$, $\widetilde{\mathbf{n}} = (\cos 2(\theta + \theta_0), \sin 2(\theta + \theta_0))$ *for arbitrary* $\theta_0 \in [0, 2\pi)$.

(d) For $\mathrm{k} > 2$, $\widetilde{\mathbf{n}} = (\cos(\phi_\alpha(\theta) + \theta), \sin(\phi_\alpha(\theta) + \theta))$, $\alpha \in \mathbb{R}^1$, *where* ϕ_α *is the unique solution of*

$$\phi'(\theta) = \sqrt{1 + \frac{c}{\cos^2\phi}}$$

$$\phi(0) = \alpha$$

and c *is the unique constant such that*

$$\mathrm{F(c)} = \frac{\pi}{\mathrm{k} - 1},$$

$$\mathrm{F(c)} = \int_0^\pi \frac{\mathrm{d}\zeta}{\sqrt{1 + \frac{c}{\cos^2\zeta}}}.$$

(e) For $\mathrm{k} < 1$, $\widetilde{\mathbf{n}} = (\cos(\phi_\alpha(\theta) + \theta), \sin(\phi_\alpha(\theta) + \theta))$, $\alpha \in \mathbb{R}^1$ *where* ϕ_α *is the unique solution of*

$$\phi'(\theta) = -\sqrt{1 + \frac{c}{\cos^2\phi}}$$

$$\phi(0) = \alpha$$

and c *is the unique constant such that* $\mathrm{F(c)} = \frac{\pi}{1-\mathrm{k}}$.

Proof. Let

$$\hat{\mathrm{A}}_{\mathrm{k}}^{\Phi} = \{\Phi \in \mathrm{W}^{1,1}(0, 2\pi) \,|\, \Phi(2\pi) - \Phi(0) = 2\mathrm{k}\pi\}$$

$$\hat{\mathrm{A}}_{\mathrm{k}}^{\phi} = \{\Phi \in \mathrm{W}^{1,1}(0, 2\pi) \,|\, \phi(2\pi) - \phi(0) = 2(\mathrm{k} - 1)\pi\}$$

In the case $\mathrm{k}_1 = 1, \mathrm{k}_3 = 0$ we have

$$\mathrm{J}(\Phi) = 2\int_0^{2\pi} \Phi_\theta^2 \cos^2(\Phi - \theta)\,\mathrm{d}\theta,$$

$$\widetilde{J}(\phi) = 2\int_0^{2\pi} (\phi_\theta + 1)^2\ \cos^2\phi\,d\theta,$$

while in the case $k_1 = 0, k_3 = 1$ we have

$$J(\Phi) = 2\int_0^{2\pi} \Phi_\theta^2 \sin^2(\Phi - \theta)\,d\theta,$$

$$\widetilde{J}(\phi) = 2\int_0^{2\pi} (\phi_\theta + 1)^2\ \sin^2\phi\,d\theta,$$

Minimizing $J(\Phi)$, $\widetilde{J}(\phi)$ in $\hat{A}_k^\Phi$, $\hat{A}_k^\phi$ respectively is equivalent, since if Φ^* is a minimizer for the case $k_1 = 1, k_3 = 0$ then $\Phi^* + \frac{\pi}{2}$ is a minimizer for the case $k_1 = 0, k_3 = 1$ and vice versa. This also showed the equivalence of the two degenerate cases mentioned in Remark 6.2.1.

Therefore we need only consider the minimizer of

$$E(\Phi) = \frac{1}{2}J(\Phi) = \int_0^{2\pi} \Phi_\theta^2 \cos^2(\Phi - \theta)\,d\theta$$

and

$$\widetilde{E}(\phi) = \frac{1}{2}\widetilde{J}(\phi) = \int_0^{2\pi} (\phi_\theta + 1)^2\ \cos^2\phi\,d\theta.$$

Note that for $\phi \in \hat{A}_k^\phi$

$$\begin{aligned}\widetilde{E}(\phi) &= \int_0^{2\pi} (\phi_\theta^2 + 1)\cos^2\phi\,d\theta + \int_0^{2\pi} (1 + \cos 2\phi)\,\phi_\theta\,d\theta\\ &= \int_0^{2\pi} (\phi_\theta^2 + 1)\cos^2\phi\,d\theta + 2(k-1)\pi.\end{aligned}$$

Hence for $k = 1$ the minimizers are given by $\phi = (m + \frac{1}{2})\,\pi$, $m \in \mathbb{Z}$. Thus we can assume that $k \neq 1$.

The case $k = 0$ is also easy. Obviously $\Phi(\theta) = \text{const}$ is a minimizer with $J(\Phi) = 0$. But if $J(\Phi) = 0$ then we have a.e. either $\Phi_\theta = 0$ or $\cos(\Phi - \theta) = 0$. If $\cos(\Phi - \theta) = 0$ on a set E of positive measure we have

$$-\sin(\Phi - \theta)(\Phi_\theta - 1) = 0 \text{ a.e. on } E$$

and hence $\Phi_\theta = 1$ a.e. on E.

But $\int_0^{2\pi} \Phi_\theta\,d\theta = 2k\pi = 0$, which implies the measure of the set E is zero, so that $\Phi_\theta = 0$ a.e. and Φ is a constant. Hence we can proceed with $k \neq 0$.

First note that any minimizer Φ of $E(\Phi)$ in the admissible set $\hat{A}_k^{\Phi}$ is monotone. Indeed, taking the case $k > 1$, and suppose Φ is not monotone. Then there exists some $\theta_0 < \theta_1$ with $\Phi(\theta_0) = \Phi(\theta_1)$ and $\Phi(\theta) < \Phi(\theta_0)$ for all $\theta \in (\theta_0, \theta_1)$. (To see this, there is some r such that $\exists \theta' \neq \theta''$ with $\Phi(\theta') = \Phi(\theta'') = r$ and $\Phi(\theta''') < r$. Consider the open set $\Phi^{-1}(r)^c$ which is a union of disjoint open intervals and pick the one containing θ'''.)

Now consider

$$\widetilde{\Phi}(\theta) = \begin{cases} \Phi(\theta) & \theta \notin (\theta_0, \theta_1) \\ \Phi(\theta_0) & \theta \in (\theta_0, \theta_1) \end{cases}$$

Then

$$E(\Phi) - E(\widetilde{\Phi}) = \int_{\theta_0}^{\theta_1} \Phi_\theta^2 \cos^2(\Phi - \theta)\, d\theta \geq 0$$

and the argument above for $k = 0$ shows that we have $E(\Phi) > E(\widetilde{\Phi})$, hence a contradiction.

Therefore it suffices to prove the existence of a minimizer among all monotone functions in $\hat{A}_k^{\Phi}$. Again take the case $k > 1$ and let $\{\Phi^{(j)}\}_{j=1}^{\infty}$ be a minimizing sequence, so that $\Phi_\theta^{(j)} \geq 0$ a.e. Since $E(\Phi + 2m\pi) = E(\Phi)$ we can suppose $\Phi^{(j)}(0) \in [0, 2\pi)$ for all $j = 1, 2, \ldots$ and hence by the monotonicity and boundary conditions, $\{\Phi^{(j)}(\theta)\}_{j=1}^{\infty}$ is uniformly bounded in $[0, 2\pi]$.

We now show that $\{\Phi^{(j)}\}_{j=1}^{\infty}$ is equicontinuous. Suppose it is not, then there exists some $\epsilon > 0$ and $\theta_j < \theta_j'$ with $|\theta_j - \theta_j'| \to 0$ and $\Phi^{(j)}(\theta_j') - \Phi^{(j)}(\theta_j) \geq \epsilon$. We can suppose that $\theta_j \to \bar{\theta}$ and $\theta_j' \to \bar{\theta}$ for some $\bar{\theta} \in [0, 2\pi]$, such that

$$\Phi^{(j)}(\theta_j) \to \alpha,$$

and

$$\Phi^{(j)}(\theta_j') \to \beta,$$

where $\beta \geq \alpha + \epsilon$.

Then there is an interval $[\gamma, \delta] \subset (\alpha, \beta)$, with $\delta > \gamma$ and $\cos^2(\eta - \bar{\theta}) > \tau > 0$ for all $\eta \in [\gamma, \delta]$. By the monotonicity there exist $\theta_j \leq \tilde{\theta}_j < \tilde{\theta}_j' \leq \theta_j'$ with $\Phi^{(j)}(\tilde{\theta}_j) = \gamma$, $\Phi^{(j)}(\tilde{\theta}_j') = \delta$ and $\Phi^{(j)}(\theta) \in [\gamma, \delta]$ for $\theta \in [\tilde{\theta}_j, \tilde{\theta}_j']$. Hence

$$E(\Phi^{(j)}) \geq \tau \int_{\tilde{\theta}_j}^{\tilde{\theta}_j'} \Phi_\theta^{(j)}(\theta)^2\, d\theta \geq \tau \cdot \frac{1}{\tilde{\theta}_j' - \tilde{\theta}_j} \left(\int_{\tilde{\theta}_j}^{\tilde{\theta}_j'} \Phi_\theta^{(j)}\, d\theta \right)^2 = \tau \cdot \frac{1}{\tilde{\theta}_j' - \tilde{\theta}_j} (\delta - \gamma)^2 \to \infty$$

as $j \to \infty$, contradicting the fact that $\{\Phi^{(j)}\}_{j=1}^{\infty}$ is a minimizing sequence. Hence by Arzela-

Ascoli theorem we can assume that $\Phi^{(j)} \to \Phi$ uniformly for some continuous monotone function Φ satisfying the boundary conditions $\Phi(2\pi) - \Phi(0) = 2k\pi$.

Next we show that Φ is absolutely continuous, so that $\Phi \in W^{1,1}(0, 2\pi)$. To show this we use the characterization in [Rud06, p. 146] that a continuous monotone function is absolutely continuous if and only if it maps sets of measure zero to sets of measure zero.

Consider the set $K = \{\theta \in (0, 2\pi) \,|\, \Phi(\theta) - \theta = (m + \frac{1}{2})\pi$ for some $m \in \mathbb{Z}\}$. Since Φ is bounded, K is the union of finitely mange relatively closed sets and so K is relatively closed in $(0, 2\pi)$. Hence its complement is the union of countably many open intervals i.e. $K^c = \cup_{i=1}^{\infty} I_i$.

We claim that $\Phi \in W^{1,1}(I_i)$ for each i.

In fact if $a, b \in I_i$, with $a < b$, then

$$\infty > M \geq E(\Phi^{(j)}) \geq \int_a^b {\Phi_\theta^{(j)}}^2 \cos^2(\Phi^{(j)} - \theta)\, d\theta \geq c \int_a^b {\Phi_\theta^{(j)}}^2\, d\theta,$$

for some $c > 0$. Hence there exists a subsequence of $\{\Phi^{(j)}\}_{j=1}^{\infty}$ which converges weakly in $H^1(a, b)$ to some Φ, so that $\Phi \in H^1_{loc}(I_i)$. But recall that $\int_{I_i} \Phi_\theta\, d\theta$ is bounded and satisfies $\Phi \in W^{1,1}(I_i)$, and thus is absolutely continuous in the interval $\bar{I}_i$.

If Φ is not absolutely continuous then there exists a set $E \subset (0, 2\pi)$ with $\operatorname{meas} E = 0$ satisfying $\Phi(E) > 0$. Clearly we have $\operatorname{meas} \Phi(E \cap K) = 0$ and therefore $\operatorname{meas} \Phi(E \cap I_i) > 0$ for some i, which is impossible due to the result given by the result in Rudin [Rud06, p. 146], since $\Phi \in W^{1,1}(I_i)$ (that a continuous monotone function is absolutely continuous iff it maps sets of measure zero to sets of measure zero).

Now we show that Φ is a minimizer. Consider an interval I_i and choose $[a, b] \subset I_i$. Then by standard lower semicontinuity results we have

$$\begin{aligned}\int_{I_i \cap [a,b]} \Phi_\theta^2 \cos^2(\Phi - \theta)\, d\theta &\leq \underline{\lim} \int_{I_i \cap [a,b]} {\Phi_\theta^{(j)}}^2 \cos^2(\Phi^{(j)} - \theta)\, d\theta \\ &\leq \underline{\lim} \int_{I_i} {\Phi_\theta^{(j)}}^2 \cos^2(\Phi^{(j)} - \theta)\, d\theta.\end{aligned}$$

Thus by monotone convergence we have

$$\int_{I_i} \Phi_\theta^2 \cos^2(\Phi - \theta)\, d\theta \leq \underline{\lim} \int_{I_i} {\Phi_\theta^{(j)}}^2 \cos^2(\Phi^{(j)} - \theta)\, d\theta,$$

and by taking the union over all the intervals we have

$$\Sigma_{i=1}^{\infty}\int_{I_i}\Phi_\theta^2\cos^2(\Phi-\theta)\,d\theta \le \underline{\lim}\,\Sigma_{i=1}^{\infty}\int_{I_i}{\Phi_\theta^{(j)}}^2\cos^2(\Phi^{(j)}-\theta)\,d\theta,$$

hence

$$\int_0^{2\pi}\Phi_\theta^2\cos^2(\Phi-\theta)\,d\theta \le \underline{\lim}\int_0^{2\pi}{\Phi_\theta^{(j)}}^2\cos^2(\Phi^{(j)}-\theta)\,d\theta. \tag{6.2.14}$$

Since $\{\Phi^{(j)}\}_{j=1}^{\infty}$ is a minimizing sequence, and by the lower semicontinuity (6.2.14) Φ is a minimizer.

Now suppose Φ is any minimizer and by Ball & Mizel [BM85, Theorem 2.3 (ii)] we have that $\phi = \Phi - \theta$ satisfies

$$(\phi_\theta^2 - 1)\,\cos^2\phi = c \quad \theta \in (0, 2\pi) \tag{6.2.15}$$

for some constant c.

For $k > 1$, we claim that $c \ge 0$.

Suppose $c < 0$ then we have from (6.2.15) that $\phi_\theta^2 < 1$ a.e. and so

$$2\pi \le 2(k-1)\pi = \int_0^{2\pi}\phi_\theta\,d\theta \le \left(\int_0^{2\pi}1^2\,d\theta\right)^{1/2}\left(\int_0^{2\pi}\phi_\theta^2\,d\theta\right)^{1/2} < 2\pi,$$

which is a contradiction, proving $c \ge 0$.

In fact $c > 0$ if and only if $k > 2$. Since if $c > 0$ we have $\phi_\theta^2 > 1$ and by (6.2.15) we have $\phi_\theta > 1$, hence

$$\int_0^{2\pi}\phi_\theta\,d\theta = 2(k-1)\pi > 2\pi,$$

implying $k > 2$.

On the other hand if $k > 2$ we have

$$2\pi < \int_0^{2\pi}\phi\,d\theta \le (2\pi)^{1/2}\left(\int_0^{2\pi}\phi_\theta^2\,d\theta\right)^{1/2}$$

so that $\int_0^{2\pi}\phi_\theta^2\,d\theta > 2\pi$, hence $c > 0$.

(i) For $k = 2$,

we have $c = 0$ and

$$(\phi_\theta^2 - 1)\cos^2\phi = 0 \quad \text{a.e.}$$

if $\cos\phi = 0$ on a set of positive measure we have $\sin\phi \cdot \phi_\theta = 0$, which implies $\phi_\theta = 0$. So we have either $\phi_\theta^2 = 1$ or $\phi_\theta^2 = 0$ in $(0, 2\pi)$. But recall that $\int_0^{2\pi} \phi_\theta \, d\theta = 2\pi$ and hence $\phi_\theta = 1$, which implies

$$\phi = \theta + \beta,$$

where β is some constant.

(ii) For k > 2,

we have c > 0, $\phi_\theta > 1$ and so

$$\phi_\theta = \sqrt{1 + \frac{c}{\cos^2\phi}}. \tag{6.2.16}$$

Then we can proceed as in the proof of Theorem 6.1.1. The case k < 0 can be proved similarly by using the same argument but taking different signs in (6.2.15) and (6.2.16), completing the proof.

□

Remark 6.2.3. *For* k < 0, $k > 2$ *the minimizer is not smooth.*

We learnt from E.G. Virga of Dzyaloshinskii's paper [Dzy70] after completion of this thesis and it has some overlap with our results stated in Chapter 6.1 and 6.2, although our approach appears to be more rigorous.

6.3 Radius-dependent case

We have studied in Theorem 6.1.1 to Theorem 6.2.2 the energy minimization problem when the planar vector field $\mathbf{n}$ is radius-independent in the one circular domain $\Omega = (a, b)\times \mathbb{S}^1$, in the case of general Oseen-Frank energy (without one-constant approximation). A natural question is whether the minimizers $\widetilde{\mathbf{n}}$ in Theorem 6.1.1 are also minimizers for $I(\mathbf{n})$ in A_k. Equivalently, are the corresponding ϕ minimizers of (by (6.0.2))

$$\begin{aligned}\widetilde{E}(\phi) &= \int_D W(s, \theta, \phi, \phi_s, \phi_\theta)\, dx \\ &= \frac{1}{2}(k_1 + k_3)\int_D \Big((1 + \lambda\cos 2\phi)\,\phi_s^2 + (1 - \lambda\cos 2\phi)(1 + \phi_\theta)^2 + 2\lambda\phi_s\,(\phi_\theta + 1)\sin 2\phi\Big)\, dx\end{aligned} \tag{6.3.1}$$

where $D = (\log a, \log b) \times (0, 2\pi)$, for $\phi \in H^1(D; \mathbb{S}^1)$ satisfying

$$\phi(s, 2\pi) - \phi(s, 0) = 2(k - 1)\pi \quad \text{for a.e. } s \in (\log a, \log b).$$

Without loss of generality we can omit the constant $\frac{1}{2}(k_1 + k_3)$. This is obviously true in the one-constant case $\lambda = 0$ ($k_1 = k_3$), since the integrand $W(s, \theta, \phi, \phi_s, \phi_\theta) \geq (1 - \lambda\cos 2\phi)(1 + \phi_\theta)^2$. However the situation is more complicated for $\lambda \neq 0$. We illustrate this by the case $k = 2$, when the minimizers in Theorem 6.1.1 are given by $\phi(\theta) = \theta + \theta_0$.

Given a smooth $u : \overline{D} \to \mathbb{R}$ with $u(s, 2\pi) - u(s, 0) = 0$ for all $s \in [\log a, \log b]$, the second variation of $\widetilde{E}$ at ϕ is given by

$$\begin{aligned}\delta^2\widetilde{E}(\phi)(u, u) &= \frac{1}{2}\frac{d^2}{d\epsilon^2}\widetilde{E}(\phi + \epsilon u)\Big|_{\epsilon=0} \\ &= \int_{\log a}^{\log b}\int_0^{2\pi} \Big((1 + \lambda\cos 2(\theta + \theta_0))u_s^2 + (1 - \lambda\cos 2(\theta + \theta_0))u_\theta^2 \\ &\quad + 2\lambda\sin 2(\theta + \theta_0)u_\theta u_s + 8\lambda\cos 2(\theta + \theta_0)uu_s\Big)\, d\theta\, ds.\end{aligned}$$

If $u = 0$ for $s = \log a$ and $s = \log b$ then the last term in the integral integrates to zero, while the remaining terms give a positive definite quadratic form, so that

$$\delta^2\widetilde{E}(\phi)(u, u) \geq \mu \int_D (u_s^2 + u_\theta^2)\, dx$$

for some constant $\mu > 0$. Thus ϕ is a weak local minimizer of E subject to its own boundary conditions. If however, we do not restrict the values of u on ∂D then we can choose $u = v(s)\cos(\theta + \theta_0)$, when a calculation shows that

$$\delta^2\widetilde{E}(\phi)(u, u) = \pi \int_{\log a}^{\log b} \Big((1 + \frac{\lambda}{2})(v^2 + v_s^2) + 3\lambda vv_s\Big)\, ds.$$

When $v = e^s$ this gives

$$\delta^2 \widetilde{E}(\phi)(u, u) = 2\pi (1 + 2\lambda)(b - a)$$

which is negative for $-1 < \lambda < -\frac{1}{2}$, when ϕ is not a local minimizer. This implies that the minimizers for the radius-independent case given in Theorem 6.1.1 are not necessarily minimizers for the radius-dependent case.

Chapter 7

Conclusion and future direction

In this thesis we have shown:

(i) the existence and uniqueness of the minimization problem of a modified one-constant Oseen-Frank elastic energy

$$I(\mathbf{n}) = \frac{1}{2}\int_{\Omega}\left(|\nabla\mathbf{n}|^2 - \frac{k^2}{|x|^2}\right)dx$$

on a 2D one-circular domain $\Omega = \{x \in \mathbb{R}^2 \,|\, 0 < a < |x| < \infty\}$ (as well as shown its asymptotic behaviour at ∞ of the minimizer);

(ii) by introducing homotopy classes for vector fields subject to given boundary conditions and degree constraints, the existence and uniqueness of minimizers in each homotopy class for the same modified Oseen-Frank energy on two-dimensional N-connected domains $\Omega = \mathbb{R}^2 \setminus \cup_{j=1}^{N}\bar{\omega}_j$, where each of $\{\omega_j\}_{j=1}^{N}$ is bounded, simply connected and having Lipschitz boundary. Also we have shown the asymptotic behaviour of each of these minimizers as well as the fact that there exists a minimizer in the collection of all these homotopy classes though it may not be unique;

(iii) by introducing an auxiliary vector field and defining homotopy classes for line fields, the existence and uniqueness minimizing line field in each homotopy class for two-dimensional N-connected domain $\Omega = \mathbb{R}^2 \setminus \cup_{j=1}^{N}\bar{\omega}_j$ with non-orientable line field boundary data (for orientable boundary data, this proves to be equivalent to (ii));

(iv) the existence of minimizers (and what they are) to the Oseen-Frank elastic energy (with general Frank elastic constants, namely without one-constant approximation) for radius-independent vector fields (in a bounded 2D one-circular domain) subject to prescribed degree and its own boundary conditions on the circular boundary.

(v) for non-equal constant case, the minimizers to the Oseen-Frank elastic energy (with general Frank elastic constants) for radius-independent vector fields subject to prescribed degree on the boundary in a bounded 2D circular domain are not necessarily minimizers for the radius-dependent case.

Our original aim was to study these problems in 3D domain for general elastic constants (i.e. not under the one-constant approximation where $k_1 = k_2 = k_3$, $k_4 = 0$). This proved to be too hard, so we tried the 2D domain and derived the above results. In the future we may work on the following directions:

(1) the existence and uniqueness of minimizers to the general Oseen-Frank elastic energy for radius-dependent vector fields in a bounded 2D one-circular domain or N-connected domain (obviously more difficult) subject to prescribed degree conditions on the boundary;

(2) the existence and uniqueness of minimizers for some modified Oseen-Frank free energy on 3D domains (either with or without holes) for vector fields subject to prescribed boundary conditions as well as degree constraints, or line fields subject to given orientable or non-orientable line fields (orientability in 3D domain seems apparently more complicated than the 2D case).

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