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Robust Framework for Quantifying the Value of Information in Pricing and Hedging*

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Abstract. We investigate asymmetry of information in the context of the robust approach to pricing and hedging of financial derivatives. We consider two agents, one who only observes the stock prices and another with some additional information, and investigate when the pricing-hedging duality for the former extends to the latter. We introduce a general framework to express the superhedging and market model prices for an informed agent. Our key insight is that an informed agent can be seen as a regular agent who can restrict her attention to a certain subset of possible paths. We use results of Hou and Obłój [*Finance Stoch.*, 22 (2018), pp. 511–567], on the robust approach with beliefs to establish the pricing-hedging duality for an informed agent. Our results cover a number of scenarios, including information arriving before trading starts, arriving after the static position in European options is formed but before dynamic trading starts, or arriving at some point before maturity. For the latter we show that the superhedging value satisfies a suitable dynamic programming principle, which is of independent interest. Finally, we explore how our results allow us to develop robust valuation of information.

Key words. robustness superhedging, pricing-hedging duality, informed investor, asymmetry of information, filtration enlargement, path restrictions, dynamic programming principle, modeling with beliefs

AMS subject classifications. 91G20, 91B24, 91B44, 91B70, 90C46, 60G44

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1. Introduction.

The robust approach to pricing and hedging has been an active field of research in mathematical finance in recent years. In this approach, instead of choosing a single probabilistic model, one considers superhedging simultaneously under a family of models, or pathwise on a set of feasible trajectories. Typically dynamic trading in stocks and static (i.e., buy and hold) trading in some European options are allowed. This setup was pioneered in the seminal work of Hobson (1998), who obtained robust pricing and hedging bounds for lookback options. Therein, all martingale models which calibrate to the given market option prices, and superhedging for all canonical paths, were considered. Similar approach and setups, both in continuous and in discrete time, were used to study other derivatives and abstract

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pricing-hedging duality questions; see, e.g., Acciaio et al. (2016a); Beiglböck, Henry-Labordère, and Penkner (2013); Biagini et al. (2017); Brown, Hobson, and Rogers (2001); Cox and Obłój (2011b); Burzoni, Frittelli, and Maggis (2016); Cox and Obłój (2011a); Cox and Wang (2013); Dolinsky and Soner (2014, 2015) and the references therein. Other papers, often intertwined with the previous stream, focused on the case when superhedging is only required under a given family of probability measures or, more recently, on a given set of feasible paths; see, e.g., Lyons (1995); Avellaneda, Levy, and Parás (1995); Mykland (2003); Denis and Martini (2006); Bouchard and Nutz (2015); Hou and Obłój (2018) and the references therein.

The prevailing focus in the literature so far has been on the case when trading strategies are adapted to the natural filtration $\mathbb{F}$ of the price process S. In contrast, our main interest in this paper is to understand what happens when a larger filtration $\mathbb{G}$ is considered. One can think of $\mathbb{F}$ as the filtration of an unsophisticated or small agent and $\mathbb{G}$ as the filtration of a sophisticated agent who invests in acquiring additional information. Equally, $\mathbb{G}$ may correspond to the filtration of an insider. In general, the two filtrations model an asymmetry of information between the agents. We are primarily interested in the case when the additional information does not lead to instant arbitrage opportunities but does offer an advantage and we wish to quantify this advantage in the context of robust pricing and hedging of derivatives.

We develop a pathwise approach. Our key insight is that the pricing and hedging problem for the agent with information $\mathbb{G}$ can often be reduced to that of the standard agent who only considers a subset of the pathspace.[1] This allows us to use the duality results with beliefs obtained by Hou and Obłój (2018). Specifically, we consider the price process as the canonical process on a restriction of the space of $\mathbb{R}^n$-valued continuous functions on $[0,T]$. The price process represents assets, stocks or options, which are traded continuously, subject to the usual admissibility constraints; see Definition 2.4 below. We further allow static positions in a given set of options $\mathcal{X}$ whose market prices $\mathcal{P}$ are known. These are less liquid options which are not assumed to be traded after time zero. The additional information could arrive both before and/or after the static trading is executed. To account for this we add to $\mathbb{G}$ an additional element $\mathcal{G}_{-1}$, $\{\emptyset,\Omega\}\subset\mathcal{G}_{-1}\subset\mathcal{G}_0$ and require that the static position α is $\mathcal{G}_{-1}$-measurable. The initial cost of such a position is $\alpha\mathcal{P}(\mathcal{X})$ and hence the superhedging cost of a derivative with payoff ξ, for an agent with filtration $\mathbb{G}$, is given by

$$V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega) := \inf\Big\{\alpha(\omega)\mathcal{P}(\mathcal{X}) : \exists\ \mathbb{G}\text{-admissible } (\alpha,\gamma) \text{ s.t. } \alpha\mathcal{X}+\int_0^T \gamma_u dS_u \geq \xi \text{ on } \Omega\Big\},$$

where α also includes a position in cash. The pricing counterpart is obtained via

$$P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega) := \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}} \mathbb{E}_{\mathbb{P}}[\xi|\mathcal{G}_{-1}](\omega),$$

where the supremum is taken over $\mathbb{G}$-martingale measures calibrated to the market prices $\mathcal{P}$ of options $\mathcal{X}$ and where we take a suitable version of the conditional expectations.

[1] When working on this paper we were made aware of a related forthcoming paper by Acciaio, Cox, and Huesmann (2016b) which also considers informed agents using pathwise restrictions. However, the technical setup therein is closely related to the pathwise approach developed in Beiglböck et al. (2017b), based on Vovk's outer measure, and the paper develops a monotonicity principle in a similar sprit to Beiglböck, Cox, and Huesmann (2017a).

Our first main contribution is to make the above key insight and discussion precise. In section 3, we provide suitable definitions and show that, for a generic $\mathbb{G}$, both $V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)$ and $P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)$ are well defined and constant on atoms of $\mathcal{G}_{-1}$. We then discuss how prices, e.g., the superhedging price, in $\mathbb{F}$ and $\mathbb{G}$ can be used to quantify the value of information. Naturally, such valuation will depend on the agent's problem. We develop two cases but note that our methods could be applied to other setups. First, we consider an agent superhedging a fixed ξ who is interested in a robust (worst-case) analysis. Second, we consider an agent who faces risk controls in the form of a superhedging price budget but who otherwise is able to decide on the composition of her portfolio and who is an expected utility maximizer. Worked-out examples for both scenarios are provided in subsection 4.4.

The key consequence of the above characterization of $V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)$ and $P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)$ is that the pricing-hedging duality in $\mathbb{G}$ can be established using the duality in $\mathbb{F}$ for subsets of the pathspace. We explore this reasoning, focusing on the case of an initial enlargement: $\mathcal{G}_0 = \sigma(Z)$, for some $\mathcal{F}_T$-measurable random variable Z. First, we treat the case when $\mathcal{G}_{-1} = \mathcal{G}_0$ and show that the duality for the informed agent is the same as duality for uninformed agents with beliefs of the form $\{\omega : Z(\omega) = c\}$. The latter can be ensured using the results in Hou and Obłój (2018), subject to the assumptions therein. We discuss two examples: a specific information $Z = \sup_{t\in[0,T]} |S_t - 1|$ and a binary information $Z = \mathbb{1}_{\{S_t\in(a,b)\ \forall t\in[0,T]\}}$. The former is considered in the case without statically traded options and the latter both with and without such options. Subsequently, we show that the duality also extends to the case of trivial $\mathcal{G}_{-1}$, under mild technical assumptions on Z and in the case with no statically traded options.

Second, we focus on the case when the additional information is disclosed at an intermediate fixed time $T' \in (0, T)$, i.e., the filtration $\mathbb{G}$ is of the form

$$\mathcal{G}_t = \mathcal{F}_t \text{ for } t \in [0, T') \text{ and } \mathcal{G}_t = \mathcal{F}_t \vee \sigma(Z) \text{ for } t \in [T', T].$$

To prove the pricing-hedging duality we establish a dynamic programming principle for both the superhedging cost $V(\xi)$ and the market model price $P(\xi)$. These are of independent interest even in the case of $\mathbb{F} = \mathbb{G}$. These results are obtained in the case without any statically traded options. Indeed, in line with the findings in Aksamit et al. (2019), we should not expect the dynamic programming principle to hold when trading at time $t = 0$ involves more assets than at subsequent times $t \in (0, T)$.

We note that the pricing-problem for an informed agent was also examined recently in Acciaio and Larsson (2017). However, the focus therein is very different to ours. The authors do not study the pricing-hedging duality but instead focus on the pricing aspect $P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)$ with a trivial $\mathcal{G}_{-1}$. They show that it is enough to optimize over extreme measures and characterize these, in analogy to the seminal work of Jacod and Yor (1977), as the ones under which semistatic completeness holds, i.e., perfect hedging of all suitably integrable ξ using the underlying assets and statically traded derivatives in $\mathcal{X}$. Under some further assumptions on $\mathbb{G}$, semistatic completeness under $\mathbb{Q}$ is equivalent to filtrations $\mathbb{F}$ and $\mathbb{G}$ coinciding under $\mathbb{Q}$.

Our paper is organized as follows. In section 2 we define the robust pricing and hedging setup we work in. In section 3 we define the relevant pricing and hedging notions for the informed agents and establish their characterizations via pathspace restrictions. Then, in subsection 3.2, we discuss how these results can be used to quantify the value of information. In section 4 we present our main results on the pricing-hedging duality under an initially enlarged filtration. Theorem 4.6 treats the case of instantly available information and Theorem 4.8 the case when the information may not be used to decide on the initial capital. Subsection 4.4 discusses examples for both the pricing-hedging duality and the valuation of information. In section 5 we study the dynamic situation when the additional information arrives at some time $T' \in (0,T)$. We establish the relevant dynamic programming principles in Propositions 5.1 and 5.2, and then we obtain a pricing-hedging duality result in Theorem 5.4. Finally, in subsection 5.3, we discuss how the value of information is affected by the timing of its arrival.

2. General setup. The space of nonnegative continuous functions from $[0,T]$ into $\mathbb{R}^n_+$ is denoted $C([0,T],\mathbb{R}^n_+)$ and is endowed with the sup norm, $||f|| := \sup_{t\le T}|f(t)|$ with $|f(t)| := \sup_{1\le i\le n}|f^i(t)|$. The filtration generated by its coordinate process is denoted $\mathbb{F} := (\mathcal{F}_t)_{t\le T}$. Note that it is not a right-continuous filtration but each element $\mathcal{F}_t$, or more generally $\mathcal{F}_\rho$ for a stopping time ρ, is countably generated as the Borel σ-field on a Polish space. The dimension n will always be clear from the context. Further, we will mostly work with a closed subset $\Omega \subset C([0,T],\mathbb{R}^n_+)$ and we then *implicitly restrict* $\mathbb{F}$ *to* Ω. Note that $(\Omega, ||\cdot||)$ is Polish.

2.1. Traded assets. We consider a financial market with $d+1$ underlying assets: a numeraire and $d \in \mathbb{N}$ risky underlying assets. We work in a frictionless setting with no transaction costs. The prices are denominated in the units of a numeraire, e.g., bank account, whose price is thus normalized to 1. We suppose that the prices of the d risky underlying assets are continuous and, with no loss of generality, their initial prices are normalized to 1. Apart from the risky underlying assets, there may be derivative products traded on the market. Some of these could be particularly liquid—we will assume they trade dynamically—and others may be less liquid and only available for static trading, i.e., buy and hold strategies at the initial time. We only consider European derivatives which can be seen as $\mathcal{F}_T$-measurable functions $X : C([0,T],\mathbb{R}^d_+) \to \mathbb{R}$. In line with Hou and Obłój (2018), we further assume all the payoffs X are bounded and uniformly continuous. We let $\mathcal{X} = (\mathcal{X}^\lambda)_{\lambda\in\Lambda}$, where Λ is a set of an arbitrary cardinality, be the vector of the payoffs of market options available for static trading. All $X \in \mathcal{X}$ are assumed to have a well-defined price at time zero given by $\mathcal{P}(X)$. To simplify the notation, we assume that $0 \in \Lambda$ and $\mathcal{X}^0$ is a unit of the numeraire, $\mathcal{X}^0(\omega) \equiv 1$ and $\mathcal{P}(\mathcal{X}^0) = 1$. There are further K options, $K \ge 0$, with nonnegative payoffs $X^{(c)}_1, \dots, X^{(c)}_K$, which are traded dynamically. We normalize their payoffs so that their initial prices are equal to 1 and we model this situation by augmenting the set of risky assets. Thus, we consider $d+K$ assets that may be traded at any time, d underlyings and K options, with paths in $C([0,T],\mathbb{R}^{d+K}_+)$. Specifically, as the prices at maturity T must be consistent with payoffs, only the following subset of $C([0,T],\mathbb{R}^{d+K}_+)$ is considered:

$$\Omega := \left\{\omega \in C([0,T],\mathbb{R}^{d+K}_+) \colon \omega_0 = (1,\dots,1),\ \omega^{(d+i)}_T = X^{(c)}_i(\omega^{(1)},\dots,\omega^{(d)})/\mathcal{P}(X^{(c)}_i) \quad \forall i \le K\right\}.$$

The space Ω was called the *information space* in Hou and Obłój (2018) since it encodes the information about the initial prices and the payoffs of continuously traded options.

2.2. Information. It is convenient to introduce a separate notation for the canonical process on Ω so we let $S_t(\omega) := \omega_t$. Recall that $\mathbb{F} := (\mathcal{F}_t)_{t\le T}$ is the filtration generated by S, i.e., $\mathcal{F}_t := \sigma(S_s : s \leq t)$ for each $t \in [0,T]$. We also consider an enlarged filtration $\mathbb{G} := (\mathcal{G}_t)_{t\leq T}$ defined as $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$, for each $t \in [0,T]$, where $\mathbb{H} := (\mathcal{H}_t)_{t\leq T}$ is another filtration with $\mathcal{H}_T = \mathcal{F}_T$. We emphasize here that we are only interested in an enlarged filtration $\mathbb{G}$ such that $\mathcal{G}_T = \mathcal{F}_T$. In other words, we consider additional information which is directly related to the canonical process S and is known to all agents at time T. We think about the additional information which is exogenous with respect to S as irrelevant or insignificant for our pricing and hedging problems.[2]

In the literature, e.g., Jeulin (1980), Jeulin and Yor (1978), Mansuy and Yor (2006), typically two special cases of the filtration $\mathbb{H}$ are studied. First, the filtration $\mathbb{H}$ can be taken constant, $\mathcal{H}_t = \sigma(Z)$, where Z is a random variable, and $\mathbb{G}$ is then called the initial enlargement of $\mathbb{F}$ with Z. Second, the filtration $\mathbb{H}$ can be taken as $\mathcal{H}_t = \sigma(\mathbb{1}_{\{\rho\leq s\}} : s \leq t)$, where ρ is a random time (a nonnegative random variable), and $\mathbb{G}$ is then called the progressive enlargement of $\mathbb{F}$ with a random time ρ. In our setup we would consider $\mathcal{F}_T$-measurable random variable Z for an initial enlargement and $\mathcal{F}_T$-measurable random time ρ for a progressive enlargement.

In our considerations it is important to specify *when* the additional information arrives—some initial trading decisions may have to be taken before and some after the additional information is acquired. To model such situations, we add to each filtration $\mathbb{G}$ an additional element $\mathcal{G}_{-1}$ with $\{\emptyset, \Omega\} \subset \mathcal{G}_{-1} \subset \mathcal{G}_0$. For a given arbitrary filtration $\mathbb{G}$ we denote by $\mathbb{G}^+$ the filtration such that $\mathcal{G}^+_{-1} = \mathcal{G}_0$ and $\mathcal{G}^+_t = \mathcal{G}_t$ for $t \in [0,T]$. Similarly, we denote by $\mathbb{G}^-$ the filtration such that $\mathcal{G}^-_{-1} = \{\emptyset, \Omega\}$ and $\mathcal{G}^-_t = \mathcal{G}_t$ for $t \in [0,T]$. We note that for the natural filtration of the price process $\mathbb{F}$ the only choice is $\mathcal{F}_{-1} = \mathcal{F}_0 = \{\emptyset, \Omega\}$. Finally, we make the following assumption.

Standing Assumption 2.1. *All σ-fields $\mathcal{G}_t$, $t \in \{-1\} \cup [0,T]$, in the enlarged filtration $\mathbb{G}$ are countably generated.*

2.3. Trading strategies. We now discuss the notion of atoms of a σ-field and introduce the right class of trading strategies with respect to a general filtration $\mathbb{G}$. We refer to (Dellacherie and Meyer (1975), Chap. 1, sect. 9–12) for useful details. For a measurable space $(\Omega, \mathcal{F}_T)$ and a sub σ-field $\mathcal{G} \subset \mathcal{F}_T$ we introduce the following equivalence relation.

Definition 2.2. *Let ω and $\widetilde{\omega}$ be two elements of Ω, and let $\mathcal{G} \subset \mathcal{F}_T$ be a σ-field. Then we say that ω and $\widetilde{\omega}$ are $\mathcal{G}$-equivalent, and write $\omega \sim_{\mathcal{G}} \widetilde{\omega}$, if for each $G \in \mathcal{G}$ we have $\mathbb{1}_G(\omega) = \mathbb{1}_G(\widetilde{\omega})$.*

We call $\mathcal{G}$-atoms the equivalence classes in Ω with respect to this relation. We denote by A^ω the atom which contains ω:

$$A^\omega = \bigcap \{A : A \in \mathcal{G}, \omega \in A\}.$$

If $\mathcal{G} = \sigma(Z)$, then $A^\omega = Z^{-1}(\{Z(\omega)\})$ and $\omega \sim_{\sigma(Z)} \widetilde{\omega}$ if and only if $Z(\omega) = Z(\widetilde{\omega})$. Also, in our setting, $\omega \sim_{\mathcal{F}_t} \widetilde{\omega}$ if and only if $\omega_u = \widetilde{\omega}_u$ for each $u \leq t$ (or equivalently for each rational u in $[0,t]$). Note that if $\mathcal{G}$ is countably generated, i.e., there exists a countable sequence $(B_n)_{n\geq 1}$

[2]We refer the reader to Biagini and Zhang (2019), where such a situation was considered for the purpose of $\mathcal{Q}$-quasi-sure reduced form modeling. Incorporating exogenous information was tackled by enlarging the space Ω to $\Omega \times \Omega'$ where $(\Omega', \mathcal{G}) = ([0,1], \mathcal{B}([0,1]))$ and imposing $\widetilde{\mathcal{Q}} = \{\widetilde{\mathbb{Q}} := \mathbb{Q} \times \mathcal{U}([0,1]) : \mathbb{Q} \in \mathcal{Q}\}$.

such that $\mathcal{G} = \sigma(B_n : n \geq 1)$, then each atom is an element of $\mathcal{G}$ as $A^\omega = \bigcap_n C_n$ is a countable intersection, where $C_n = B_n$ if $\omega \in B_n$ and $C_n = B_n^c$ if $\omega \in B_n^c$. Also, it is then enough to check the relation from Definition 2.2 on the generators of $\mathcal{G}$.

In Definitions 2.4 and 2.5 we shall define trading in such a way that strategies can be chosen separately on each atom of $\mathcal{G}_0$ for dynamic trading and each atom of $\mathcal{G}_{-1}$ for static trading. We believe it is an important feature. Our approach is inspired by Billingsley (1995), Dubra and Echenique (2004), and Hervés-Beloso and Monteiro (2013) and we now briefly recall their ideas. We fix a countably generated $\mathcal{G}$. In particular, there exists a $\mathcal{G}$-measurable random variable $Z : \Omega \to [0,1]$ such that $\mathcal{G} = \{Z^{-1}(B) : B \in \mathcal{B}([0,1])\}$. For a decision-making agent, having information $\mathcal{G}$ is equivalent to having information about the *signal* Z. Consider the partition of Ω induced by Z, i.e., $\Delta(Z) := \{Z^{-1}(x) : x \in [0,1]\}$. Following Hervés-Beloso and Monteiro (2013), we say that a subset $\mathfrak{P}$ of Ω is an *informed set* if

$$\text{for each } A \in \Delta(Z) \text{ we have that } A \subset \mathfrak{P} \text{ or } A \subset \mathfrak{P}^c.$$

The set $\mathscr{I}(Z)$ of all informed sets is called the *informational content* of the signal Z and is a σ-field. We have $\sigma(Z) \subset \mathscr{I}(Z)$ and in general the inclusion is strict since $\mathscr{I}(Z)$ allows for uncountable unions. For an agent who can observe the outcome of Z it is reasonable to assume that she can adjust her decisions accordingly. More specifically, a mapping is said to use information contained in $\mathcal{G}$, or equivalently in the signal Z, if it is constant on atoms of $\mathcal{G}$ or, equivalently, if it is $\mathscr{I}$-measurable. The set of all such functions is given by

$$\mathcal{U}(\mathcal{G}) := \{M : \Omega \to \mathbb{R} \text{ s.t. } \omega \sim_{\mathcal{G}} \widetilde{\omega} \text{ implies } M(\omega) = M(\widetilde{\omega})\} \tag{2.1}$$

and is, in general, larger than the set of $\mathcal{G}$-measurable functions, as the following example demonstrates.

Example 2.3. Let us consider the measurable space $([0,1], \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra. Note that atoms of $\mathcal{B}$ are single points and therefore $\mathcal{U}(\mathcal{B})$ consists of all real functions on $[0,1]$. Clearly, the class $\mathcal{U}(\mathcal{B})$ is a much larger set that the class of real Borel functions on $[0,1]$.

Definitions 2.4 and 2.5 are aligned with the informational content of a signal. In the above, we set $\mathcal{G} = \mathcal{G}_t$, for some fixed $t \in \{-1\} \cup [0,T]$, and recall that by Standing Assumption 2.1 $\mathcal{G}$ is countably generated. Information provided by the σ-field $\mathcal{G}$ is seen as a signal or, equivalently, as a partition of Ω into atoms. An agent observing the signal can restrict their attention to one atom at a time. Accordingly, after receiving the signal, the agent choses a superhedging strategy which satisfies the usual admissibility constraints: it is nonanticipative and generates a wealth which is bounded from below. However, these admissibility constraints are considered on an atom-by-atom basis.

We only consider trading strategies γ which are of finite variation. This allows us, similarly to Dolinsky and Soner (2014) and Hou and Obłój (2018), to define integrals pathwise simply via the integration by parts formula:

$$\int_0^t \gamma_u d\omega_u := \gamma_t \omega_t - \gamma_0 \omega_0 - \int_0^t \omega_u d\gamma_u, \quad \omega \in \Omega.$$

The above integration provides dynamic trading in continuously traded assets as defined below.

Definition 2.4. (i) *The mapping* $\gamma : \Omega \times [0,T] \to \mathbb{R}^{d+K}$ *is called*

(a) *càdlàg if for each* $\omega \in \Omega$, $\gamma(\omega) : [0,T] \to \mathbb{R}^{d+K}$ *is càdlàg,*

(b) $\mathbb{G}$*-progressively measurable on atoms of* $\mathcal{G}_0$ *if for each atom* A^ω *of* $\mathcal{G}_0$ *the mapping* $\gamma \mathbb{1}_{A^\omega}$ *is* $\mathbb{G}$*-progressively measurable.*

(ii) *The mapping* $\gamma : \Omega \times [0,T] \to \mathbb{R}^{d+K}$ *is called* $(\mathbb{G}, M)$*-admissible for* $M \in \mathcal{U}(\mathcal{G}_0)$*, where* $\mathcal{U}(\mathcal{G}_0)$ *is defined in* (2.1)*, if it is càdlàg,* $\mathbb{G}$*-progressively measurable on atoms of* $\mathcal{G}_0$*, and of finite variation, satisfying*

$$\int_0^t \gamma(\omega)_u dS_u(\omega) \geq -M(\omega) \quad \forall\, t \in [0,T],\ \omega \in \Omega\,. \tag{2.2}$$

The set of all $(\mathbb{G}, M)$*-admissible strategies is denoted* $\mathcal{A}^M(\mathbb{G})$*. The set of all* $\mathbb{G}$*-admissible strategies is defined by*

$$\mathcal{A}(\mathbb{G}) := \bigcup_{M \in \mathcal{U}(\mathcal{G}_0)} \mathcal{A}^M(\mathbb{G}). \tag{2.3}$$

In addition to the dynamic trading, we allow for static trading in options in $\mathcal{X}$. The static position is taken w.r.t. the information in $\mathcal{G}_{-1}$. This leads to the following definition.

Definition 2.5. (i) *Let* $M \in \mathcal{U}(\mathcal{G}_0)$*. A* $(\mathbb{G}, M)$*-admissible semistatic strategy is a pair* (α, γ) *where* $\gamma \in \mathcal{A}^M(\mathbb{G})$ *and* $\alpha = (\alpha^\lambda)_{\lambda \in \Lambda}$ *such that for each* $\lambda \in \Lambda$, $\alpha^\lambda \in \mathcal{U}(\mathcal{G}_{-1})$ *and for each atom* A^ω *of* $\mathcal{G}_{-1}$ *there exists a finite subset* $\Lambda_0^\omega \subset \Lambda$ *such that* $\alpha^\lambda(\omega) = 0$ *for each* $\lambda \notin \Lambda_0^\omega$*.*

(ii) *The set of all* $(\mathbb{G}, M)$*-admissible semistatic strategies is denoted by* $\mathcal{A}^M_{\mathcal{X}}(\mathbb{G})$ *and all* $\mathbb{G}$*-admissible semistatic strategies are given by*

$$\mathcal{A}_{\mathcal{X}}(\mathbb{G}) := \bigcup_{M \in \mathcal{U}(\mathcal{G}_0)} \mathcal{A}^M_{\mathcal{X}}(\mathbb{G}). \tag{2.4}$$

Note that if $\mathcal{G} \subset \widetilde{\mathcal{G}}$, then $\mathcal{U}(\mathcal{G}) \subset \mathcal{U}(\widetilde{\mathcal{G}})$. Importantly, it follows that $\mathcal{A}_{\mathcal{X}}(\mathbb{G}) \subset \mathcal{A}_{\mathcal{X}}(\widetilde{\mathbb{G}})$ whenever $\mathbb{G} \subset \widetilde{\mathbb{G}}$. Finally, let A^ω denote an atom of $\mathcal{G}_{-1}$ and (α, γ) be a $\mathbb{G}$-admissible trading strategy. Note that, for each $\widetilde{\omega} \in A^\omega$, the initial cost of (α, γ) equals

$$\alpha(\omega)\mathcal{P}(\mathcal{X}) = \alpha(\widetilde{\omega})\mathcal{P}(\mathcal{X}) = \alpha^0(\widetilde{\omega}) + \sum_{\lambda \in \Lambda} \alpha^\lambda(\widetilde{\omega})\mathcal{P}(\mathcal{X}^\lambda) = \alpha^0(\widetilde{\omega}) + \sum_{\lambda \in \Lambda_0^\omega} \alpha^\lambda(\widetilde{\omega})\mathcal{P}(\mathcal{X}^\lambda).$$

For each $\omega = (\omega^{(1)}, \ldots, \omega^{(d)}, \omega^{(d+1)}, \ldots, \omega^{(d+K)}) \in \Omega$ the final payoff of (α, γ) is given by

$$(\gamma \circ S)_T(\omega) + (\alpha\mathcal{X})(\omega) = \int_0^T \gamma(\omega)_u dS_u(\omega) + \sum_{\lambda \in \Lambda} \alpha^\lambda(\omega)\mathcal{X}^\lambda(\omega^{(1)}, \ldots, \omega^{(d)}). \tag{2.5}$$

3. Pathspace approach to information quantification.

We are now in a position to develop our general framework to express the superhedging and market model prices for an informed agent. We work with a general filtration $\mathbb{G}$ and this induces further difficulties, as compared with the case of the natural filtration $\mathbb{F}$. Our key insight is to see the informed agent as a regular agent who can restrict her attention to a certain subset of possible paths. We then argue how the framework can be used to quantify the value of the information for an agent.

3.1. Robust pricing and hedging for an informed agent. We start with a superhedging problem.

Definition 3.1. *Let $A \subset \Omega$. The $\mathbb{G}$-superhedging cost of ξ on A is given by*

$$V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A}(\xi)(\omega) := \inf \big\{\alpha(\omega)\mathcal{P}(\mathcal{X}) : \exists\, (\alpha, \gamma) \in \mathcal{A}_{\mathcal{X}}(\mathbb{G}) \text{ such that } (\gamma \circ S)_T(\widetilde{\omega}) + (\alpha\mathcal{X})(\widetilde{\omega}) \geq \xi(\widetilde{\omega}) \text{ for each } \widetilde{\omega} \in A\big\}, \quad \omega \in A,$$

where $(\gamma \circ S)_T + (\alpha\mathcal{X})$ is defined in (2.5).

Thus the $\mathbb{G}$-superhedging cost of ξ on A is the pathwise infimum over all initial costs $\alpha\mathcal{P}(\mathcal{X})$ of $\mathbb{G}$-admissible semistatic strategies $(\alpha, \gamma) \in \mathcal{A}_{\mathcal{X}}(\mathbb{G})$ which superreplicate ξ on A over $[0, T]$, i.e., $\alpha\mathcal{X} + (\gamma \circ S)_T \geq \xi$ on A. Recall that $\mathcal{A}_{\mathcal{X}}(\mathbb{G}) \subset \mathcal{A}_{\mathcal{X}}(\widetilde{\mathbb{G}})$ whenever $\mathbb{G} \subset \widetilde{\mathbb{G}}$. In particular, the following inequalities are clear:

$$V^{\mathbb{G}^+}_{\mathcal{X},\mathcal{P},A}(\xi) \leq V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A}(\xi) \leq V^{\mathbb{G}^-}_{\mathcal{X},\mathcal{P},A}(\xi) \leq V^{\mathbb{F}}_{\mathcal{X},\mathcal{P},A}(\xi). \tag{3.1}$$

As soon as $\mathcal{G}_{-1}$ is not trivial, $\alpha\mathcal{P}(\mathcal{X})$ is random and hence $V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A}(\xi)$ is random as well and its measurability is not clear a priori. However, the following result shows that the superhedging cost is constant on the atoms of $\mathcal{G}_{-1}$. In particular, when $\mathcal{G}_{-1}$ has at most countably many atoms, e.g., is generated by a discrete random variable, then it follows that the superhedging cost is $\mathcal{G}_{-1}$-measurable.

Proposition 3.2. *The $\mathbb{G}$-superhedging cost of ξ on Ω, defined in Definition 3.1, is constant on atoms of $\mathcal{G}_{-1}$. Specifically, for any $\omega \in \Omega$ we have*

$$V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega) = V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega') = V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}(\xi)(\omega') \quad \forall \omega' \in A^\omega,$$

where A^ω denotes the $\mathcal{G}_{-1}$-atom containing ω.

Proof. Fix $\omega \in \Omega$ and $\omega' \in A^\omega$. It follows from $\alpha^\lambda \in \mathcal{U}(\mathcal{G}_{-1})$ for each $\lambda \in \Lambda$ that $\alpha(\omega)\mathcal{P}(\mathcal{X}) = \alpha(\omega')\mathcal{P}(\mathcal{X})$ for any $\mathbb{G}$-admissible strategy $(\alpha, \gamma) \in \mathcal{A}_{\mathcal{X}}(\mathbb{G})$ which superreplicates ξ on A^ω. This in turn implies that

$$V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}(\xi)(\omega) = V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}(\xi)(\omega').$$

It remains to argue that these are also equal to $V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega')$. Clearly the latter can only be larger as the superhedging is required on a larger set. As for the reverse inequality, note that any $\mathbb{G}$-admissible semistatic strategy (α, γ) which superreplicates ξ on A^ω can be extended to a strategy $(\bar{\alpha}, \bar{\gamma})$ superreplicating ξ on Ω by taking $\bar{\alpha} = \alpha$ on A^ω, and $\bar{\alpha}^0 = ||\xi||$ and $\bar{\alpha}^\lambda = 0$ for $\lambda \neq 0$ otherwise, $\bar{\gamma} = \gamma \mathbb{1}_{A^\omega}$. ∎

With a slight abuse of notation, we write $V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}(\xi)$ for both the constant function, extended to Ω, as well as its value equal to $V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)_{|A^\omega}$.

We now turn to the pricing problem. In the classical approach markets with no-arbitrage are modeled using martingale measures. We denote by $\mathcal{M}^{\mathbb{G}}$ the set of probability measures on $(\Omega, \mathcal{F}_T)$ such that S is a $\mathbb{G}$-martingale. For a given $A \in \mathcal{F}_T$, we look at possible classical market models which calibrate to market prices of options and are supported on A:

$$\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A} := \Big\{\mathbb{P} \in \mathcal{M}^{\mathbb{G}} : \mathbb{P}(A) = 1 \text{ and } \mathbb{E}_{\mathbb{P}}[\mathcal{X}^\lambda|\mathcal{G}_{-1}] = \mathcal{P}(\mathcal{X}^\lambda)\ \forall\, \lambda \in \Lambda,\ \mathbb{P}\text{-a.s.}\Big\}.$$

We emphasize that any measure $\mathbb{P} \in \mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A}$ is calibrated to the initial prices of options in $\mathcal{X}$. In particular, for any atom A^ω of $\mathcal{G}_{-1}$ one has $\mathbb{E}_\mathbb{P}[\mathcal{X}^\lambda \mathbb{1}_{A^\omega}] = \mathbb{E}_\mathbb{P}[\mathcal{P}(\mathcal{X}^\lambda)\mathbb{1}_{A^\omega}]$ for each $\lambda \in \Lambda$.

To consider the pricing problem we need to look at "$\sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}} \mathbb{E}_\mathbb{P}[\xi|\mathcal{G}_{-1}]$." However, unless $\mathcal{G}_{-1}$ is a trivial σ-field, the conditional expectation $\mathbb{E}_\mathbb{P}[\xi|\mathcal{G}_{-1}]$ is a random variable which is determined only $\mathbb{P}$-a.s. As measures in the set $\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}$ may be mutually singular we have to be careful about choosing a good version of these conditional expectations.

Lemma 3.3. *Let $\mathbb{P} \in \mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}$. Then, there exists a set $\Omega^\mathbb{P} \in \mathcal{G}_{-1}$ with $\mathbb{P}(\Omega^\mathbb{P}) = 1$ and a version $\{\mathbb{P}_\omega\}$ of the regular conditional probabilities of $\mathbb{P}$ with respect to $\mathcal{G}_{-1}$ such that for each $\omega \in \Omega^\mathbb{P}$, $\mathbb{P}_\omega \in \mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}$, where A^ω is the $\mathcal{G}_{-1}$-atom containing ω.*

The proof, based on standard existence results for regular conditional probabilities, is reported in the appendix. Suppose now that we fix a representation $\Omega^\mathbb{P}$ in Lemma 3.3 for each $\mathbb{P} \in \mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}$ and use these to define the market model price as follows.

Definition 3.4. *Let $A \in \mathcal{F}_T$. The $\mathbb{G}$-market model price of ξ on A is defined by*

$$P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A}(\xi)(\omega) := \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A}} \bar{\mathbb{E}}_{\mathbb{P}_\omega}[\xi], \quad \omega \in \Omega,$$

where $\bar{\mathbb{E}}_{\mathbb{P}_\omega}[\xi] = \mathbb{E}_{\mathbb{P}_\omega}[\xi]$ for $\omega \in \Omega^\mathbb{P}$ and $\bar{\mathbb{E}}_{\mathbb{P}_\omega}[\xi] = -\infty$ for $\omega \in \Omega\backslash\Omega^\mathbb{P}$, where $\Omega^\mathbb{P}$ is a set from Lemma 3.3.

We now show that, as in the case of superhedging, the model price is constant on atoms of $\mathcal{G}_{-1}$ and is uniquely defined, irrespective of the choice of $\Omega^\mathbb{P}$ above. It follows that, while in general the measurability of $P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)$ is not clear, when $\mathcal{G}_{-1}$ has at most countably many atoms then $P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)$ is $\mathcal{G}_{-1}$-measurable.

Proposition 3.5. *The $\mathbb{G}$-market model price of ξ on Ω is uniquely defined on Ω and is constant on atoms of $\mathcal{G}_{-1}$. Specifically, for $\omega \in \Omega$ we have*

$$P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega) = P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega') = P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}(\xi)(\omega') = \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}} \mathbb{E}_\mathbb{P}[\xi] \quad \forall \omega' \in A^\omega,$$

where A^ω is the $\mathcal{G}_{-1}$-atom containing ω.

Proof. Let $\mathbb{P} \in \mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}$, and let $\Omega^\mathbb{P}$ be any set satisfying the conditions from Lemma 3.3 for measure $\mathbb{P}$. In particular $\Omega^\mathbb{P}$ is $\mathcal{G}_{-1}$-measurable and $\mathbb{P}(\Omega^\mathbb{P}) = 1$. Hence, since A^ω is an atom of $\mathcal{G}_{-1}$ and $\mathbb{P}(A^\omega) = 1$, we have that $A^\omega \subset \Omega^\mathbb{P}$ and $\mathbb{P}_{\omega'} = \mathbb{P}$ for all $\omega' \in A^\omega$. Therefore, $P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}(\xi)(\omega') = \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}} \mathbb{E}_\mathbb{P}[\xi]$ for all $\omega' \in A^\omega$, i.e., $P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}(\xi)$ is well defined and constant on A^ω.

Further, since additionally the obvious inclusion $\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega} \subset \mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}$ holds, we have that $\sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}} \mathbb{E}_\mathbb{P}[\xi] \le \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}} \bar{\mathbb{E}}_{\mathbb{P}_{\omega'}}[\xi]$ for all $\omega' \in A^\omega$. On the other hand, let $\mathbb{P} \in \mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}$ and $\Omega^\mathbb{P}$ be a set from Lemma 3.3. Let $\omega' \in A^\omega$. If $A^\omega \subset \Omega^\mathbb{P}$, then $\mathbb{P}_{\omega'} \in \mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}$; otherwise if $A^\omega \cap \Omega^\mathbb{P} = \emptyset$, then $\bar{\mathbb{E}}_{\mathbb{P}'_\omega}[\xi] = -\infty$, which shows that $\sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}} \mathbb{E}_\mathbb{P}[\xi] \ge \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}} \bar{\mathbb{E}}_{\mathbb{P}_{\omega'}}[\xi]$ for all $\omega' \in A^\omega$. In consequence,

$$P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega') = P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}(\xi)(\omega'), \quad \omega' \in A^\omega,$$

as required. In particular, $P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)$ is well defined and constant on atoms and does not depend on the choice of $\Omega^{\mathbb{P}}$ for $\mathbb{P}\in\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}$. Finally, if for some $\omega\in\Omega$, $\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}=\emptyset$, then, for any $\mathbb{P}\in\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}$, $\Omega^{\mathbb{P}}\cap A^\omega=\emptyset$ and the definitions agree giving

$$\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\omega)=-\infty. \qquad \blacksquare$$

With a slight abuse of notation, we write $P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}(\xi)$ both for the constant function, extended to Ω, as well as its value equal to $\sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}}\mathbb{E}_{\mathbb{P}}[\xi]=P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)_{|A^\omega}$.

Using Proposition 3.5 and the fact that the families of martingale measures decrease with filtration enlargement, we note that the following inequalities hold:

$$P^{\mathbb{G}^+}_{\mathcal{X},\mathcal{P},A}(\xi)\le P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A}(\xi)\le P^{\mathbb{G}^-}_{\mathcal{X},\mathcal{P},A}(\xi)\le P^{\mathbb{F}}_{\mathcal{X},\mathcal{P},A}(\xi).$$

Remark 3.6. We note that Proposition 3.5 implies that when computing the $\mathbb{G}$-market model price of ξ on Ω, or on some set in $\mathcal{G}_{-1}$, it is sufficient to maximize over measures supported on a single atom of $\mathcal{G}_{-1}$ but we consider all the measures in $\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}$ and not only the extreme ones. More generally, the measures considered are not necessarily the extreme measures in $\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}$, which were the focus in Acciaio and Larsson (2017).

3.2. The value of information in pricing and hedging. Having defined the superheding cost and the market model price of a payoff ξ with respect to different informational contents allows us to quantify the advantage of acquiring additional information in the context of robust pricing and hedging of ξ. It is captured via

$$V^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)-V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega)\quad\text{or by}\quad P^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)-P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega).$$

The two quantities have similar properties and are equal when the robust pricing-hedging duality (for both the regular and the informed agent) holds. We focus on $V^{\mathbb{F}}-V^{\mathbb{G}}$ as it has a direct interpretation in terms of trading. Note that this is a random quantity. However, using a suitable functional of it, the agent can quantify the value of information. We focus here on two different situations which lead to different functionals. Clearly, other functionals may be relevant in other contexts. The discussion below is mainly formal as it explores situations in which the methodology introduced above can be employed to quantify the value of information. We then present some examples where we work both approaches in detail; see Example 4.10(b), (c), and (d) and Example 4.11(b).

First, we consider a trader who uses the robust approach to hedge a short position in a contract with payoff ξ. At price $V^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)$, she can set up the robust hedge which accounts for any uncertainty in the markets and protects from transaction costs associated with delta-vega hedging. The difference between the market price of ξ and $V^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)$ is a lower bound on the trader's loss and such control over the lower tail of P&L may be attractive to a risk-averse trader even under mild model uncertainty and/or transaction costs; see Obłój and Ulmer (2012). The trader may employ an equally conservative valuation of the information and consider

$$\mathfrak{A}^{\mathbb{G},\mathbb{F}}_V[\xi]:=\inf_{\omega\in\Omega}\Big(V^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)-V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega)\Big)=V^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)-\sup_{\omega\in\Omega}V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega). \tag{3.2}$$

At this price, or lower, acquiring the information is always advantageous and improves the lower bound on the trader's loss. We note that this quantity is nonnegative and in Example 4.10(b) and (c) and Example 4.11(b) we show that it may be zero but may also be strictly positive.

The second scenario we wish to consider is of a trader who makes her own trading decisions but is subject to regulatory requirements, either internal or external, which impose a risk budget on her and the risk is measured using the superhedging functional. We think of a trader who manages a whole book, via dynamic trading, trading in options in $\mathcal{X}$ as well as over-the-counter contracts, and is hence able to attain a desired final payoff structure ξ from among a large family, $\xi \in \Xi$. We assume that Ξ is stable under cash shifts, that is, $\xi \in \Xi$ implies that $(x + \xi) \in \Xi$ for any $x \in \mathbb{R}$. The trader is a classical utility maximizer and her optimization problem without the additional information is given by

$$u^{\mathbb{F}}(R) = \sup_{\xi \in \Xi \,:\, V^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\Omega}(\xi) \le R} \mathbb{E}_{\mathbb{P}}\left[U\left(\xi\right)\right],$$

where $\mathbb{P}$ is a probability measure reflecting the trader's subjective modeling view, U is her utility function, and $R > 0$ is her risk budget. This problem, following the approach of Basak and Shapiro (2001) and Gundel and Weber (2008), is a variant on the classical optimal investment problem of Merton (1971), but with the budget constraint replaced by a risk-budget constraint.

When the additional information is considered, the value function becomes

$$u^{\mathbb{G}}(R) = \mathbb{E}_{\mathbb{P}}\left[u^{\mathbb{G}}(\omega, R)\right], \quad \text{where } u^{\mathbb{G}}(\omega, R) := \sup_{\xi \in \Xi \,:\, V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega) \le R} \mathbb{E}_{\mathbb{P}_\omega}\left[U\left(\xi\right)\right],$$

and $\{\mathbb{P}_\omega\}$ are the regular conditional probabilities of $\mathbb{P}$ with respect to $\mathcal{G}_{-1}$, as in Lemma 3.3. The value function $u^{\mathbb{G}}(\omega, R)$ is random and depends on the information which is revealed in $\mathcal{G}_{-1}$ and is then, in line with the usual Bellman principle approach, averaged out using $\mathbb{P}$. Note that, since $V^{\mathbb{G}}(\omega) \le V^{\mathbb{F}}$, it follows that $u^{\mathbb{F}} \le u^{\mathbb{G}}$. In this setup, a natural way to quantify the value of information, relative to the investment opportunities defined via Ξ and R, would be through the certainty equivalence, i.e., via

$$\inf\{x \ge 0 : u^{\mathbb{F}}(x + R) \ge u^{\mathbb{G}}(R)\}. \tag{3.3}$$

We refer to classical papers on the evaluation of information under fixed probability measure via indifference pricing, including LaValle (1968), Morris (1974), Willinger (1989), and Amendinger, Becherer, and Schweizer (2003). We note that this discussion is formal as it is not clear when $u^{\mathbb{G}}(\cdot, R)$ is measurable. In Example 4.10(d) below, we work things out in a concrete setting (there measurability of $u^{\mathbb{G}}(\cdot, R)$ is straightforward), and compute the value of information. Finally, we note that we have considered the simplest case of an expected utility maximizing agent. Using the superhedging price as a risk measure is very conservative and may suggest a highly uncertain environment. We might then also expect the agent to account for a degree of model uncertainty and to consider all measures $\mathbb{P}$ in a certain set of priors $\mathcal{P}$. This leads to a max-min problem, or variational preferences in the sense of Gilboa and Schmeidler (1989). A similar problem with risk-budget was considered in Gundel and Weber (2007).

4. Pricing-hedging duality under initially enlarged filtration. Our first contribution in this paper, developed in Propositions 3.2 and 3.5 above, was to describe the superhedging cost and the pricing problem for an agent with additional information. We turn now to our second main contribution: understanding when pricing-hedging duality for a regular agent carries over to the informed one. It is straightforward to see that an inequality holds in general.

Lemma 4.1. *The $\mathbb{G}$-superhedging cost $V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)$ and the $\mathbb{G}$-market model price $P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)$ of ξ on Ω satisfy*

$$V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega) \geq P^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega) \quad \forall \omega \in \Omega\,.$$

Thanks to the results of section 3, the proof is carried out on atoms of $\mathcal{G}_{-1}$ and is reported in the appendix. Our goal in this section is to provide sufficient conditions for equality in the above inequality. We start with the case when $\mathbb{G}$ is an initial enlargement of $\mathbb{F}$. Then, γ is $\mathbb{G}$-progressively measurable if and only if $\gamma(\omega)_t = \gamma(\widetilde{\omega})_t$ whenever $\omega|_{[0,t]} = \widetilde{\omega}|_{[0,t]}$ and $\omega \sim_{\mathcal{G}_0} \widetilde{\omega}$.

4.1. Preliminaries: Pricing-hedging duality with beliefs. We start by recalling notions and results from Hou and Obłój (2018). As explained before, we use their setup with beliefs to *zoom in* on the part of the pathspace considered by an informed agent. For $\mathfrak{P} \in \mathcal{F}_T$ let $\widetilde{V}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\mathfrak{P}}(\xi)$ be the approximate $\mathbb{F}$-superhedging cost of ξ on $\mathfrak{P}$, i.e.,

$$\begin{aligned}\widetilde{V}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\mathfrak{P}}(\xi) := \inf\{\alpha\mathcal{P}(\mathcal{X}) : \quad & \exists(\alpha,\gamma) \in \mathcal{A}_{\mathcal{X}}(\mathbb{F}) \text{ such that} \\ & (\gamma \circ S)_T(\omega) + (\alpha\mathcal{X})(\omega) \geq \xi(\omega) \ \ \forall\, \omega \in \mathfrak{P}(\varepsilon) \quad \text{for some} \quad \varepsilon > 0\},\end{aligned}$$

where

$$\mathfrak{P}(\varepsilon) = \{\omega \in \Omega : \inf_{\widetilde{\omega} \in \mathfrak{P}} ||\omega - \widetilde{\omega}|| \leq \varepsilon\}. \tag{4.1}$$

Similarly, let $\widetilde{P}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\mathfrak{P}}(\xi)$ be the approximate $\mathbb{F}$-market model price of ξ, i.e.,

$$\widetilde{P}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\mathfrak{P}}(\xi) := \lim_{\eta \searrow 0} \sup_{\mathbb{P} \in \mathcal{M}^{\mathbb{F},\eta}_{\mathcal{X},\mathcal{P},\mathfrak{P}}} \mathbb{E}_{\mathbb{P}}[\xi]$$

with $\mathcal{M}^{\mathbb{F},\eta}_{\mathcal{X},\mathcal{P},\mathfrak{P}} := \{\mathbb{P} \in \mathcal{M}^{\mathbb{F}}_{\Omega} : \mathbb{P}(\mathfrak{P}(\eta)) > 1 - \eta \text{ and } |\mathbb{E}_{\mathbb{P}}[\mathcal{X}^\lambda] - \mathcal{P}(\mathcal{X}^\lambda)| < \eta \text{ for all } \lambda \in \Lambda\ \}$.

The following assumption says that the vector $\mathcal{X}$ is not too large, and initial prices of dynamically traded options are not "on the boundary of the no-arbitrage region."

Assumption 4.2. (i) *$Lin_1(\mathcal{X})$ is a compact subset of $C(\Omega,\mathbb{R})$, where $Lin_N(\mathcal{X})$ is given by*

$$\left\{\sum_{\lambda\in\Lambda} \alpha^\lambda \mathcal{X}^\lambda : \alpha = (\alpha^\lambda)_{\lambda\in\Lambda} \in \mathbb{R}^\Lambda \textit{ with finitely many } \alpha^\lambda \neq 0 \textit{ and } \sum_{\lambda\in\Lambda} |\alpha^\lambda| \leq N\right\}. \tag{4.2}$$

(ii) *Either $K = 0$, i.e., there are no continuously traded options, or there exists an $\varepsilon > 0$ such that for any $(p_k)_{k\leq K}$ with $|\mathcal{P}(X^{(c)}_k) - p_k| \leq \varepsilon$ for all $k \leq K$, $\mathcal{M}^{\mathbb{F}}_{\widetilde{\Omega}} \neq \emptyset$, where*

$$\widetilde{\Omega} = \{\omega \in \Omega^{d+K} : S^{(d+i)}_T(\omega) = X^{(c)}_i(\omega)/p_i \quad \forall\, i \leq K\}.$$

Theorem 4.3 (Hou and Obłój (2018, Theorem 3.2)). *Suppose that Assumption 4.2 holds. Let* $\mathfrak{P} \in \mathcal{F}_T$ *be such that* $\mathcal{M}^{\mathbb{F},\eta}_{\mathcal{X},\mathcal{P},\mathfrak{P}} \neq \emptyset$ *for any* $\eta > 0$*. Then, for any uniformly continuous and bounded* ξ*, the approximate pricing-hedging duality holds:*

$$\widetilde{V}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\mathfrak{P}}(\xi) = \widetilde{P}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\mathfrak{P}}(\xi).$$

The first and by far most technical part of the proof of the above theorem is the case of duality on Ω without options, i.e., $P^{\mathbb{F}}_{\Omega}(\xi) = V^{\mathbb{F}}_{\Omega}(\xi)$. This is then extended to the general setting with variational arguments and an application of a min-max theorem; see section 4.1 in Hou and Obłój (2018). The same reasoning can be applied for a generic filtration and gives us the following result.

Corollary 4.4. *Let* $\mathbb{G}$ *be an* arbitrary *filtration such that* $\mathcal{G}_{-1}$ *is a trivial* σ*-field. Suppose that Assumption 4.2 holds with* $\mathbb{G}$ *in place of* $\mathbb{F}$ *and that* $\mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega} \neq \emptyset$*. Moreover assume that the pricing-hedging duality for the model without options holds, i.e.,* $P^{\mathbb{G}}_{\Omega}(\xi) = V^{\mathbb{G}}_{\Omega}(\xi)$ *for all uniformly continuous and bounded* ξ*. Then*

$$\widetilde{P}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi) = \widetilde{V}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi) = V^{\mathbb{G}}_{\mathcal{X},\mathcal{P},\Omega}(\xi) \quad \text{for all uniformly continuous and bounded } \xi.$$

4.2. Duality in an enlarged filtration: Case of $\mathbb{G}^+$. We now link the pricing-hedging duality in the enlarged filtration and in the canonical filtration for $\mathfrak{P}$ representing atoms in $\mathcal{G}^+_{-1} = \mathcal{G}_0$. We start with the following assumption.

Assumption 4.5. *The set* $\mathfrak{P} \in \mathcal{F}_T$ *is such that* $\mathcal{M}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\mathfrak{P}} \neq \emptyset$*, and*

$$P^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\mathfrak{P}}(\xi) = V^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\mathfrak{P}}(\xi) \quad \forall\, \xi \in \Psi, \tag{4.3}$$

where Ψ *is a given family of payoffs.*

This assumption on absence of duality gap in the original filtration $\mathbb{F}$ for a suitable choice of $\mathfrak{P}$ and Ψ plays a crucial role in the following developments. Specifically, we use it for $\mathfrak{P}$ ranging through the elementary informed sets: the elements of $\Delta(Z)$, the partition of Ω generated by Z.

Theorem 4.6. *Let* Z *be a random variable,* $\mathbb{G} := \mathbb{F} \vee \sigma(Z)$ *and* Ψ *a fixed family of* $\mathcal{F}_T$*-measurable functions. Assume that for each value* $c \in Z(\Omega)$ *either* $\{Z = c\}$ *satisfies Assumption 4.5 or* $\mathcal{M}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=c\}} = \emptyset$*. Then, there is no duality gap in* $\mathbb{G}^+$ *on the set* $\{\omega : \mathcal{M}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=Z(\omega)\}} \neq \emptyset\}$*, i.e.,* $V^{\mathbb{G}^+}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega) = P^{\mathbb{G}^+}_{\mathcal{X},\mathcal{P},\Omega}(\xi)(\omega)$ *holds for any* ω *such that* $\mathcal{M}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=Z(\omega)\}} \neq \emptyset$ *and any* $\xi \in \Psi$*.*

Proof. In the first step we prove that for each value $c \in Z(\Omega)$ such that $\mathcal{M}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=c\}} \neq \emptyset$ we have

$$P^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=c\}}(\xi) = P^{\mathbb{G}^+}_{\mathcal{X},\mathcal{P},\{Z=c\}}(\xi) \leq V^{\mathbb{G}^+}_{\mathcal{X},\mathcal{P},\{Z=c\}}(\xi) \leq V^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=c\}}(\xi). \tag{4.4}$$

Note that the last inequality holds by (3.1). To show the first equality, it suffices to show that $\mathcal{M}^{\mathbb{G}^+}_{\mathcal{X},\mathcal{P},\{Z=c\}} = \mathcal{M}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=c\}}$. For any $\mathbb{P} \in \mathcal{M}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=c\}}$ and $0 \leq s \leq t \leq T$ we have

$$\mathbb{E}_{\mathbb{P}}[S_t|\mathcal{G}^+_s] = \mathbb{E}_{\mathbb{P}}[S_t|\mathcal{F}_s \vee \sigma(Z)] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[S_t|\mathcal{F}^{\mathbb{P}}_s]|\mathcal{F}_s \vee \sigma(Z)] = \mathbb{E}_{\mathbb{P}}[S_s|\mathcal{F}_s \vee \sigma(Z)] = S_s,$$

where $\mathcal{F}_s^{\mathbb{P}}$ is a $\mathbb{P}$-completion of $\mathcal{F}_s$, showing $\mathbb{P} \in \mathcal{M}^{\mathbb{G}+}_{\mathcal{X},\mathcal{P},\{Z=c\}}$. The reverse inclusion is clear. Finally, the middle inequality in (4.4) is implied by Lemma 4.1 as the atoms of $\mathcal{G}^+_{-1}$ are exactly $\{Z=c\}$.

By representations given in Propositions 3.2 and 3.5, the proof is complete since we get that $V^{\mathbb{G}+}_{\mathcal{X},\mathcal{P},\{Z=c\}}(\xi) = P^{\mathbb{G}+}_{\mathcal{X},\mathcal{P},\{Z=c\}}(\xi)$ for any value $c \in Z(\Omega)$ for which $\mathcal{M}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=c\}} \neq \emptyset$ and any $\xi \in \Psi$. ∎

Remark 4.7. Under Assumption 4.2, thanks to Theorem 4.3, we have

$$\widetilde{V}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=c\}}(\xi) = \widetilde{P}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=c\}}(\xi) \quad \text{for all uniformly continuous and bounded } \xi. \tag{4.5}$$

Hence, if Ψ is a subset of all uniformly continuous and bounded payoffs, then, combining equality (4.5) with the general inequalities

$$P^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=c\}}(\xi) \leq V^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=c\}}(\xi) \leq \widetilde{V}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=c\}}(\xi),$$

we conclude that in Theorem 4.6, instead of assuming that (4.3) holds, it is enough to assume that

$$P^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=c\}}(\xi) = \widetilde{P}^{\mathbb{F}}_{\mathcal{X},\mathcal{P},\{Z=c\}}(\xi) \ \forall\, \xi \in \Psi$$

and that Assumption 4.2 holds.

4.3. Duality in an enlarged filtration: Case of $\mathbb{G}^-$.

We turn now to the pricing-hedging duality in $\mathbb{G}^-$ in the case $\Lambda = \{0\}$. We recall that $\mathcal{G}^-_{-1} = \{\emptyset, \Omega\}$ which models the situation when the static position in $\mathcal{X}^0$ has to be determined before acquiring any additional information.

Theorem 4.8. *Assume that $\Lambda = \{0\}$. Let Z be a random variable such that for each $c \in Z(\Omega)$ the set $\{Z = c\}$ satisfies Assumption 4.5. Define $\mathbb{G} = \mathbb{F} \vee \sigma(Z)$. Assume moreover that $\mathcal{M}^{\mathbb{G}^-}_{\mathcal{X},\mathcal{P},\Omega} \neq \emptyset$. Then pricing-hedging duality holds in $\mathbb{G}^-$:*

$$V^{\mathbb{G}^-}_{\Omega}(\xi) = P^{\mathbb{G}^-}_{\Omega}(\xi) \qquad \forall\, \xi \in \Psi.$$

Proof. We will prove the following sequence of equalities:

$$\begin{aligned} V^{\mathbb{G}^-}_{\Omega}(\xi) &= \sup_{c\in Z(\Omega)} V^{\mathbb{G}+}_{\{Z=c\}}(\xi) = \sup_{c\in Z(\Omega)} P^{\mathbb{G}+}_{\{Z=c\}}(\xi) \\ &= \sup_{\mathbb{P}\in\bigcup_{c\in Z(\Omega)} \mathcal{M}^{\mathbb{G}+}_{\{Z=c\}}} \mathbb{E}_{\mathbb{P}}[\xi] = \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{G}^-}_{\Omega}} \mathbb{E}_{\mathbb{P}}[\xi] = P^{\mathbb{G}^-}_{\Omega}. \end{aligned} \tag{4.6}$$

Let us start with the first equality. Since for each $c \in Z(\Omega)$, $V^{\mathbb{G}+}_{\{Z=c\}}(\xi) \leq V^{\mathbb{G}^-}_{\Omega}(\xi)$, we get that $\sup_{c\in Z(\Omega)} V^{\mathbb{G}+}_{\{Z=c\}}(\xi) \leq V^{\mathbb{G}^-}_{\Omega}(\xi)$. To show the reverse inequality, fix $\varepsilon > 0$. Then, for each $c \in Z(\Omega)$, there exists $\gamma^c \in \mathcal{A}(\mathbb{G})$ such that

$$V^{\mathbb{G}+}_{\{Z=c\}}(\xi) + \varepsilon + (\gamma^c \circ S)_T \geq \xi \quad \text{on } \{Z=c\}.$$

Define the strategy γ as $\gamma(\omega) = \sum_{c\in Z(\Omega)} \gamma^c(\omega) \mathbb{1}_{\{Z(\omega)=c\}}$, which belongs to $\mathcal{A}(\mathbb{G})$ and satisfies

$$\sup_{c\in Z(\Omega)} V^{\mathbb{G}+}_{\{Z=c\}}(\xi) + \varepsilon + (\gamma \circ S)_T \geq \xi \quad \text{on } \Omega.$$

Then, from the definition of superhedging cost, we conclude that

$$\sup_{c\in Z(\Omega)} V^{\mathbb{G}^+}_{\{Z=c\}}(\xi) + \varepsilon \geq V^{\mathbb{G}^-}_{\Omega}(\xi).$$

As $\varepsilon > 0$ was arbitrary one has $\sup_{c\in Z(\Omega)} V^{\mathbb{G}^+}_{\{Z=c\}}(\xi) \geq V^{\mathbb{G}^-}_{\Omega}(\xi)$.

By our assumption $\{\omega : \mathcal{M}^{\mathbb{F}}_{\{Z=Z(\omega)\}} \neq \emptyset\} = \Omega$ and pricing-hedging duality holds on $\{Z = c\}$ for each $c \in Z(\Omega)$. Thus the second equality in (4.6) follows, whereas the third and fifth equalities in (4.6) hold by definition. To show the fourth one, note that one inequality is immediate since for any c, $\mathcal{M}^{\mathbb{G}^+}_{\{Z=c\}} \subset \mathcal{M}^{\mathbb{G}^-}_{\Omega}$, and the other inequality then follows by Lemma 3.3. ■

Remark 4.9. By the proof of Theorem 4.8, as shown in (4.6), in the case $\Lambda = \{0\}$, we have that

$$V^{\mathbb{G}^-}_{\Omega}(\xi) = \sup_{c\in Z(\Omega)} V^{\mathbb{G}^+}_{\{Z=c\}}(\xi) \quad \text{and} \quad P^{\mathbb{G}^-}_{\Omega}(\xi) = \sup_{c\in Z(\Omega)} P^{\mathbb{G}^+}_{\{Z=c\}}(\xi).$$

We would like to emphasize that the above representation typically does not hold, neither for $V^{\mathbb{G}^-}_{\Omega}$ nor for $P^{\mathbb{G}^-}_{\Omega}$, when we consider the case of $\Lambda \neq \{0\}$. The static hedging position may have the opposite direction on different atoms and therefore simple aggregation is not possible. Similarly, calibration on each atom separately is a much more restrictive condition than unconditional calibration.

4.4. Examples.

We present now examples which illustrate the results of this section, as well as the notions of robust valuation of information outlined in subsection 3.2. Our informed agent has additional knowledge about the price process at the terminal date. We consider two rather different situations: one when the informed agent gets detailed information and there is a continuum of atoms (see Example 4.10) and another when the informed agent gets binary information (and $\mathcal{G}_{-1}$ only has two atoms); see Examples 4.11 and 4.12. In Examples 4.10 and 4.11, we consider the case with no static trading, $\Lambda = \{0\}$, and such that pricing-hedging duality in $\mathbb{G}^+$ holds. Then, in Example 4.12, we show how to extend the setup of Example 4.11 to also include statically traded options. The following simple observation is used a number of times. In a one-dimensional setting with no statically or dynamically traded options, i.e., $d = 1$, $K = 0$, and $\Lambda = \{0\}$, if $\xi(\omega) = \zeta(\omega_T)$ for a uniformly continuous and bounded function ζ, by Hou and Obłój (2018, Theorem 5.1), or an explicit construction of martingale measures with S_T supported on two points, and comparison with a one-period model superhedging, e.g., Carassus, Obłój, and Wiesel (2019), we have $P^{\mathbb{F}}_{\Omega}(\xi) = V^{\mathbb{F}}_{\Omega}(\xi) = \hat{\zeta}(1)$, where $\hat{\zeta}$ is the concave envelope of ζ.

Example 4.10. Consider a one-dimensional setting with no statically or dynamically traded options: $d = 1$, $K = 0$, and $\Lambda = \{0\}$. The informed agent acquires detailed knowledge of the stock price process: namely she knows the maximum deviation over time interval $[0, T]$ of the stock price from the initial price. Naturally, the agent does not know the sign of the deviation as this would give an instant arbitrage. This situation corresponds to taking $Z = \sup_{t\in[0,T]} |S_t - 1|$.

(a) Consider first the duality $\mathbb{G}^+$ with Ψ the family of all uniformly continuous and bounded payoffs. For $c > 1$, the market model price $P^{\mathbb{G}^+}_{\{Z=c\}}(\xi) = -\infty$ as any martingale measure $\mathbb{P} \in \mathcal{M}^{\mathbb{G}^+}_{\{Z=c\}}$ would satisfy

$$1 = \mathbb{P}\left(\sup_{t\in[0,T]} S_t \geq 1+c\right) \leq \frac{1}{1+c} < 1;$$

thus the set of martingale measures $\mathcal{M}^{\mathbb{G}^+}_{\{Z=c\}}$ must be empty. Likewise the superhedging cost $V^{\mathbb{G}^+}_{\{Z=c\}}(\xi) = -\infty$ as a long position in the stock generates arbitrage.

Fix $c \leq 1$. Our proof is similar to Example 3.9 in Hou and Obłój (2018). For each $N \in \mathbb{N}$, take a measure $\mathbb{P}^N \in \mathcal{M}^{\mathbb{F},1/N}_{\{Z=c\}}$ such that

$$\mathbb{E}_{\mathbb{P}^N}[\xi] \geq \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{F},1/N}_{\{Z=c\}}} \mathbb{E}_{\mathbb{P}}[\xi] - \frac{1}{N}.$$

Fix a large M and define $\tau_M := \inf\{t : S_t = M\}$. Doob's inequality gives

$$|\mathbb{E}_{\mathbb{P}^N}[\xi(S) - \xi(S^{\tau_M})]| \leq \frac{2||\xi||}{M}.$$

Let π^N be the distribution of $S^{\tau_M}_T$ under $\mathbb{P}^N$. Since $\pi^N([0,M]) = 1$ for each N, this is a tight family of probability measures and therefore a converging subsequence $(\pi^{N_k})_k$ exists. Denote the limit of $(\pi^{N_k})_k$ by π and note that, since each measure π^N has mean equal to one, so does π.

Observe that under any $\mathbb{P} \in \mathcal{M}^{\mathbb{F}}_{\{Z=c\}}$, S is a uniformly integrable martingale with $S_T = S^{\tau_c}_T \in \{1-c, 1+c\}$ $\mathbb{P}$-a.s., where $\tau_c := \inf\{t : S_t \in \{1-c, 1+c\}\}$. In particular, letting

$$U_N := \left[1-c-\frac{1}{N}, 1-c+\frac{1}{N}\right] \cup \left[1+c-\frac{1}{N}, 1+c+\frac{1}{N}\right],$$

we have

$$\mathbb{P}^N(S^{\tau_M}_T \in U_N) = \mathbb{P}^N(||S|| \leq M,\ S_T \in U_N) \geq \mathbb{P}^N\left(||S|| \leq 1+c+\frac{1}{N},\ S_T \in U_N\right) \geq 1 - \frac{1}{N}.$$

An application of the portemanteau theorem shows that $\pi(U_k) = 1$ for all k and hence $\pi(\{1-c\}) = \pi(\{1+c\}) = 1/2$ so that $\mathbb{P} \in \mathcal{M}^{\mathbb{F}}_{\{Z=c\}}$ if and only if $\mathbb{P} \in \mathcal{M}^{\mathbb{F}}$ and $S_T \sim \pi$ under $\mathbb{P}$.

Finally,

$$\begin{aligned}
\widetilde{P}^{\mathbb{F}}_{\{Z=c\}}(\xi) &= \lim_{N\to\infty} \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{F},1/N}_{\{Z=c\}}} \mathbb{E}_{\mathbb{P}}[\xi(S)] \leq \lim_{N\to\infty}\left(\mathbb{E}_{\mathbb{P}^N}[\xi(S)] + \frac{1}{N}\right) \\
&\leq \limsup_{k\to\infty} \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{F}} \text{ s.t. } \mathcal{L}(S_T)=\pi^{N_k}} \mathbb{E}_{\mathbb{P}}[\xi(S^{\tau_M})] + \frac{1}{N_k} + \frac{2\|\xi\|}{M} \\
&\leq \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{F}} \text{ s.t. } \mathcal{L}(S_T)=\pi} \mathbb{E}_{\mathbb{P}}[\xi(S^{\tau_M})] + \frac{2\|\xi\|}{M} = \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{F}}_{\{Z=c\}}} \mathbb{E}_{\mathbb{P}}[\xi(S)] + \frac{2\|\xi\|}{M},
\end{aligned}$$

where the third inequality holds by Lemma 4.3 in Hou and Obłój (2018). Since M was arbitrary we obtain that $\widetilde{P}^{\mathbb{F}}_{\{Z=c\}} \leq P^{\mathbb{F}}_{\{Z=c\}}$ and by Theorem 4.6 and Remark 4.7 we conclude that there is no duality gap in $\mathbb{G}^+$.

(b) Consider now payoff ξ_b satisfying $\xi_b(\omega) := \zeta(\omega_T)$ for some nonnegative and continuous function ζ such that $\zeta(\omega_T) = 0$ if $\omega_T > 2$, and $\zeta(\omega_T) = \zeta(2-\omega_T)$ if $0 \le \omega_T \le 2$. Then $P^{\mathbb{G}}_{\Omega}(\xi_b)(\omega) = P^{\mathbb{F}}_{\{Z=Z(\omega)\}}(\xi_b) = \zeta(\omega_T) = \xi_b(\omega)$ and $P^{\mathbb{F}}_{\Omega}(\xi_b) = ||\xi_b||$. This makes the advantage of acquiring additional information in the context of robust pricing and hedging of ξ_b, as discussed in subsection 3.2, explicit:

$$V^{\mathbb{F}}_{\Omega}(\xi_b) - V^{\mathbb{G}}_{\Omega}(\xi_b)(\omega) = P^{\mathbb{F}}_{\Omega}(\xi_b) - P^{\mathbb{G}}_{\Omega}(\xi_b)(\omega) = ||\xi_b|| - \xi_b(\omega),$$

where we used the fact that the duality holds for this enlargement of filtration, as shown above. In particular, in this example, a conservative or robust assessment of this advantage in (3.2) is zero, $\mathfrak{A}^{\mathbb{G},\mathbb{F}}_V[\xi_b] = 0$. The case when this quantity is strictly positive is exemplified next.

(c) Consider another payoff ξ_c satisfying $\xi_c(\omega) = \psi(\omega_T)$ for a nonnegative and continuous function ψ defined as

$$\psi(x) = \begin{cases} 1-2x, & x \le \frac{1}{2}, \\ x - \frac{1}{2}, & x \in [\frac{1}{2}, \frac{3}{2}], \\ 4-2x, & x \in [\frac{3}{2}, 2], \\ 0, & x \ge 2. \end{cases}$$

Note that the concave envelope of ψ takes value 1 at one so, by the simple observation above, $V^{\mathbb{F}}_{\Omega}(\xi_c) = 1$. On the other hand, ψ is constructed so that for any $c \in (0,1)$, the line connecting the points $(1-c, \psi(1-c))$ and $(1+c, \psi(1+c))$ goes through $(1, 1/2)$ which readily gives $V^{\mathbb{G}}_{\Omega}(\xi_c)(\omega) = V^{\mathbb{F}}_{\{Z=Z(\omega)\}}(\xi_c) = \frac{1}{2}$ if $Z(\omega) \le 1$. This makes the advantage of acquiring an additional information in the context of robust pricing and hedging of ξ_c, as discussed in subsection 3.2, explicit:

$$V^{\mathbb{F}}_{\Omega}(\xi_c) - V^{\mathbb{G}}_{\Omega}(\xi_c)(\omega) = P^{\mathbb{F}}_{\Omega}(\xi_c) - P^{\mathbb{G}}_{\Omega}(\xi_c)(\omega) = \frac{1}{2},$$

where we used the fact that the duality holds for this enlargement of filtration, as shown above. In particular, in this example, a conservative or robust assessment of this advantage in (3.2) is strictly positive: $\mathfrak{A}^{\mathbb{G},\mathbb{F}}_V[\xi] = 1/2$.

(d) Consider the second problem from subsection 3.2 and an agent with a simple family Ξ of available final positions, with each $\xi \in \Xi$ satisfying $\xi(\omega) = f(\omega_T)$, for some continuous function f with $f(x) = 0$ for $x \ge 2$, and $f(1) \ge f(x) \ge 0$ for each $x \ge 0$. Fix a probability measure $\mathbb{P}$ on Ω satisfying $\mathbb{P}(\{\omega : \omega_T < 1\}) = 1$. Then, for any concave increasing utility function U, $u^{\mathbb{F}}(R) = U(R)$ and $u^{\mathbb{G}}(R) = U(2R)$. Hence, information encoded in the signal Z is worth R given capital R.

Example 4.11. Assume $\Lambda = \{0\}$, $d = 1$, and $K = 0$. Take $Z = \mathbb{1}_{\{S_t \in (a,b)\ \forall t \in [0,T]\}}$, where $a < 1 < b$.

(a) We use Theorem 4.6 to show that the pricing-hedging duality holds in $\mathbb{G}^+$. By the same type of arguments as in the proof of Example 3.9 in Hou and Obłój (2018), we have

$$P^{\mathbb{F}}_{\{Z=1\}}(\xi) = \lim_{\varepsilon \searrow 0} P^{\mathbb{F}}_{\{S_t \in [a+\varepsilon, b-\varepsilon] \forall t \in [0,T]\}}(\xi) = \lim_{\varepsilon \searrow 0} \widetilde{P}^{\mathbb{F}}_{\{S_t \in [a+\varepsilon, b-\varepsilon] \forall t \in [0,T]\}}(\xi) = \widetilde{P}^{\mathbb{F}}_{\{Z=1\}}(\xi)$$

for any uniformly continuous and bounded ξ on Ω.

Consider now the set $\{Z=0\}=\{\tau_{a,b}\le T\}$, where $\tau_{a,b}=\inf\{t\in[0,T]:S_t\notin(a,b)\}$ and restrict to $\Psi=\{\xi(\omega)=\zeta(\omega_T):\zeta$ uniformly continuous and bounded$\}$. Conditioning on the value of $S_{\tau_{a,b}}$, using the simple observation at the beginning of the section, we see that

$$P^{\mathbb{F}}_{\{Z=0\}}(\xi)=\frac{b-1}{b-a}\hat\zeta(a)+\frac{1-a}{b-a}\hat\zeta(b),$$

where $\xi(\omega)=\zeta(\omega_T)$ and $\hat\zeta$ is the concave envelope of ζ. Fix large N and let $\tau_N=\inf\{t\in[0,T]:S_t\notin(a+1/N,b-1/N)\}$. Then, for any $\mathbb{P}\in\mathcal{M}^{\mathbb{F},1/N}_{\{Z=c\}}$, we have $\mathbb{P}(\tau_N<T)\ge 1-1/N$ and on $\{\tau_N<T\}$ we can compute the superhedging price as above. It follows that

$$P^{\mathbb{F}}_{\{Z=0\}}(\xi)\le\widetilde{P}^{\mathbb{F}}_{\{Z=0\}}(\xi)\le\frac{b-1/N-1}{b-a-2/N}\hat\zeta(a+1/N)+\frac{1-a+1/N}{b-a-2/N}\hat\zeta(b-1/N)+\frac{\|\xi\|}{N},$$

and we obtain the equality $P^{\mathbb{F}}_{\{Z=0\}}(\xi)=\widetilde{P}^{\mathbb{F}}_{\{Z=0\}}(\xi)$ by letting $N\to\infty$.

Note that, for such a ξ, $P^{\mathbb{F}}_{\{Z=1\}}(\xi)$ is the value at one of the concave envelope of ζ on $[a,b]$. It is very easy to construct examples of ζ for which either $P^{\mathbb{F}}_{\{Z=1\}}(\xi)$ and $\mathbb{P}^{\mathbb{F}}_{\{Z=0\}}(\xi)$ are both smaller than $\hat\zeta(1)$ and hence $\mathfrak{A}^{\mathbb{G},\mathbb{F}}_V[\xi]>0$, or at least one is equal to $\hat\zeta(1)$ and hence $\mathfrak{A}^{\mathbb{G},\mathbb{F}}_V[\xi]=0$. We note that extending the duality on $\{Z=0\}$ to all uniformly continuous and bounded ξ on Ω would require novel arguments. We believe this might be possible through an extension of the dynamic programming principle established in section 5 from deterministic times to first exit times. We leave this topic for further research. For a given particular payoff ξ one can hope to build a direct and tailored argument. The subsequent point showcases this for a type of a barrier option.

(b) Consider now ξ of a barrier type payoff such that $\xi(\omega)=\zeta(\overline{\omega})$, where M is large and fixed, $\overline{\omega}=(\max_{t\le T}\omega_t-b+\frac{1}{M})^+\vee(a+\frac{1}{M}-\min_{t\le T}\omega_t)^+$, and

$$\zeta(x)=\begin{cases}1-Mx, & x\in[0,\frac{1}{M}),\\ 0, & x\in[\frac{1}{M},\infty).\end{cases}$$

Then $\widetilde{P}^{\mathbb{F}}_{\{Z=0\}}(\xi)\le\frac{1}{N}+\frac{M}{N}$ for any N large enough. Consequently, $\widetilde{P}^{\mathbb{F}}_{\{Z=0\}}(\xi)=0=P^{\mathbb{F}}_{\{Z=0\}}(\xi)$ and $\widetilde{P}^{\mathbb{F}}_{\{Z=1\}}(\xi)=P^{\mathbb{F}}_{\{Z=1\}}(\xi)=1$, as by the general argument above. It follows that $P^{\mathbb{G}}_{\Omega}(\xi)(\omega)=P^{\mathbb{F}}_{\{Z=Z(\omega)\}}(\xi)=Z(\omega)$. Moreover $P^{\mathbb{F}}_{\Omega}(\xi)=1$. Using the duality, by Theorem 4.6 and Remark 4.7, we have

$$V^{\mathbb{F}}_{\Omega}(\xi)-V^{\mathbb{G}}_{\Omega}(\xi)(\omega)=P^{\mathbb{F}}_{\Omega}(\xi)-P^{\mathbb{G}}_{\Omega}(\xi)(\omega)=1-Z(\omega).$$

In particular, the robust assessment of the value of the information is $\mathfrak{A}^{\mathbb{G},\mathbb{F}}_V[\xi]=0$.

Example 4.12. We extend the setup of Example 4.11 to include statically traded options. It is easy to do so explicitly since, as compared to Example 4.10, the additional information here is much less precise and $\mathcal{G}_{-1}$ has only two atoms. Let us start with two martingale measures under which S_T takes only two values. Consider $\epsilon>0$ and let $x_1=1-\epsilon\in(a,1)$, $x_0=\epsilon\in(0,a)$, and $y_1=(1+b)/2\in(1,b)$,

$$y_0=1-x_1+\frac{x_1}{1-x_1}(y_1-x_1)=\epsilon+\frac{1-\epsilon}{\epsilon}(y_1-x_1)\in(b,\infty).$$

Let $\mathbb{P}_i$, $i = 0, 1$, be a martingale measure under which $S_T \in \{x_i, y_i\}$. Then, in the setup of Example 4.11 with $\Lambda = \{0\}$, we have $\mathbb{P}_i \in \mathcal{M}^{\mathbb{F}}_{\{Z=i\}}$, $i = 0, 1$. Note also that the points were selected in such a way that $x_0 = 1 - x_1$ and $\mathbb{P}_1(S_T = x_1) = \mathbb{P}_0(S_T = x_0)$.

We now add statically traded options: $\mathcal{X} = \{\mathcal{X}^\lambda : \lambda = 0, 1, \ldots\}$ with $\mathcal{X}^0(\omega) = 1$, as usual, $\mathcal{X}^1(\omega) = (\omega_T - K)^+$, and for $\lambda \in \Lambda \setminus \{0\}$, $\mathcal{X}^\lambda(\omega) = f_\lambda(\omega_T)$, where $K > y_0$ and f_λ are arbitrary continuous functions such that $f_\lambda(x) = 0$ for $x \notin (0, 1)$ and $f_\lambda(x) = f_\lambda(1 - x)$ for $x \in [0, 1]$. An example of such a payoff is given by $f(x) = (\frac{1}{2} - |x - \frac{1}{2}|)^+$. Set $\mathcal{P}(\mathcal{X}^0) = 1$, $\mathcal{P}(\mathcal{X}^1) = 0$, and

$$\mathcal{P}(\mathcal{X}^\lambda) = \mathbb{E}_{\mathbb{P}_1}[\mathcal{X}^\lambda] = f_\lambda(x_1)\mathbb{P}_1(S_T = x_1) = f_\lambda(x_0)\mathbb{P}_0(S_T = x_0) = \mathbb{E}_{\mathbb{P}_0}[\mathcal{X}^\lambda], \quad \lambda \geq 2.$$

By definition, both measures are calibrated to the new options and $\mathbb{P}_i \in \mathcal{M}^{\mathbb{F}}_{\mathcal{X},\{Z=i\}}$, $i = 0, 1$, and in particular these sets are nonempty. Consider now $\xi(\omega) = \zeta(\omega_T)$, with a continuous bounded ζ and $\mathbb{P}_N \in \mathcal{M}^{\mathbb{F},1/N}_{\mathcal{X},\{Z=0\}}$ such that

$$\mathbb{E}_{\mathbb{P}^N}[\xi] \geq \sup_{\mathbb{P} \in \mathcal{M}^{\mathbb{F},1/N}_{\mathcal{X},\{Z=0\}}} \mathbb{E}_{\mathbb{P}}[\xi] - \frac{1}{N}.$$

Note that $\mathbb{E}_{\mathbb{P}^N}[(S_T - K)^+] \leq 1/N$ so that the family π^N of the distribution of S_T under $\mathbb{P}^N$ is tight. It admits a subsequence converging weakly to some π. It is easy to see that π dominates in the convex order the two-point measure on $\{a, b\}$ centred in 1, so that we can construct $\mathbb{P} \in \mathcal{M}^{\mathbb{F}}_{\mathcal{X},\{Z=0\}}$ with $S_T \sim \pi$ under $\mathbb{P}$. It follows that $P^{\mathbb{F}}_{\mathcal{X},\{Z=0\}}(\xi) = \widetilde{P}^{\mathbb{F}}_{\mathcal{X},\{Z=0\}}(\xi)$. The case of $\{Z = 1\}$ is dealt with in analogy to Examples 4.10 and 4.11.

5. Timing of information arrival, dynamic programming principle, and the pricing-hedging duality. To extend the initial enlargement perspective we study now the case where the additional information is disclosed at a time $T' \in (0, T)$, i.e., the filtration $\mathbb{G}$ is of the form

$$\mathcal{G}_t = \mathcal{F}_t \text{ for } t \in [0, T') \text{ and } \mathcal{G}_t = \mathcal{F}_t \vee \sigma(Z) \text{ for } t \in [T', T]. \tag{5.1}$$

We also consider the case $T' = 0$ for comparison. We divide our problem into two time intervals and use the results from the previous sections. First we look at the pricing and hedging problems on $[T', T]$ and then on $[0, T']$. We consider the case when the additional information only pertains to the evolution of prices after time T' irrespectively of the prices on, or before, time T'. This is encoded in the following structural assumption on Z:

$$Z(\omega) = \widetilde{Z}\left(\kappa^{T'}(\omega)\right), \tag{5.2}$$

where $\widetilde{Z} : C([0, 1], \mathbb{R}^d_+) \to \mathbb{R}$ is measurable and $\kappa^{T'} : C([0, T], \mathbb{R}^d_+) \to C([0, 1], \mathbb{R}^d_+)$ is given by

$$\kappa^{T'}(\omega) = \kappa^{T'}((\omega^1, \ldots, \omega^d)) = \left(\left(\frac{\omega^1_{t(T-T')+T'}}{\omega^1_{T'}}\right)_{t\in[0,1]}, \ldots, \left(\frac{\omega^d_{t(T-T')+T'}}{\omega^d_{T'}}\right)_{t\in[0,1]}\right)$$

with the convention $\frac{x}{0} = 1$.

Finally, we stress that *throughout this section we assume that there are no dynamically traded options, i.e., $K = 0$, and no statically traded options, i.e., $\Lambda = \{0\}$.*

5.1. Dynamic programming principle for $V^{\mathbb{G}}$ and $P^{\mathbb{G}}$ for fixed times. We begin with two propositions where we develop the dynamic programming principle for the superhedging cost and the market model price. Note that the case $Z \equiv \text{const}$ and $\mathbb{F} = \mathbb{G}$ is also of interest, as it gives the dynamic programming principle under $\mathbb{F}$.

To formulate the dynamic programming principle for the superhedging cost we naturally extend the notions introduced in Definition 2.4 on the time interval $[0,T]$ to a subinterval $[T',T''] \subset [0,T]$ (with $T' \leq T''$) and let $\mathcal{A}^M(\mathbb{G},[T',T''])$ denote $(\mathbb{G},[T',T''],M)$-admissible strategies and $\mathcal{A}(\mathbb{G},[T',T''])$ their union over $M \in \mathcal{U}(\mathcal{G}_0)$.

Similarly, we define the set of measures $\mathcal{M}_A^{\mathbb{G},[T',T]}$ concentrated on $A \in \mathcal{F}_T$ as follows:

$$\mathcal{M}_A^{\mathbb{G},[T',T]} := \{\mathbb{P} : S \text{ is a } \mathbb{G}\text{-martingale on } [T',T] \text{ and } \mathbb{P}(A) = 1\}.$$

The following two results establish suitable regularity and dynamic programming principle for the superhedging cost and the pricing operator.

Proposition 5.1. *Recall that $\Lambda = \{0\}$ and $K = 0$. Let B^ω denote the $\mathcal{F}_{T'}$-atom containing ω. Then for a bounded uniformly continuous ξ the following hold:*

(i) *The mapping $V_\Omega^{\mathbb{G},[T',T]}(\xi) : \Omega \to \mathbb{R}$ defined as*

$$V_\Omega^{\mathbb{G},[T',T]}(\xi)(\omega) := \inf\left\{x \in \mathbb{R} : \exists\, \gamma \in \mathcal{A}(\mathbb{G},[T',T]) \text{ such that } x + \int_{T'}^T \gamma_t dS_t \geq \xi \text{ on } B^\omega\right\}$$

is uniformly continuous and $\mathcal{F}_{T'}$-measurable.

(ii) *The dynamic programming principle for fixed times holds in the sense that*

$$V_\Omega^{\mathbb{G},[0,T]}(\xi) = V_\Omega^{\mathbb{F},[0,T']}\left(V_\Omega^{\mathbb{G},[T',T]}(\xi)\right).$$

We note that part (i) shows that uniform continuity of the final payoff ξ propagates backward in time. One would then naturally expect the superhedging price to satisfy the dynamic programming principle in a given filtration, i.e., $V_\Omega^{\mathbb{G},[0,T]}(\xi) = V_\Omega^{\mathbb{G},[0,T']}(V_\Omega^{\mathbb{G},[T',T]}(\xi))$ and then part (ii) follows naturally since the two filtrations $\mathbb{F}$ and $\mathbb{G}$ coincide before time T'. We now state an analogous result, preserving the same structure of the statement, for the pricing operator. Observe that, for both results, the absence of traded options, $\Lambda = \{0\}$, is crucial. Indeed, otherwise the dynamic programming principle is likely to fail, as recently discussed in Aksamit et al. (2019).

Proposition 5.2. *Recall that $\Lambda = \{0\}$ and $K = 0$. Let B^ω denote the $\mathcal{F}_{T'}$-atom containing ω and assume that $\mathcal{M}_{B^\omega}^{\mathbb{G},[T',T]} \neq \emptyset$ for each ω. Then for a bounded uniformly continuous ξ the following hold:*

(i) *The mapping $P_\Omega^{\mathbb{G},[T',T]}(\xi)$ defined as $P_\Omega^{\mathbb{G},[T',T]}(\xi)(\omega) := \sup_{\mathbb{P} \in \mathcal{M}_{B^\omega}^{\mathbb{G},[T',T]}} \mathbb{E}_\mathbb{P}[\xi]$ is uniformly continuous and $\mathcal{F}_{T'}$-measurable.*

(ii) *The dynamic programming principle for fixed times holds in the form*

$$P_\Omega^{\mathbb{G},[0,T]}(\xi) = P_\Omega^{\mathbb{F},[0,T']}(P_\Omega^{\mathbb{G},[T',T]}(\xi)).$$

Remark 5.3. The dynamic programming principle for P stated in Proposition 5.2(ii) is linked to conditional sublinear expectations studied in Nutz and van Handel (2013, Theorem 2.3). Since there is more structure in our setup we prove it relying on uniform continuity of ξ instead of a general analytic selection argument.

While both Propositions 5.1 and 5.2 seem natural, their proofs are longer than one might expect and require certain technical details. We present them in the appendix. In particular, we note that assumption (5.2) is important in the proofs and one would not expect such results to hold for an arbitrary Z. Indeed, consider for example $Z = |\ln S_T - \ln S_{T'}|\mathbb{1}_{\{S_{T'}=c\}\cap\{S_T>0\}}$, which violates (5.2). It is easy to see that in this case we cannot guarantee the uniform continuity of $V_\Omega^{\mathbb{G},[T',T]}(\xi)$ or $P_\Omega^{\mathbb{G},[T',T]}(\xi)$.

5.2. Pricing-hedging duality. We show now that pricing-hedging duality holds for an agent with information flow $\mathbb{G}$. This is done using arguments analogous to those in sections 3 and 4 and treating independently each atom B^ω of $\mathcal{F}_{T'}$. First, as in Theorem 4.6 for the corresponding $\mathbb{G}^+$ filtration, we look at the intersections with each level set of the random variable Z, namely at the sets $B^\omega \cap \{Z = c\}$, which form the atoms of $\mathcal{G}_{T'}$. Second, we aggregate over Z, as in Theorem 4.8 for the corresponding $\mathbb{G}^-$ filtration. The described operation reduces the problem to the $[0, T']$ interval, where $\mathbb{G}$ coincides with $\mathbb{F}$. We conclude by use of the dynamic programming principle.

Theorem 5.4. *Recall that we consider the case with no statically or dynamically traded options, i.e., $\Lambda = \{0\}$ and $K = 0$. Let Z be a random variable such that for each $c \in Z(\Omega)$ and each $\mathcal{F}_{T'}$-atom B^ω the set $\{Z = c\} \cap B^\omega$ satisfies Assumption 4.5 on $[T', T]$. Assume moreover that $\mathcal{M}_\Omega^{\mathbb{G}} \neq \emptyset$. Then for any bounded uniformly continuous ξ*

$$V_\Omega^{\mathbb{G},[T',T]}(\xi)(\omega) = P_\Omega^{\mathbb{G},[T',T]}(\xi)(\omega) \quad \forall\omega \quad \textit{and} \quad V_\Omega^{\mathbb{G}}(\xi) = P_\Omega^{\mathbb{G}}(\xi).$$

Proof. We use the previous subsection to show the following equalities:

$$\begin{aligned} V_\Omega^{\mathbb{G},[0,T]}(\xi) &= V_\Omega^{\mathbb{F},[0,T']}(V_\Omega^{\mathbb{G},[T',T]}(\xi)) = V_\Omega^{\mathbb{F},[0,T']}(P_\Omega^{\mathbb{G},[T',T]}(\xi)) \\ &= P_\Omega^{\mathbb{F},[0,T']}(P_\Omega^{\mathbb{G},[T',T]}(\xi)) = P_\Omega^{\mathbb{G},[0,T]}(\xi). \end{aligned}$$

After applying Propositions 5.1 and 5.2, it remains to show the second and third equalities. To prove the second equality stating that $P_\Omega^{\mathbb{G},[T',T]}(\xi) = V_\Omega^{\mathbb{G},[T',T]}(\xi)$ first we remark some analogies with section 3. Defining the $(\mathbb{G}^+, [T', T])$-superhedging cost $V_A^{\mathbb{G}^+,[T',T]}$ as

$$V_A^{\mathbb{G}^+,[T',T]} := \inf\left\{x \in \mathcal{G}_{T'} : \exists\, \gamma \in \mathcal{A}(\mathbb{G}, [T', T]) \text{ such that } x + \int_{T'}^T \gamma_u dS_u \geq \xi \text{ on } A\right\},$$

analogously to Proposition 3.2 we have that

$$V_{B^\omega}^{\mathbb{G}^+,[T',T]}(\omega') = V_{B^\omega\cap\{Z=Z(\omega')\}}^{\mathbb{G}^+,[T',T]} \quad \text{for each } \omega \text{ and each } \omega' \in B^\omega.$$

Analogously to Proposition 3.5, we deduce that

$$P_{B^\omega}^{\mathbb{G}^+,[T',T]}(\omega') = P_{B^\omega\cap\{Z=Z(\omega')\}}^{\mathbb{G}^+,[T',T]} \quad \text{for each } \omega \text{ and each } \omega' \in B^\omega.$$

Then, by mimicking the proof of Theorem 4.8, we pass from "$\mathbb{G}^+$ to $\mathbb{G}^-$" and derive that

$$V_\Omega^{\mathbb{G},[T',T]}(\xi)(\omega) = P_\Omega^{\mathbb{G},[T',T]}(\xi)(\omega) \quad \forall\omega.$$ ■

The third equality follows by general duality results on $[0,T']$ (see Hou and Obłój (2018) and Dolinsky and Soner (2015)).

Example 5.5. Analogously to Example 4.10 let us look at additional information in the dynamic setup. We consider a one-dimensional setting with no statically or dynamically traded options: $d = 1$, $K = 0$, and $\Lambda = \{0\}$. At time T', the informed agent acquires detailed knowledge of the stock prices process of the form

$$Z = \sup_{t\in[T',T]} |\ln S_t - \ln S_{T'}|.$$

Note that for each $c \in Z(\Omega)$ and each $\mathcal{F}_{T'}$-atom B^ω, one has that

$$\mathcal{M}^{\mathbb{G},[T',T]}_{\{Z=c\}\cap B^\omega} = \mathcal{M}^{\mathbb{F},[T',T]}_{\{Z=c\}\cap B^\omega} \neq \emptyset,$$

where the first equality holds since each $\{Z = c\} \cap B^\omega$ is a $\mathcal{G}_{T'}$-atom. To show that

$$P^{\mathbb{F},[T',T]}_{\{Z=c\}\cap B^\omega} = \widetilde{P}^{\mathbb{F},[T',T]}_{\{Z=c\}\cap B^\omega}$$

we use the same arguments as in Example 4.10. Thus Assumption 4.5 with no options ($\Lambda = \{0\}$) on $[T',T]$ is satisfied for each each set $\{Z = c\} \cap B^\omega$. Moreover note that, by a concatenation of measure argument, $\mathcal{M}^{\mathbb{G}}_\Omega \neq \emptyset$ and, since $\Lambda = \{0\}$, Assumption 4.2 is also satisfied. We now apply Theorem 5.4 to show that there is no duality gap in $\mathbb{G}$.

Example 5.6. Analogously to Example 4.11 we consider the additional information in a dynamic setup of binary type. As before we consider a one-dimensional setting with no statically or dynamically traded options: $d = 1$, $K = 0$, and $\Lambda = \{0\}$. At time T' the informed agent acquires knowledge of the form

$$Z = \mathbb{1}_{\left\{a<\frac{S_t}{S_{T'}}<b \ \ \forall\, t\in[T',T]\right\}}, \qquad \text{where} \qquad a < 1 < b.$$

Along the lines of Example 5.5, applying Theorem 5.4, we deduce that there is no duality gap in $\mathbb{G}$.

5.3. The value in timing of information. We continue the discussion on value of information from subsection 3.2 focusing on the question of whether the timing of the arrival of such information makes a difference and how. We recall that the additional information only pertains to the evolution of prices after time T' irrespective of the prices at, or before, time T'. Moreover, our structural condition (5.2) implies that the information is a function of paths which are uniformly reparametrized to the unit interval $t \in [0,1]$. Many payoffs have such a property: they only depend on properties of path, e.g., hitting of a barrier level, which are invariant under continuous time changes. This hints at two likely consequences. First, for a particular payoff ξ, the relevance of additional information may be highly dependent on the timing of its arrival. Second, information "arriving too late" for one payoff ξ may "arrive just in time" to hedge a different payoff ξ'. We make both of these properties precise below. The first one is illustrated in Example 5.8 and the second one is established in Proposition 5.7.

To emphasize the dependence on the arrival time T' of the information, we write $\mathbb{G}^{T'}$ and $\mathcal{G}_t^{T'}$. We also let Ξ denote the set of uniformly continuous functions $\xi : \Omega \to [0,1]$ and recall the conservative assessment of the value of information $\mathfrak{A}_V^{\mathbb{G}^{T'},\mathbb{F}}[\xi]$ in (3.2).

Proposition 5.7. *Assume that $\Lambda = \{0\}$, $K = 0$ and recall that the additional information is given by (5.1)–(5.2), where $T' \in (0,T)$. Then, for any $\xi \in \Xi$,*

(i) *there exists $\zeta \in \Xi$ such that $\mathfrak{A}_V^{\mathbb{G}^{T'},\mathbb{F}}[\zeta] = \mathfrak{A}_V^{\mathbb{G}^{0},\mathbb{F}}[\xi]$,*

(ii) *for any time $T'' \in (0,T)$, there exists $\zeta \in \Phi$ such that $\mathfrak{A}_V^{\mathbb{G}^{T'},\mathbb{F}}[\xi] = \mathfrak{A}_V^{\mathbb{G}^{T''},\mathbb{F}}[\zeta]$.*

Proof. (i) For a fixed T' and ξ let us define ζ by

$$\zeta(\omega) := \xi\left(\left(\frac{\omega_{t(T-T')/T+T'}}{\omega_{T'}}\right)_{t\in[0,T]}\right),$$

where $\frac{0}{0} = 1$. Then, by the dynamic programming principle of Proposition 5.1

$$V_\Omega^{\mathbb{G}^{T'}}(\zeta) = V_\Omega^{\mathbb{F},[0,T']}\left(V_\Omega^{\mathbb{G}^{T'},[T',T]}(\zeta)\right).$$

By definition of ζ and $\mathbb{G}^{T'}$, we have that $V_\Omega^{\mathbb{G}^{T'},[T',T]}(\zeta) = V_\Omega^{\mathbb{G}^{0-}}(\xi)$, hence

$$V_\Omega^{\mathbb{G}^{T'}}(\zeta) = V_\Omega^{\mathbb{F},[0,T']}\left(V_\Omega^{\mathbb{G}^{0-}}(\xi)\right) = V_\Omega^{\mathbb{G}^{0-}}(\xi).$$

As a consequence we also obtain that $V_\Omega^{\mathbb{F}}(\zeta) = V_\Omega^{\mathbb{F}}(\xi)$. It remains to note that, by Remark 4.9, $\mathfrak{A}_V^{\mathbb{G}^0,\mathbb{F}}[\xi] = V_\Omega^{\mathbb{F}}(\xi) - V_\Omega^{\mathbb{G}^{0-}}(\xi) = V_\Omega^{\mathbb{F}}(\zeta) - V_\Omega^{\mathbb{G}^{T'}}(\zeta) = \mathfrak{A}_V^{\mathbb{G}^{T'},\mathbb{F}}[\zeta]$.

(ii) For fixed T', T'', and ξ let us define $\zeta \in \Xi$ by $\zeta := \xi \circ \kappa^{-1}$, where $\kappa : C([0,T],\mathbb{R}_+^d) \to C([0,T],\mathbb{R}_+^d)$ is $\kappa(\omega) = (\omega_{f(t)})_{t\in[0,T]}$ with the following time change:

$$f(t) = t\frac{T'}{T''}\mathbb{1}_{\{t\in[0,T'']\}} + \frac{T'(T-t)+T(t-T'')}{T-T''}\mathbb{1}_{\{t\in(T'',T]\}}.$$

Then, by Proposition 5.1 and the form of additional information Z, it holds that

$$\begin{aligned} V_\Omega^{\mathbb{G}^{T'}}(\xi) &= V_\Omega^{\mathbb{F},[0,T']}\left(V_\Omega^{\mathbb{G}^{T'},[T',T]}(\xi)\right) = V_\Omega^{\mathbb{F},[0,T']}\left(V_\Omega^{\mathbb{G}^{T''},[T',T]}(\zeta)\circ\kappa\right) \\ &= V_\Omega^{\mathbb{F},[0,T'']}\left(V_\Omega^{\mathbb{G}^{T''},[T'',T]}(\zeta)\right) = V_\Omega^{\mathbb{G}^{T''}}(\zeta). \end{aligned}$$ ■

One interesting implication of the above result is that if an agent had the capacity to design her payoff, then the timing of the information arrival makes no difference, as long as it happens strictly between 0 and T. As an example, we could consider FX markets where barrier options are widely traded. If the information is helpful to understand level crossings of the stock process, the agent would wait for its arrival and then trade in barrier contracts for the remaining maturity. In this scenario, the above analysis may apply and, other things being equal, the agent may not have a strong preference between arrival times T' and T''.

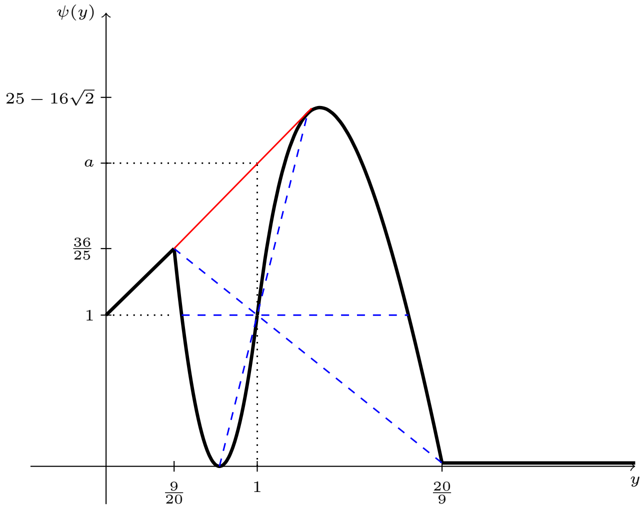

Figure 1. *Thick solid line is the graph of a specific function ψ from Example 5.8. Thin (red) solid line is a part of the upper concave envelope of this ψ in the entire domain. Dashed (blue) lines are upper concave envelopes for this ψ restricted to three distinctive atoms. Dotted lines show values of upper concave envelopes at 1.*

Example 5.8. Consider additional information given as in (5.2) by $\widetilde{Z} : C([0,1], \mathbb{R}^d_+) \to \mathbb{R}$, $\widetilde{Z} = \sup_{t\in[0,\frac{1}{3}]} |\ln S_t|$. Hence, in particular,

$$Z^0 := \widetilde{Z} \circ \kappa^0 = \sup_{t\in[0,\frac{T}{3}]} |\ln S_t| \quad \text{and} \quad Z^{\frac{T}{2}} := \widetilde{Z} \circ \kappa^{\frac{T}{2}} = \sup_{t\in[\frac{T}{2},\frac{2T}{3}]} |\ln S_t - \ln S_{\frac{T}{2}}|.$$

Now fix payoffs ξ and ζ such that $\xi(\omega) = \psi(\omega_{\frac{2T}{3}}/\omega_{\frac{T}{2}})$ and $\zeta(\omega) = \psi(\omega_{\frac{T}{3}})$, where ψ satisfies $\psi(y)\frac{1}{1+y} + \psi(\frac{1}{y})\frac{y}{1+y} = 1$, $\psi(y) \geq 0$, and there exist $x < 1 < z$ such that $\max_{y\in[0,1]} \psi(y) = \psi(x) > 1$ and $\max_{y\in[1,\infty)} \psi(y) = \psi(z) > 1$. Such functions can be constructed in the setup of Example 5.5 using ideas from Example 4.10(c); a particular instance is plotted in Figure 1 and given by

$$\psi(y) = \begin{cases} \frac{44}{45}y + 1, & y \in [0, \frac{9}{20}), \\ 16(y - \frac{3}{4})^2, & y \in [\frac{9}{20}, 1), \\ 25 - 8y - \frac{16}{y}, & y \in [1, \frac{20}{9}), \\ \frac{1}{45}, & y \in [\frac{20}{9}, \infty). \end{cases}$$

In this case $z = \sqrt{2}$ with $\psi(\sqrt{2}) = 25 - 16\sqrt{2}$ and $x = \frac{9}{20}$ with $\psi(\frac{9}{20}) = \frac{36}{25}$.

Let a be the value at 1 of the upper concave envelope of ψ. Observe that $V_\Omega^{\mathbb{F},[T/2,T]}(\xi) = a > 1$. In comparison, the information arriving at time $\frac{T}{2}$, say, $\{Z^{\frac{T}{2}} = c\}$, implies that $\omega_{\frac{2T}{3}}/\omega_{\frac{T}{2}} \in \{e^{-c}, e^c\}$ and hence, by construction, $V_\Omega^{\mathbb{G}^{\frac{T}{2}},[T/2,T]}(\xi) = 1$. In contrast, information arriving at time 0 is irrelevant, i.e., the the superhedging cost of ξ remains a. In consequence, we have $\mathfrak{A}_V^{\mathbb{G}^{\frac{T}{2}},\mathbb{F}}(\xi) = a - 1 > 0 = \mathfrak{A}_V^{\mathbb{G}^0,\mathbb{F}}(\xi)$.

On the other hand, information arriving at time 0 reduces the superhedging cost of ζ compared to the reference filtration, from a to 1, and information arriving at time $\frac{T}{2}$ is irrelevant, i.e., the superhedging cost of ζ remains a. In this case, we have $\mathfrak{A}_V^{\mathbb{G}^0,\mathbb{F}}(\zeta) = a-1 > 0 = \mathfrak{A}_V^{\mathbb{G}^{\frac{T}{2}},\mathbb{F}}(\zeta)$.

Appendix A. Proofs of Lemmas 3.3 and 4.1 and Propositions 5.1 and 5.2.

Proof of Lemma 3.3. Existence of $\mathbb{P}_\omega$ and its properties are all classical results; see Stroock and Varadhan (2007, pp. 12–16). Since, by Standing Assumption 2.1, $\mathcal{G}_{-1} = \sigma(B_n^{-1}, n \geq 1)$, there exists a set $\Omega_{-1} \in \mathcal{G}_{-1}$ such that $\mathbb{P}(\Omega_{-1}) = 1$ and $\mathbb{P}_\omega(A^\omega) = 1$ for each $\omega \in \Omega_{-1}$.

Fix $t \geq s$ and $G \in \mathcal{G}_s$. Then, since $G_0 := \{\mathbb{E}_{\mathbb{P}_\omega}[(S_t - S_s)\mathbb{1}_G] > 0\} \in \mathcal{G}_{-1}$, we get

$$0 = \mathbb{E}_{\mathbb{P}}[(S_t - S_s)\mathbb{1}_{G\cap G_0}] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[(S_t - S_s)\mathbb{1}_G|\mathcal{G}_{-1}]\mathbb{1}_{G_0}] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}_\omega}[(S_t - S_s)\mathbb{1}_G]\mathbb{1}_{G_0}],$$

which implies that $\mathbb{P}$-a.s. $\mathbb{E}_{\mathbb{P}_\omega}[(S_t - S_s)\mathbb{1}_G] \leq 0$. In the same way we prove that $\mathbb{P}$-a.s. $\mathbb{E}_{\mathbb{P}_\omega}[(S_t - S_s)\mathbb{1}_G] \geq 0$. So finally, $\mathbb{P}$-a.s. $\mathbb{E}_{\mathbb{P}_\omega}[(S_t - S_s)\mathbb{1}_G] = 0$ and therefore there exists a set $\Omega_{s,t,G} \in \mathcal{G}_{-1}$ such that $\mathbb{P}(\Omega_{s,t,G}) = 1$ and $\mathbb{E}_{\mathbb{P}_\omega}[(S_t - S_s)\mathbb{1}_G] = 0$ for each $\omega \in \Omega_{s,t,G}$. To conclude that there exists a set $\Omega_m \in \mathcal{G}_{-1}$ such that $\mathbb{P}(\Omega_m) = 1$ and S is a $(\mathbb{P}_\omega, \mathbb{G})$-martingale for every $\omega \in \Omega_m$ we use continuity of paths of S and Standing Assumption 2.1. Finally we note that

$$\mathbb{P}\text{-a.s.} \quad \forall \lambda \in \Lambda \quad \mathcal{P}(\mathcal{X}^\lambda) = \mathbb{E}_{\mathbb{P}}[\mathcal{X}^\lambda|\mathcal{G}_{-1}] = \mathbb{E}_{\mathbb{P}_\omega}[\mathcal{X}^\lambda]$$

and therefore there exists a set $\Omega_{\mathcal{X}} \in \mathcal{G}_{-1}$ such that $\mathbb{P}(\Omega_{\mathcal{X}}) = 1$, and for every $\omega \in \Omega_{\mathcal{X}}$ it holds that $\mathcal{P}(\mathcal{X}^\lambda) = \mathbb{E}_{\mathbb{P}_\omega}[\mathcal{X}^\lambda]$ for each $\lambda \in \Lambda$. To complete the proof it is enough to take $\Omega^{\mathbb{P}} = \Omega_{-1} \cap \Omega_m \cap \Omega_{\mathcal{X}}$. ■

Proof of Lemma 4.1. Using Propositions 3.2 and 3.5, it is enough to show the asserted inequality separately on each atom A^ω of the σ-field $\mathcal{G}_{-1}$. The proof then follows by a classical argument. Take any $\mathbb{G}$-admissible superreplicating portfolio $(\alpha, \gamma) \in \mathcal{A}_{\mathcal{X}}(\mathbb{G})$ on A^ω and any measure $\mathbb{P} \in \mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A^\omega}$. Let $\{\mathbb{P}_v\}$ denote regular conditional probabilities of $\mathbb{P}$ with respect to $\mathcal{G}_0$. Thus, by Lemma 3.3, $\mathbb{P}$-a.s., $\mathbb{P}_v \in \mathcal{M}^{\mathbb{G}}_{B^v}$, where $\{B^v\}$ are $\mathcal{G}_0$-atoms containing v. Note that $\{B^v\}$ form a finer partition than $\{A^\omega\}$. Then, since $\mathbb{P}$-a.s., $\mathbb{P}_v(M \equiv \text{const}) = 1$ and $\mathbb{P}_v(\gamma = \gamma\mathbb{1}_{B^v})$, where $\gamma\mathbb{1}_{B^v}$ is jointly measurable, we deduce that

$$\mathbb{E}_{\mathbb{P}_v}[\xi] \leq \mathbb{E}_{\mathbb{P}_v}\left[\alpha\mathcal{X} + \int_0^T \gamma_u dS_u\right] \leq \mathbb{E}_{\mathbb{P}_v}[\alpha\mathcal{X}], \quad \mathbb{P}\text{-a.s.}$$

as $\int_0^\cdot \gamma_u \mathbb{1}_{B^v} dS_u$ is a $\mathbb{G}$-local martingale bounded from below and thus a $\mathbb{G}$-supermartingale. Since $\{\mathbb{P}_v\}$ are regular conditional probabilities of $\mathbb{P}$ and $\mathbb{P}(\alpha^\lambda = \alpha^\lambda(\omega))$ for each $\lambda \in \Lambda$, taking expectations under $\mathbb{P}$, we have

$$\mathbb{E}_{\mathbb{P}}[\xi] \leq \mathbb{E}_{\mathbb{P}}[\alpha\mathcal{X}] = \alpha(\omega)\mathcal{P}(\mathcal{X}),$$

which completes the proof. ■

In preparation for the proofs of Propositions 5.1 and 5.2 we need the following gluing operator and its measurability.

Lemma A.1. *Let $\Omega_+ = \{\omega \in \Omega : \omega_{T'} > 0\}$. For a fixed $v \in \Omega_+$ define the mapping $\mathfrak{a}^v$ on $\Omega_+ \times \Omega_+$ with values in Ω_+ by*

$$\mathfrak{a}^v(\widetilde{v}, \omega) := \begin{cases} v|_{[0,T']} \otimes \frac{v_{T'}}{\widetilde{v}_{T'}} \omega|_{[T',T]}, & \omega \in B^{\widetilde{v}}, \\ \widetilde{v}|_{[0,T']} \otimes \frac{\widetilde{v}_{T'}}{v_{T'}} \omega|_{[T',T]}, & \omega \in B^{v}, \\ \omega, & \omega \notin B^v \cup B^{\widetilde{v}}, \end{cases} \tag{A.1}$$

where $\lambda\omega$ is a multiplicative modification of ω by λ in Ω and $v|_{[0,T']} \otimes \lambda\omega|_{[T',T]}$ means that the path is equal to v on $[0, T']$ and to $\lambda\omega$ on $[T', T]$; B^v and $B^{\widetilde{v}}$ denote the $\mathcal{F}_{T'}$-atoms containing respectively v and $\widetilde{v}$.

Then, $\mathfrak{a}^v$ is $\mathcal{F}_T \otimes \mathcal{F}_T$-measurable.

Proof of Lemma A.1. Note that $\mathfrak{a}^v$ can be written as

$$\mathfrak{a}^v(\widetilde{v}, \omega) = \begin{cases} \mathfrak{a}_1(\widetilde{v}, \omega) & \text{on } \Omega_+ \times \Omega_+ \backslash \{(\widetilde{v}, \omega) : \omega|_{[0,T']} = \widetilde{v}|_{[0,T']} \ \vee \ \omega|_{[0,T']} = v|_{[0,T']}\}, \\ \mathfrak{a}_2(\omega) & \text{on } \{(\widetilde{v}, \omega) : \omega|_{[0,T']} = \widetilde{v}|_{[0,T']}\}, \\ \mathfrak{a}_3(\widetilde{v}) & \text{on } \{(\widetilde{v}, \omega) : \omega|_{[0,T']} = v|_{[0,T']}\}, \end{cases}$$

where $\mathfrak{a}_i$ for $i \in \{1, 2, 3\}$ are given by $\mathfrak{a}_1(\widetilde{v}, \omega) = \omega$, $\mathfrak{a}_2(\omega) = \mathfrak{a}^v(\omega, \omega)$, and $\mathfrak{a}_3(\omega) = \mathfrak{a}^v(\omega, v)$ with $\mathfrak{a}^v$ defined in (A.1). Thus, since sets $\{(\widetilde{v}, \omega) : \omega|_{[0,T']} = \widetilde{v}|_{[0,T']}\}$ and $\{(\widetilde{v}, \omega) : \omega|_{[0,T']} = v|_{[0,T']}\}$ are $\mathcal{F}_T \otimes \mathcal{F}_T$-measurable, it is now enough to show that each $\mathfrak{a}_i$ is measurable. The map $\mathfrak{a}_1$ is measurable since it is simply a projection.

Note that $\mathfrak{a}_2(\omega) = v|_{[0,T']} \otimes \frac{v_{T'}}{\omega_{T'}} \omega|_{[T',T]}$ and $\mathfrak{a}_3(\omega) = \omega|_{[0,T']} \otimes \frac{\omega_{T'}}{v_{T'}} v|_{[T',T]}$. Hence the mappings $\mathfrak{a}_2$ and $\mathfrak{a}_3$ are measurable since they are constructed via measurable transformations such as time change, scaling, stopping, and pasting. ■

Proof of Proposition 5.1. In the proof we denote $\widetilde{\xi} := V_\Omega^{\mathbb{G},[T',T]}(\xi)$.
(i) Let $v := (v^1, \ldots, v^d) \in \Omega$ and $\widetilde{v} := (\widetilde{v}^1, \ldots, \widetilde{v}^d) \in \Omega$. Note that

$$\left|\widetilde{\xi}((v^1, \ldots, v^d)) - \widetilde{\xi}((\widetilde{v}^1, \ldots, \widetilde{v}^d))\right| \leq \sum_{k=1}^{d} \Big| \,\widetilde{\xi}((\widetilde{v}^1, \ldots, \widetilde{v}^{k-1}, v^k, \ldots, v^d)) - \widetilde{\xi}((\widetilde{v}^1, \ldots, \widetilde{v}^k, v^{k+1}, \ldots, v^d))\Big|.$$

Thus, to establish uniform continuity of $\widetilde{\xi}$, it is enough to consider v and $\widetilde{v}$ which differ on one coordinate only and, without loss of generality, we may assume that $d = 1$.

Consider a small $\delta > 0$. Suppose that $||v - \widetilde{v}|| \leq \delta$, $|v_{T'} - \widetilde{v}_{T'}| = D \geq 0$, $v_{T'} > 0$, and $\widetilde{v}_{T'} > 0$.

In the first step we show that $\widetilde{\xi}(\widetilde{v}) \leq \widetilde{\xi}(v) + \varepsilon$ for an appropriately chosen ε, depending only on ξ and δ. For each $\eta > 0$ there exists a strategy γ such that

$$\widetilde{\xi}(v) + \int_{T'}^{T} \gamma_t dS_t \geq \xi - \eta \text{ on } B^v.$$

Let $\lambda = v_{T'}/\widetilde{v}_{T'} \in (0, \infty)$ and define the path modification mapping $\mathfrak{a}$ by $\mathfrak{a}(\omega) := \mathfrak{a}^v(\widetilde{v}, \omega)$, where $\mathfrak{a}^v$ is given in (A.1). Note that $\mathfrak{a}$ is a bijection satisfying $\mathfrak{a} = \mathfrak{a}^{-1}$. Introduce a stopping time

$$\widetilde{\tau}(\omega) := \tau^{v,\widetilde{v}}(\omega) := \begin{cases} \inf\{t > T' : \omega_t - \widetilde{v}_{T'} \geq \widetilde{v}_{T'} D^{-\frac{1}{2}}\} \wedge T, & \omega \in B^{\widetilde{v}}, \\ \inf\{t > T' : \omega_t - v_{T'} \geq v_{T'} D^{-\frac{1}{2}}\} \wedge T, & \omega \in B^{v}, \\ T', & \omega \notin B^v \cup B^{\widetilde{v}}. \end{cases} \tag{A.2}$$

To show that $\widetilde{\xi}(\widetilde{v}) \leq \widetilde{\xi}(v) + \varepsilon$ we will consider a strategy $\lambda\gamma \circ \mathfrak{a} + \frac{D^{\frac{1}{4}}}{\widetilde{v}_{T'}} \mathbb{1}_{[T',\widetilde{\tau})}$ on $B^{\widetilde{v}}$. The second term of this strategy is clearly $\mathbb{G}$-adapted. To show that the first term is $\mathbb{G}$-adapted as well, it is enough to show that $Z \circ \mathfrak{a}$ is $\sigma(Z)$-measurable. The last is true since

$$Z \circ \mathfrak{a}(\omega) = \widetilde{Z}\left(\frac{\mathfrak{a}(\omega)|_{[T',T]}}{\mathfrak{a}(\omega)_{T'}}\right) = \widetilde{Z}\left(\frac{\omega|_{[T',T]}}{\omega_{T'}}\right) = Z(\omega).$$

Then, we obtain

$$\begin{aligned} &\widetilde{\xi}(v) + \lambda \int_{T'}^{T} \gamma \circ \mathfrak{a}(\omega)_t dS_t(\omega) + \frac{D^{\frac{1}{4}}}{\widetilde{v}_{T'}}(\omega_{\widetilde{\tau}} - \widetilde{v}_{T'}) \\ &= \widetilde{\xi}(v) + \int_{T'}^{T} \gamma \circ \mathfrak{a}(\omega)_t dS_t \circ \mathfrak{a}(\omega) + \frac{D^{\frac{1}{4}}}{\widetilde{v}_{T'}}(\omega_{\widetilde{\tau}} - \widetilde{v}_{T'}) \\ &\geq \xi \circ \mathfrak{a}(\omega) - \eta + \frac{D^{\frac{1}{4}}}{\widetilde{v}_{T'}}(\omega_{\widetilde{\tau}} - \widetilde{v}_{T'}), \end{aligned}$$

where the first equality is due to our definition of integration. In the case that $\widetilde{\tau}(\omega) = T$ one has

$$\frac{D^{\frac{1}{4}}}{\widetilde{v}_{T'}}(\omega_{\widetilde{\tau}} - \widetilde{v}_{T'}) \geq -\frac{D^{\frac{1}{4}}}{\widetilde{v}_{T'}}\widetilde{v}_{T'} = -D^{1/4}$$

and

$$\|\mathfrak{a}(\omega) - \omega\| \leq \delta \vee (\lambda - 1)\widetilde{v}_{T'}(D^{-\frac{1}{2}} + 1) \leq 2\delta^{1/2}. \tag{A.3}$$

Thus, for $\widetilde{\tau}(\omega) = T$, it follows that

$$\xi \circ \mathfrak{a}(\omega) - \eta + \frac{D^{\frac{1}{4}}}{\widetilde{v}_{T'}}(\omega_{\widetilde{\tau}} - \widetilde{v}_{T'}) \geq \xi(\omega) - e_\xi(2\delta^{1/2}) - \eta - D^{1/4},$$

where e_ξ is modulus of continuity of ξ. Hence, for $\widetilde{\tau}(\omega) = T$, we deduce $\widetilde{\xi}(\widetilde{v}) \leq \widetilde{\xi}(v) + e_\xi(2\delta^{1/2}) + D^{1/4}$. In the case that $\widetilde{\tau}(\omega) < T$ one has

$$\frac{D^{\frac{1}{4}}}{\widetilde{v}_{T'}}(\omega_{\widetilde{\tau}} - \widetilde{v}_{T'}) = \frac{D^{\frac{1}{4}}}{\widetilde{v}_{T'}}\widetilde{v}_{T'} D^{-\frac{1}{2}} = D^{-1/4}$$

and

$$\xi \circ \mathfrak{a}(\omega) - \eta + \frac{D^{\frac{1}{4}}}{\widetilde{v}_{T'}}(\omega_{\widetilde{\tau}} - \widetilde{v}_{T'}) \geq -||\xi|| - \eta + D^{-1/4},$$

which, for D small enough ($D \leq (2||\xi||)^{-4}$), dominates $\xi(\omega)$. We deduce that $\widetilde{\xi}(\widetilde{v}) \leq \widetilde{\xi}(v) + e_\xi(2\delta^{1/2}) + D^{1/4}$ and conclude that $\widetilde{\xi}$ is uniformly continuous on $\{\omega \in \Omega : \|\omega\| > 0\}$.

To complete the proof, we now consider the case where $\widetilde{v}_{T'} = 0$. Let, for some small $\delta > 0$, $||v - \widetilde{v}||_{[0,T']} \le \delta$ and $v_{T'} = D > 0$. First notice that $\widetilde{v}$ must satisfy $\widetilde{\xi}(\widetilde{v}) = \xi(\widetilde{v}|_{[0,T']} \otimes 0|_{[T',T]})$ since we can buy any amount of stock at price 0 at time T', thus only the constant path is relevant, and therefore $\widetilde{\xi}(\widetilde{v}) \le \widetilde{\xi}(v) + e_\xi(\delta)$. Now consider the strategy γ for $\omega \in B^v$ defined as

$$\gamma(\omega) := \delta^{-1/2}\mathbb{1}_{[T',\sigma(\omega))}, \quad \text{where} \quad \sigma(\omega) := \inf\{t > T' : \omega_t - v_{T'} \ge \delta^{1/4}\}.$$

Then, whenever $\sigma(\omega) < T$,

$$\widetilde{\xi}(\widetilde{v}) + \int_{T'}^{T} \gamma(\omega)_t dS_t(\omega) = \widetilde{\xi}(\widetilde{v}) + \delta^{-1/2}(\omega_\sigma - v_{T'}) = \widetilde{\xi}(\widetilde{v}) + \delta^{-1/4},$$

which, for d small enough, majorates $\xi(\omega)$. Otherwise, if $\sigma(\omega) = T$

$$\widetilde{\xi}(\widetilde{v}) + \int_{T'}^{T} \gamma(\omega)_t dS_t(\omega) \ge \widetilde{\xi}(\widetilde{v}) - \delta^{1/2} \ge \xi(\omega) - e_\xi(2\delta^{1/4}) - \delta^{1/2}$$

since $||\widetilde{v} - \omega|| \le 2\delta^{1/4}$. Therefore, $\widetilde{\xi}(v) \le \widetilde{\xi}(\widetilde{v}) + e_\xi(2\delta^{1/4}) + \delta^{1/2}$.

(ii) Let $V^1 := V_\Omega^{\mathbb{G},[0,T]}(\xi)$ and $V^2 := V_\Omega^{\mathbb{G},[0,T']}(\widetilde{\xi})$. For each $\eta > 0$ there exists a strategy $\gamma \in \mathcal{A}(\mathbb{G}, [0,T])$ such that

$$V^1 + \int_0^{T'} \gamma_t dS_t + \int_{T'}^{T} \gamma_t dS_t \ge \xi - \eta \quad \text{on} \quad \Omega.$$

Let $\rho(S) := \inf\{t > 0 : V^1 + \int_0^t \gamma_u dS_u \ge \sup_{\omega\in\Omega} \xi(\omega) - \eta\} \wedge T$. It is a stopping time. Hence $\widetilde{\gamma} := \gamma \mathbb{1}_{[0,\rho]} \in \mathcal{A}(\mathbb{G}, [0,T])$ and satisfies that

$$V^1 + \int_0^{T'} \widetilde{\gamma}_t dS_t + \int_{T'}^{T} \widetilde{\gamma}_t dS_t \ge \xi - \eta \quad \text{on} \quad \Omega.$$

Moreover, for any $t \ge T'$,

$$\int_{T'}^{t} \widetilde{\gamma}_u dS_u \ge \xi - \sup_{\omega\in\Omega} \xi(\omega) \quad \text{on} \quad \Omega,$$

and therefore $\widetilde{\gamma} \in \mathcal{A}(\mathbb{G}, [T',T])$.

In particular, for a fixed $\omega \in \Omega$, the superhedging holds on B^v. Since $V^1 + \int_0^{T'} \widetilde{\gamma}_t dS_t$ is constant on B^ω, we deduce that $V^1 + \int_0^{T'} \widetilde{\gamma}_t dS_t \ge \widetilde{\xi}$ on $\Omega|_{[0,T']}$ and therefore $V^1 \ge V^2$.

To prove the reverse inequality take $z > V^2$. First, there exists $\gamma^1 \in \mathcal{A}(\mathbb{F}, [0,T'])$ such that $z + \int_0^{T'} \gamma_t^1 dS_t \ge \widetilde{\xi}$ on Ω. If, for each $\eta > 0$, there exists a strategy $\gamma^2 \in \mathcal{A}(\mathbb{G}, [T',T])$ such that γ^2 is jointly measurable and $z + \int_0^{T'} \gamma_t^1 dS_t + \int_{T'}^{T} \gamma_t^2 dS_t \ge \xi - \eta$, then clearly $z \ge V^1$.

We now show the existence of such γ^2 for every $\eta > 0$. Let $\{\omega^n\}_n$ be a countable dense subset of $\Omega|_{[0,T']}$ and $B^n := B^{\omega^n}$, and denote the closed ball around ω^n of radius δ by $\widetilde{B}^n(\delta) := \{\omega : \sup_{t\in[0,T']} |\omega_t - \omega_t^n| \le \delta\}$. Define the path modification mapping $\mathfrak{a}^{n,\omega}$ by $\mathfrak{a}^{n,\omega} := \mathfrak{a}^{\omega^n}(\omega, \cdot)$, where $\mathfrak{a}^{\omega^n}(\omega,\cdot)$ is given in (A.1). Note that $\mathfrak{a}^{n,\omega}$ is a bijection satisfying $\mathfrak{a}^{n,\omega}=(\mathfrak{a}^{n,\omega})^{-1}$. We now take $\{\gamma^n\}_n$, a set of strategies in $\mathcal{A}(\mathbb{G},[T',T])$, such that

$$\widetilde{\xi}(\omega^n)+\int_{T'}^{T}\gamma^n_u(S)dS_u\geq \xi-\delta \quad \text{on } B^n. \tag{A.4}$$

Let us consider $\widetilde{\gamma}^n:\Omega\to$ defined by $\widetilde{\gamma}^n(\omega)=0$ if $\omega\notin\widetilde{B}^n(D)$ and for $\omega\in\widetilde{B}^n(D)$

$$\widetilde{\gamma}^n(\omega)=\begin{cases}\frac{\omega^n_{T'}}{\omega_{T'}}\gamma^n\circ\mathfrak{a}^{n,\omega}+\frac{D^{\frac{1}{4}}}{\omega_{T'}}\mathbb{1}_{[T',\tau^{\omega^n,\omega})} & \text{if } \omega^n_{T'}\geq\delta,\\ \delta^{-1/2}\mathbb{1}_{[T',\sigma(\omega))} & \text{if } \omega^n_{T'}<\delta,\end{cases}$$

where $\sigma(\omega):=\inf\{t>T' : \omega_t-\omega_{T'}\geq\delta^{1/4}\}$. It follows from above that there exists a constant $\epsilon(D,\delta)$ which depends on D and δ with $\epsilon(D,\delta)\to 0$ as $D,\delta\to 0$, such that

$$\widetilde{\xi}(S)+\int_{T'}^{T}\widetilde{\gamma}^n_u(S)dS_u\geq\xi(S)-\epsilon(D,\delta).$$

The strategy $\widetilde{\gamma}^n$ is clearly $\mathcal{F}_T$-measurable. We also notice that it is adapted to $\mathcal{F}$ on $[T',T]$ since it is straightforward to see that for any $\omega,\upsilon\in\Omega$ such that $\omega_u=\upsilon_u$ for any $u\leq[t,T]$ with $t\geq T'$, $\widetilde{\gamma}^n_u(\omega)=\widetilde{\gamma}_u(\upsilon)$ on $[T',t]$. Hence, $\widetilde{\gamma}^n\in\mathring{A}([T',T])$. In addition, we know that for any n, $t\in[T',T]$, and $S\in B^n$, there exists $\widetilde{S}$ such that $\widetilde{S}_u=S_u$ for any $u\leq t$ and $\widetilde{S}_u=S_t$ for any $u\geq t$, and therefore

$$\int_{T'}^{t}\gamma^n_u(S)dS_u=\int_{T'}^{T}\gamma^n_u(\tilde{S})d\tilde{S}_u\geq\xi(\widetilde{S})-\delta-\widetilde{\xi}(\omega^n)\geq 2\inf_{\omega\in\Omega}\xi(\omega)-1. \tag{A.5}$$

Let us now define $\widetilde{\gamma}^\varepsilon$ by

$$\widetilde{\gamma}^\varepsilon(\omega):=\sum_n\mathbb{1}_{C^n}(\omega)\widetilde{\gamma}^n(\omega)\quad\text{where}\quad C^n:=\widetilde{B}^n\backslash\bigcup_{k=1}^{n-1}\widetilde{B}^k.$$

It is then straightforward to see that $\widetilde{\gamma}^\varepsilon$ is progressively measurable and satisfies the admissibility condition in (2.2). ■

Proof of Proposition 5.2. In the proof we denote $\widehat{\xi}:=P^{\mathbb{G},[T',T]}_{\Omega}(\xi)$.
(i) Let $\upsilon:=(\upsilon^1,\ldots,\upsilon^d)\in\Omega$ and $\widetilde{\upsilon}:=(\widetilde{\upsilon}^1,\ldots,\widetilde{\upsilon}^d)\in\Omega$. Note that

$$\left|\widehat{\xi}((\upsilon^1,\ldots,\upsilon^d))-\widehat{\xi}((\widetilde{\upsilon}^1,\ldots,\widetilde{\upsilon}^d))\right|\leq\sum_{k=1}^{d}\left|\,\widehat{\xi}((\widetilde{\upsilon}^1,\ldots,\widetilde{\upsilon}^{k-1},\upsilon^k,\ldots,\upsilon^d))\right.$$
$$\left.-\widehat{\xi}((\widetilde{\upsilon}^1,\ldots,\widetilde{\upsilon}^k,\upsilon^{k+1},\ldots,\upsilon^d))\right|.$$

Thus, to prove uniform continuity of $\widetilde{\xi}$, it is enough to consider υ and $\widetilde{\upsilon}$ which differ on one coordinate only and, without loss of generality, we may assume that $d=1$.

Suppose that $||\upsilon-\widetilde{\upsilon}||_{[0,T']}\leq\delta$ and $|\upsilon_{T'}-\widetilde{\upsilon}_{T'}|=D\geq 0$. It is enough to show that $\widehat{\xi}(\widetilde{\upsilon})\leq\widehat{\xi}(\upsilon)+\varepsilon$ for an appropriately chosen ε depending only on ξ and δ. Take $\mathbb{P}\in\mathcal{M}^{\mathbb{G},[T',T]}_{B^{\widetilde{\upsilon}}}$, i.e., $\mathbb{P}(B^{\widetilde{v}}) = 1$, where $B^{\widetilde{v}} := \{\omega : \omega_t = \widetilde{v}_t \text{ for } t \in [0, T']\}$, $\mathbb{P}(S_{T'} = \widetilde{v}_{T'}) = 1$, and $\mathbb{E}_{\mathbb{P}}[S_t \mathbb{1}_G] = \mathbb{E}_{\mathbb{P}}[S_s \mathbb{1}_G]$ for each $T' \le s \le t \le T$ and $G \in \mathcal{G}_s$. Define measure $\bar{\mathbb{P}}$ as $\bar{\mathbb{P}} = \mathbb{P} \circ \mathfrak{a}$ with path modification $\mathfrak{a}$ given in (A.1). Then $\bar{\mathbb{P}}$ is an element of $\mathcal{M}^{\mathbb{G},[T',T]}_{B^v}$ since $\bar{\mathbb{P}}(B^v) = \mathbb{P}(\mathfrak{a}(B^v)) = \mathbb{P}(B^{\widetilde{v}}) = 1$, and it is a martingale measure on $[T', T]$ as for $T' \le s \le t \le T$ and $G \in \mathcal{G}_s$

$$\mathbb{E}_{\bar{\mathbb{P}}}[S_t \mathbb{1}_G] = \mathbb{E}_{\mathbb{P}}[(S \circ \mathfrak{a})_t \mathbb{1}_{\mathfrak{a}(G)}] = \mathbb{E}_{\mathbb{P}}[(S \circ \mathfrak{a})_s \mathbb{1}_{\mathfrak{a}(G)}] = \mathbb{E}_{\bar{\mathbb{P}}}[S_s \mathbb{1}_G],$$

where the second equality follows by $\mathfrak{a}(G) \in \mathcal{G}_s$. The latter is true since, for any Borel set B, one has $\mathfrak{a}(\{Z \in B\} \cap B^v) = B^{\widetilde{v}} \cap \{Z \in A\}$; the σ-field $\mathcal{F}_s$ coincides with the trivial σ-field up to $\mathbb{P}$-null sets and up to $\bar{\mathbb{P}}$-null sets; the general case follows from the monotone class argument. Hence, with $\widetilde{\tau}$ defined in (A.2),

$$\begin{aligned} |\mathbb{E}_{\mathbb{P}}[\xi] - \mathbb{E}_{\bar{\mathbb{P}}}[\xi]| &= |\mathbb{E}_{\mathbb{P}}[\xi] - \mathbb{E}_{\mathbb{P}}[\xi \circ \mathfrak{a}]| \\ &= \mathbb{E}_{\mathbb{P}}[(\xi - \xi \circ \mathfrak{a}) \mathbb{1}_{\{\widetilde{\tau} = T\}}] + \mathbb{E}_{\mathbb{P}}[(\xi - \xi \circ \mathfrak{a}) \mathbb{1}_{\{\widetilde{\tau} < T\}}] \\ &\le e_\xi(2\delta^{1/2}) + 2||\xi|| \frac{D}{D + D^{1/2}}, \end{aligned} \tag{A.6}$$

where in the last inequality we used (A.3), Doob's inequality, and the fact that

$$\mathbb{P}(\widetilde{\tau} < T) = \mathbb{P}\left(\sup_{t \in [T', T]} S_t \ge \widetilde{v}_{T'}(1 + D^{-1/2}) \right) \le \frac{\widetilde{v}_{T'}}{\widetilde{v}_{T'}(1 + D^{-1/2})} = \frac{D}{D + D^{1/2}}.$$

(ii) To prove that $P^{\mathbb{G},[0,T]}_\Omega(\xi) \le P^{\mathbb{F},[0,T']}_\Omega(P^{\mathbb{G},[T',T]}_\Omega(\xi))$ it is enough to note that

$$\sup_{\mathbb{P} \in \mathcal{M}^{\mathbb{G},[0,T]}_\Omega} \mathbb{E}_{\mathbb{P}}[\xi] = \sup_{\mathbb{P} \in \mathcal{M}^{\mathbb{G},[0,T]}_\Omega} \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}_\omega}[\xi]] \le \sup_{\mathbb{P} \in \mathcal{M}^{\mathbb{G},[0,T]}_\Omega} \mathbb{E}_{\mathbb{P}} \left[\sup_{\bar{\mathbb{P}} \in \mathcal{M}^{\mathbb{G},[T',T]}_{B^\omega}} \mathbb{E}_{\bar{\mathbb{P}}}[\xi] \right],$$

where $\{\mathbb{P}_\omega\}$ is regular conditional distribution with respect to $\mathcal{F}_{T'}$ and where in the last step we used measurability implied by assertion (i).

Now we will show the remaining inequality. Let $\{\omega^n\}_n$ be a countable dense subset of $\Omega|_{[0,T']}$ and $B^n := B^{\omega^n}$. Define the path modification mapping $\mathfrak{a}^{n,\omega}$ by $\mathfrak{a}^{n,\omega} := \mathfrak{a}^{\omega^n}(\omega, \cdot)$ where $\mathfrak{a}^{\omega^n}(\omega, \cdot)$ is given in (A.1). Note that $\mathfrak{a}^{n,\omega}$ is a bijection satisfying $\mathfrak{a}^{n,\omega} = (\mathfrak{a}^{n,\omega})^{-1}$. For any $\mathbb{P}_n \in \mathcal{M}^{\mathbb{G},[T',T]}_{B^n}$ the measure $\mathbb{P}_n \circ \mathfrak{a}^{n,\omega}$ belongs to $\mathcal{M}^{\mathbb{G},[T',T]}_{B^\omega}$. Moreover, similarly to (A.6), we obtain that

$$|\mathbb{E}_{\mathbb{P}_n}[\xi] - \mathbb{E}_{\mathbb{P}_n \circ \mathfrak{a}^{n,\omega}}[\xi]| \le e_\xi(2\delta^{1/2}) + 2||\xi|| \frac{\delta}{\delta + \delta^{1/2}}$$

whenever $||\omega^n - \omega||_{[0,T']} \le \delta$. Let us consider probability kernel $N_n : \Omega \to \mathcal{M}^{\mathbb{G},[T',T]}$ defined by $N_n(\omega) := \mathbb{P}_n \circ \mathfrak{a}^{n,\omega}$. The kernel N_n is $\mathcal{F}_T$-measurable, i.e., $N_n(\omega, F) = \mathbb{P}_n \circ \mathfrak{a}^{n,\omega}(F)$ is $\mathcal{F}_T$-measurable for any $F \in \mathcal{F}_T$, since $(\omega, \widetilde{\omega}) \to \mathbb{1}_F \circ \mathfrak{a}^{n,\omega}(\widetilde{\omega})$ is $\mathcal{F}_T \otimes \mathcal{F}_T$-measurable and bounded, thus $\mathbb{E}_{\mathbb{P}_n}[\mathbb{1}_F \circ \mathfrak{a}^{n,\omega}]$ is $\mathcal{F}_T$-measurable (see Bogachev (2007, section 3.3)). Measurability of

$\mathfrak{a}^n(\widetilde{\omega},\omega)$ was shown in Lemma A.1. Then, since N_n is constant on atoms of $\mathcal{F}_{T'}$, we deduce from Blackwell's theorem (see Cohn (1980, Theorem 8.6.7) and/or Dellacherie and Meyer (1975, Chapter III, section 26, pp.80–81)) that N_n is an $\mathcal{F}_{T'}$-measurable probability kernel.

Denoting the closed ball around ω^n of radius δ by $\widetilde{B}^n(\delta) := \{\omega : \sup_{t\in[0,T']} |\omega_t - \omega^n_t| \le \delta\}$, we observe that

$$\sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{G},[T',T]}_{\widetilde{B}^n(\delta)}} \mathbb{E}_{\mathbb{P}}[\xi] = \sup_{\omega\in\widetilde{B}^n(\delta)} \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{G},[T',T]}_{B^\omega}} \mathbb{E}_{\mathbb{P}}[\xi] = \sup_{\omega\in\widetilde{B}^n(\delta)} \widehat{\xi}(\omega) \le \widehat{\xi}(\omega^n) + \varepsilon(\delta),$$

where the second equality follows from uniform continuity of $\widehat{\xi}$.

Fix $\varepsilon > 0$. Then we can choose $\delta > 0$ and a family of measures $(\mathbb{P}^\varepsilon_n)_{n\in\mathbb{N}}$ such that $\mathbb{P}^\varepsilon_n \in \mathcal{M}^{\mathbb{G},[T',T]}_{B^n}$ for each n and

$$\varepsilon/2 + \mathbb{E}_{\mathbb{P}^\varepsilon_n}[\xi] \ge \widehat{\xi}(\omega) \quad \forall\omega\in\widetilde{B}^n(\delta) \quad \text{and} \quad e_\xi(2\delta^{1/2}) + 2||\xi||\delta/(\delta+\delta^{1/2}) \le \varepsilon/2.$$

Let us now define the $\mathcal{F}_{T'}$-measurable probability kernel N^ε as

$$N^\varepsilon(\omega) := \sum_n \mathbb{1}_{C^n}(\omega)\, \mathbb{P}^\varepsilon_n \circ \mathfrak{a}^{n,\omega}, \quad \text{where} \quad C^n := \widetilde{B}^n \setminus \bigcup_{k=1}^{n-1} \widetilde{B}^k.$$

The probability kernel N^ε is constructed such that it satisfies

$$\varepsilon + \mathbb{E}_{N^\varepsilon(\omega)}[\xi] \ge \widehat{\xi}(\omega) \quad \forall\omega\in\Omega \quad \text{and} \quad N^\varepsilon(\omega)\in\mathcal{M}^{\mathbb{G},[0,T']}_{B^\omega}.$$

There exists a measure $\mathbb{P}^\varepsilon \in \mathcal{M}^{\mathbb{F},[0,T']}_\Omega$ such that $\varepsilon + \mathbb{E}_{\mathbb{P}^\varepsilon}[\widehat{\xi}] \ge \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{F},[0,T']}_\Omega} \mathbb{E}_{\mathbb{P}}[\widehat{\xi}]$. The concatenation of measures $\bar{\mathbb{P}}^\varepsilon := \mathbb{P}^\varepsilon \otimes N^\varepsilon$ (see section 3.1 in Karoui and Tan (2013)), defined, for each $F\in\mathcal{F}_T$, as

$$\bar{\mathbb{P}}^\varepsilon(F) = \mathbb{E}_{\mathbb{P}^\varepsilon}\Big[\sum_n \mathbb{1}_{C^n} N^\varepsilon(F)\Big],$$

is a probability measure. Note that regular conditional probabilities of $\bar{\mathbb{P}}^\varepsilon$ w.r.t. $\mathcal{F}_{T'}$ equal N^ε and $d\bar{\mathbb{P}}^\varepsilon|_{\mathcal{F}_{T'}} = d\mathbb{P}^\varepsilon|_{\mathcal{F}_{T'}}$. Thus, for $s\le t$ and $G_s\in\mathcal{G}_s$, we have

$$\begin{aligned}\mathbb{E}_{\bar{\mathbb{P}}^\varepsilon}\big[(S_t-S_s)\mathbb{1}_{G_s}\big] &= \mathbb{E}_{\bar{\mathbb{P}}^\varepsilon}\big[\mathbb{E}_{N^\varepsilon}[(S_t-S_s)\mathbb{1}_{G_s}]\big] = \mathbb{E}_{\bar{\mathbb{P}}^\varepsilon}\big[\mathbb{E}_{N^\varepsilon}[(S_{s\vee T'}-S_s)\mathbb{1}_{G_s}]\big] \\ &= \mathbb{E}_{\bar{\mathbb{P}}^\varepsilon}\big[(S_{s\vee T'}-S_s)\mathbb{1}_{G_s}\big] = \mathbb{E}_{\mathbb{P}^\varepsilon}\big[(S_{s\vee T'}-S_s)\mathbb{1}_{G_s}\big] = 0,\end{aligned}$$

which shows that S is a $(\bar{\mathbb{P}}^\varepsilon,\mathbb{G})$-martingale. Moreover $\bar{\mathbb{P}}^\varepsilon$ satisfies

$$\mathbb{E}_{\bar{\mathbb{P}}^\varepsilon}[\xi] = \mathbb{E}_{\mathbb{P}^\varepsilon}\big[\mathbb{E}_{N^\varepsilon}[\xi]\big] \ge \mathbb{E}_{\mathbb{P}^\varepsilon}[\widehat{\xi}] - \varepsilon \ge \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{F},[0,T']}_\Omega} \mathbb{E}_{\mathbb{P}}[\widehat{\xi}] - 2\varepsilon.$$

The proof is complete. ∎

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