

On the splitting-up method for rough (partial) differential equations

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Abstract

This article introduces the splitting method to systems responding to rough paths as external stimuli. The focus is on nonlinear partial differential equations with rough noise but we also cover rough differential equations. Applications to stochastic partial differential equations arising in control theory and nonlinear filtering are given.

1 Introduction

This article introduces the splitting-up method for systems responding to rough paths as external stimuli. We deal with (nonlinear) rough partial differential equations, RPDEs, formally written as

$$du = F(t, x, u, Du, D^2u) dt + \Lambda(t, x, u, Du) dz \text{ on } (0, T] \times \mathbb{R}^e, \quad u(0, x) = u_0(x) \quad (1)$$

but we also cover rough differential equations, RDEs, of the form

$$dy_t = V(y_t) dt + W(y_t) dz_t.$$

In both examples, \mathbf{z} is an external stimuli given as a rough path, F is a nonlinear (possibly degenerate) elliptic operator, Λ is a collection of affine linear operators, i.e.

$$\Lambda_k(t, x, r, p) = (p \cdot \sigma_k(t, x)) + r \nu_k(t, x) + g_k(t, x).$$

and σ, ν, g resp. $V, W = (W_i)$ are (collections) of vector fields on $[0, T] \times \mathbb{R}^e$ resp. \mathbb{R}^e . Consequences of our results are splitting results for (nonlinear) stochastic partial differential equations, SPDEs, that is, when \mathbf{z} is taken to be a rough path lift of a stochastic process (e.g. fractional BM with Hurstparameter $> \frac{1}{4}$); for example, linear SPDEs with Brownian noise in Stratonovich form (e.g. the Zakai equation from nonlinear filtering),

$$du = L(t, x, u, Du, D^2u) dt + \sum_{k=1}^d \Lambda_k(t, x, u, Du) \circ dB^k, \quad u(0, \cdot) = u_0(\cdot). \quad (2)$$

are covered. L is here a linear (degenerate elliptic) operator,

$$L(t, x, r, p, X) = \text{Trace}[A(t, x) \cdot X] + b(t, x) \cdot p + c(t, x, r),$$

and B a standard d -dimensional Brownian motion. The splitting-up method (which runs under many names: dimensional splitting, operator splitting, Lie-Trotter-Kato formula, Baker-Campbell-Hausdorff formula, Chernoff formula, leapfrog method, predictor-corrector method, etc.) is one of the most prominent methods for calculating solutions of (stochastic-, ordinary-, partial-) differential equations numerically; for a survey we recommend [MQ02]. For S(P)DEs a splitting-up method was introduced in [BG89] for the Zakai equation in filtering and has received much attention since. We explicitly mention [KG03] which extends the previous results to general linear SPDEs of the form (2).

Let us informally describe the general idea of splitting using (2): For $n \in \mathbb{N}$ consider the partition $D^n = \{t_i^n = in^{-1}T, i = 0, \dots, n\}$ of the interval $[0, T]$ and define the approximation u_n recursively

$$u_n(t_{i+1}^n, \cdot) := \left[\mathbf{Q}_{t_i^n t_{i+1}^n} \circ \mathbf{P}_{t_i^n t_{i+1}^n} \right] (u_n(t_i^n, \cdot))$$

with $\{\mathbf{P}_{st}, 0 \leq s \leq t \leq T\}$ and $\{\mathbf{Q}_{st}, 0 \leq s \leq t \leq T\}$ the solution operators of

$$dv = L(t, x, v, Dv, D^2v) dt, \quad v(s, x) = v(x) \quad \text{and} \quad (3)$$

$$dw = \sum_{k=1}^d \Lambda_k(t, x, w, Dw, D^2w) \circ dB_t^k, \quad w(s, x) = v(x). \quad (4)$$

That is, on each interval $[t_i^n, t_{i+1}^n]$ one solves first equation (4) on $[t_i^n, t_{i+1}^n]$ with initial value $u_n(t_i^n, \cdot)$ and then one uses its solution as initial value for the PDE (3) (so-called ‘‘predictor’’ and ‘‘corrector’’ steps in [FLG91]). Under appropriate conditions, one can show that u_n converges to u and also derive rates of convergence, [KG03].

All the above mentioned authors use (to the best of our knowledge) either semigroup theory or stochastic calculus to prove splitting results but neither are available for (1) due to the nonlinear operator F and the nonsemimartingale noise \mathbf{z} . The point of view of this article is different; loosely speaking: *splitting-up results follow from stability in a rough path sense*. We combine the method of Krylov and Gyöngy, [KG03], of stretching out the time-scale with certain stability results of RPDEs. Applications to SPDEs then follow and our results are, to the best of our knowledge, new for nonlinear PDEs with noise of above form (see also [LS98a, LS98b, LS98b, LS00a, LS00b]). Due to the generality of equation (1) we do not give rates of convergence but hope to return to this question in the future.

1.1 Some ideas from rough path and viscosity theory

Let us recall some basic ideas of (second order) viscosity theory [CIL92, FS06b] and rough path theory [LQ02, LCL07]. As for viscosity theory, consider a real-valued function $u = u(t, x)$ with $t \in [0, T], x \in \mathbb{R}^e$ and assume $u \in C^2$ is a classical subsolution,

$$\partial_t u + F(t, x, u, Du, D^2u) \leq 0,$$

where F is a (continuous) function, *degenerate elliptic* in the sense that $F(t, x, r, p, A + B) \leq F(t, x, r, p, A)$ whenever $B \geq 0$ in the sense of symmetric matrices. The idea is to consider a

(smooth) test function φ and look at a local maxima (\hat{t}, \hat{x}) of $u - \varphi$. Basic calculus implies that $Du(\hat{t}, \hat{x}) = D\varphi(\hat{t}, \hat{x})$, $D^2u(\hat{t}, \hat{x}) \leq D^2\varphi(\hat{t}, \hat{x})$ and, from degenerate ellipticity,

$$\partial_t \varphi + F(\hat{t}, \hat{x}, u, D\varphi, D^2\varphi) \leq 0. \quad (5)$$

This suggests to define a *viscosity supersolution* (at the point (\hat{x}, \hat{t})) to $\partial_t + F = 0$ as a continuous function u with the property that (5) holds for any test function. Similarly, *viscosity subsolutions* are defined by reversing inequality in (5); *viscosity solutions* are both super- and subsolutions. A different point of view is to note that $u(t, x) \leq u(\hat{t}, \hat{x}) - \varphi(\hat{t}, \hat{x}) + \varphi(t, x)$ for (t, x) near (\hat{t}, \hat{x}) . A simple Taylor expansion then implies

$$u(t, x) \leq u(\hat{t}, \hat{x}) + a(t - \hat{t}) + p \cdot (x - \hat{x}) + \frac{1}{2}(x - \hat{x})^T \cdot X \cdot (x - \hat{x}) + o(|\hat{x} - x|^2 + |\hat{t} - t|) \quad (6)$$

as $|\hat{x} - x|^2 + |\hat{t} - t| \rightarrow 0$ with $a = \partial_t \varphi(\hat{t}, \hat{x})$, $p = D\varphi(\hat{t}, \hat{x})$, $X = D^2\varphi(\hat{t}, \hat{x})$. Moreover, if (6) holds for some (a, p, X) and u is differentiable, then $a = \partial_t u(\hat{t}, \hat{x})$, $p = Du(\hat{t}, \hat{x})$, $X \leq D^2u(\hat{t}, \hat{x})$, hence by degenerate ellipticity

$$\partial_t \varphi + F(\hat{t}, \hat{x}, u, p, X) \leq 0.$$

Pushing this idea further leads to a definition of viscosity solutions based on a generalized notion of “ $(\partial_t u, Du, D^2u)$ ” for nondifferentiable u , the so-called parabolic semijets, and it is a simple exercise to show that both definitions are equivalent. The resulting theory (existence, uniqueness, stability, ...) is without doubt one of the most important recent developments in the field of partial differential equations. As a typical result¹, the initial value problem $(\partial_t + F)u = 0$, $u(0, \cdot) = u_0 \in \text{BUC}(\mathbb{R}^e)$ has a unique solution in $\text{BUC}([0, T] \times \mathbb{R}^e)$ provided $F = F(t, x, u, Du, D^2u)$ is continuous, degenerate elliptic, proper (i.e. increasing in the u variable) and satisfies a (well-known) technical condition². In fact, uniqueness follows from a stronger property known as *comparison*: assume u (resp. v) is a supersolution (resp. subsolution) and $u_0 \geq v_0$; then $u \geq v$ on $[0, T] \times \mathbb{R}^e$. A key feature of viscosity theory is what workers in the field simply call *stability properties*. For instance, it is relatively straight-forward to study $(\partial_t + F)u = 0$ via a sequence of approximate problems, say $(\partial_t + F^n)u^n = 0$, provided $F^n \rightarrow F$ locally uniformly and some apriori information on the u^n (e.g. locally uniform convergence, or locally uniform boundedness³). Note the stark contrast to the classical theory where one has to control the actual derivatives of u^n .

The notion of stability is also central to rough path theory. Let $y_0 \in \mathbb{R}^e$, $V, W = (W_i)_{i=1, \dots, d}$ be (collections of) vector fields on \mathbb{R}^e . Using rough path theory⁴, one can speak of solutions to

$$dy_t = V(y_t) d\xi + W(y_t) dz_t, \quad y(0) = y_0 \in \mathbb{R}^e \quad (7)$$

if the weak geometric rough path $\mathbf{z} \in C^{p\text{-var}}([0, T], G^{[p]}(\mathbb{R}^d))$ and $\xi \in C^{q\text{-var}}([0, T], \mathbb{R})$ have “complementary Young regularity” $\frac{1}{p} + \frac{1}{q} > 1$: then (ξ, \mathbf{z}) can be seen as “time-space” rough path, i.e. an element of $C^{\max(p, q)}([0, T], G^{[\max(p, q)]}(\mathbb{R} \oplus \mathbb{R}^d))$ since all necessary cross iterated integrals between \mathbf{z} and ξ are well-defined (using Young integration; [FV09],[LV06]). Now any sequence

¹BUC(...) denotes the space of bounded, uniformly continuous functions.

²(3.14) of the User’s Guide [CIL92].

³What we have in mind here is the *Barles–Perthame method of semi-relaxed limits* [FS06b].

⁴Young theory is actually good enough for $p < 2$; see [Lyo94].

$(\xi^n, z^n)_n \subset C^{1\text{-var}}([0, T], \mathbb{R}) \times C^{1\text{-var}}([0, T], \mathbb{R}^d)$ leads to a sequence of ODE solutions $(y^n)_n$ of

$$dy_t^n = V(y_t^n) d\xi_t^n + \sum_{i=1}^d W_i(y_t^n) dz_t^{n,i}$$

(again assuming the vector fields are regular enough) and we call any limit point in uniform topology of $(y^n)_n$ a solution of the RDE

$$dy_t = V(y_t) d\xi_t + W(y_t) dz_t,$$

if (ξ^n, z^n) converges to (ξ, \mathbf{z}) in the sense

$$\begin{aligned} \sup_n \|S_{[p]}(z^n)\|_{p\text{-var}} + \sup_n \|S_{[q]}(\xi^n)\|_{q\text{-var}} &< \infty \\ d_0(S_{[p]}(z^n), \mathbf{z}) + |\xi^n - \xi|_{\infty; [0, T]} &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (8)$$

Here, $S_{[p]}$ denotes the canonical lift to a geometric step- $[p]$ rough path (i.e. given by Riemann-Stieltjes integration),

$$S_{[p]}(z^n)_{s,t} = 1 + \int_s^t dz_{u_1}^n + \dots + \int_{s \leq u_1 \leq \dots \leq u_{[p]} \leq t} dz_{u_1}^n \otimes \dots \otimes dz_{u_{[p]}}^n. \quad (9)$$

If there exists a unique solution we denote it by $\pi_{V, (W)}(0, x; (\xi, \mathbf{z}))$ to emphasize dependence on the initial condition y_0 , the rough path (ξ, \mathbf{z}) , the vector field V and the collection of vector fields $W = (W_i)_{i=1}^d$. A variation of Lyons' limit theorem, [FV09], gives sufficient⁵ conditions for existence and uniqueness of such RDE solutions.

2 Two examples

In this section we sketch our approach on two examples, ODEs and RDEs (resp. SDEs when take the rough path lift of a stochastic process). Therefore let us introduce some notation: for fixed $\Delta > 0$ and $t \in [0, T]$ set⁶ $t_\Delta = \lfloor t/\Delta \rfloor \Delta$ and $t^\Delta = \lfloor t/\Delta \rfloor \Delta + \Delta$ (i.e. $[t_\Delta, t^\Delta)$ is the interval in the partition of $[0, T]$ of constant mesh size Δ that contains t). Motivated by [KG03] define two time changes

$$a(\Delta, t) = \begin{cases} t_\Delta + 2(t - t_\Delta), & t_\Delta \leq t \leq t_\Delta + \Delta/2 \\ t^\Delta, & t_\Delta + \Delta/2 < t \leq t^\Delta \end{cases}, \quad b(\Delta, t) = a\left(\Delta, t + \frac{\Delta}{2}\right).$$

That is, $a(\Delta, \cdot)$ runs on the first half of each interval $[t_\Delta, t^\Delta)$ with double speed from t_Δ to t^Δ and stays still in the second half, whereas $b(\Delta, \cdot)$ does this in opposite order; also the paths $(a(\Delta, \cdot), b(\Delta, \cdot))$ converge in $(1 + \varepsilon)$ -variation against the path $id_2 : t \mapsto (t, t)$ for every $\varepsilon > 0$.

Further, for given $n \geq 1$ denote by D^n the partition $\{\frac{k}{n}T, k = 0, \dots, n\}$ of $[0, T]$.

The following two examples were shown to us by Terry Lyons at the IRTG SMCP Summer School 2009 in Chorin:

⁵...and essentially sharp (see [Dav07]). We remark also that one can slightly improve on regularity assumption of the vector fields by reproving the Universal Limit theorem for RDEs with drift instead of using the "space-time" rough path (\mathbf{z}, ξ)

⁶[.] denotes the lower floor function.

2.1 Splitting ODEs

Let $V, W \in Lip^1(\mathbb{R}^e, \mathbb{R}^e)$. We are interested in splitting of the ODE

$$dy_t = V(y_t) dt + W(y_t) dt, \quad y(0) = y_0 \in \mathbb{R}^e. \quad (10)$$

As in the introduction, denote the solution of (10) as $\pi_{V,W}(0, y_0; id_2)$. Classic Lie splitting corresponds to the approximation of the path id_2 by the sequence of paths $t \mapsto (a(n^{-1}, t), b(n^{-1}, t))$. Therefore let y^n be the ODE solution of

$$dy_t^n = V(y_t) da(n^{-1}, t) + W(y_t) db(n^{-1}, t), \quad y(0) = y_0 \in \mathbb{R}^e,$$

i.e. $y^n = \pi_{V,W}(0, y_0; (a(n^{-1}, \cdot), b(n^{-1}, \cdot)))$. For brevity, define the solution operator $\{\mathbf{P}_{s,t}^{n;V}, 0 \leq s \leq t \leq T\}$ and $\{\mathbf{P}_u^V, 0 \leq u \leq T\}$ mapping points in \mathbb{R}^e to \mathbb{R}^e as

$$\mathbf{P}_{s,t}^{n;V}(x) := \pi_V(s, x; a(n^{-1}, \cdot))_t \quad \text{and} \quad \mathbf{P}_{t-s}^V(x) := \pi_V(0, x; id_1)_{t-s} = \pi_V(s, x; id_1)_t$$

(here $id_1 : t \mapsto t$ and \mathbf{P}^V is a one parameter semigroup due to homogeneity of id_1); similarly define $\mathbf{Q}^{n,W}$ and \mathbf{Q}^W . Firstly, note that by definition of a and b ,

$$y^n \left(t + \frac{1}{n} \right) = \left[\mathbf{Q}_{t,t+1/n}^{n;W} \circ \mathbf{P}_{t,t+1/n}^{n;V} \right] (y^n(t)) \quad (11)$$

for t being a multiple of T/n . Secondly, it is intuitively clear, and an easy exercise to show, that $\mathbf{P}_{s,t}^{n;V} \equiv \mathbf{P}_{t-s}^V$ resp. $\mathbf{Q}_{s,t}^{n,W} \equiv \mathbf{Q}_{t-s}^W$ for s, t points on the dissection D^n . Since $(a(n^{-1}, \cdot), b(n^{-1}, \cdot))$ converges to the path $id_2 : t \mapsto (t, t)$ in the sense of (8) (that is, $\xi^n(t) = a(n^{-1}, t)$ and $z^n(t) = b(n^{-1}, t)$, $q = p = 1$), we know that

$$\pi_{V,W}(0, y_0; (a(n^{-1}, \cdot), b(n^{-1}, \cdot))) = y^n \rightarrow y = \pi_{V,W}(0, y_0; id_2) \quad \text{as } n \rightarrow \infty$$

in $|\cdot|_{\infty;[0,T]}$ norm. Using the identity (11) one recovers the "classic Lie-splitting"

$$\left[\mathbf{Q}_{1/n}^W \circ \mathbf{P}_{1/n}^V \right]^{[t/n]}(y_0) \rightarrow y_t \quad \text{as } n \rightarrow \infty \quad \text{for every } t \in [0, T]$$

where y is the ODE solution of (10). Moreover, the convergence holds in $|\cdot|_{\infty;[0,T]}$ norm and by interpolation even in stronger $(1 + \varepsilon)$ -variation norm for every $\varepsilon > 0$.

Note that no rough path theory is needed and everything follows from continuity in the sense of (8) (with $q = 1$) which can be established by elementary computations; see [FV09, Chapter 3]. Let us remark that using Young integration theory one can push this method to driving signals of finite p -variation for $p \lesssim 2$. Paths of Brownian regularity or worse are outside the scope of Young theory but one can use rough path results.

2.2 Splitting RDEs

Motivated by the above example we can ask for splitting for an RDE with drift of the form

$$dy_t = V(y_t) d\xi + W(y_t) dz_t, \quad y(0) = y_0 \in \mathbb{R}^e, \quad (12)$$

where $\mathbf{z} \in C^{p\text{-var}}([0, T], G^{[p]}(\mathbb{R}^d))$ and $\xi \in C^{1\text{-var}}([0, T], G^{[p]}(\mathbb{R}^d))$; the path (ξ, \mathbf{z}) has trivially "complementary Young regularity" and if $V \in Lip^\gamma(\mathbb{R}^e, \mathbb{R}^e)$ and $W = (W_i) \subset Lip^{\tilde{\gamma}}(\mathbb{R}^e, \mathbb{R}^e)$ for $\gamma > 1, \tilde{\gamma} > p$ then there exists a unique solution $\pi_{V,(W)}(0, y_0; (\xi, \mathbf{z}))$ to (12). For later use we show

Lemma 1 *Let $\mathbf{z} \in C^{p\text{-var}}([0, T], G^{[p]}(\mathbb{R}^d))$, $\xi \in C^{q\text{-var}}([0, T], \mathbb{R})$. If we define $\xi^\Delta(t) = \xi(a(\Delta, t)) \in C^{q\text{-var}}([0, T], \mathbb{R})$, $\mathbf{z}^\Delta(t) = \mathbf{z}(b(\Delta, t)) \in C^{p\text{-var}}([0, T], G^{[p]}(\mathbb{R}^d))$ then*

$$\begin{aligned} \sup_{\Delta > 0} \|\mathbf{z}^\Delta\|_{p\text{-var}} + \sup_{\Delta > 0} \|\xi^\Delta\|_{q\text{-var}} &< \infty \\ d_0(\mathbf{z}^\Delta, \mathbf{z}) + \|\xi^\Delta - \xi\|_{\infty; [0, T]} &\rightarrow 0 \text{ as } \Delta \rightarrow 0. \end{aligned}$$

Proof. First note that the variation norm is invariant under reparametrisation which implies the first statement. For the second statement it is sufficient to show pointwise convergence (by interpolation pointwise convergence in combination with uniform variation bounds implies convergence in supremum norm). However, \mathbf{z} and ξ are, by assumption, both continuous paths which gives pointwise convergence. ■

Define

$$\xi^n(t) := \xi(a(n^{-1}, t)) \text{ and } \mathbf{z}_t^n := \mathbf{z}(b(n^{-1}, t)).$$

Similar to the ODE example, define the solution operator $\{\mathbf{P}_{s,t}^{n;V}, 0 \leq s \leq t \leq T\}$ mapping points in \mathbb{R}^e to \mathbb{R}^e as $\mathbf{P}_{s,t}^{n;V}(x) := \pi_V(s, x; \xi^n)_t$ and the operators $\mathbf{P}^V, \mathbf{Q}^{n,W}$ and \mathbf{Q}^W . It remains to show that

$$\mathbf{P}_{s,t}^{n;V} \equiv \mathbf{P}_{s,t}^V \text{ and } \mathbf{Q}_{s,t}^{n;W} \equiv \mathbf{Q}_{s,t}^W \text{ for } s, t \in D^n$$

(in contrast with the ODE example, \mathbf{P}^V and \mathbf{Q}^W are now two parameter semigroups due to the time-inhomogeneity of ξ and \mathbf{z}). Since $G^{[p]}(\mathbb{R}^d)$ is a geodesic space, there exists a sequence of paths (concatenations of geodesics on the sequence of dissections D^m), $(z^m)_m \subset C^{1\text{-var}}([0, T], \mathbb{R}^d)$ with $S_{[p]}(z_{s,t}^m) = \mathbf{z}_{s,t}$ for $s, t \in D^m$, such that

$$\begin{aligned} \sup_m \|S_{[p]}(z^m)\|_{p\text{-var}} &< \infty, \\ d_0(S_{[p]}(z^m), \mathbf{z}) &\rightarrow 0 \text{ as } m \rightarrow \infty, \end{aligned}$$

with $S_{[p]}$ as in (9). Now define $\mathbf{z}^{n,m}(t) := \mathbf{z}^m(a(n^{-1}, t))$ and note that \mathbf{z}^m and $\mathbf{z}^{n,m}$ have bounded 1-variation (\mathbf{z}^m by construction, $\mathbf{z}^{n,m}$ because the variation norm is invariant under reparametrisation). Hence, for m, n fix we deal with an ODE as in the example above and therefore

$$\pi_W(s, x; \mathbf{z}^{n,m})_{s,t} = \pi_W(s, x; \mathbf{z}^m)_{s,t}$$

for $s, t \in D^n$. Keeping n fixed and letting $m \rightarrow \infty$, the LHS converges to $\pi_W(s, x; \mathbf{z}^n)_{s,t}$ by lemma (1) and Lyons' limit theorem and the RHS to $\pi_W(s, x; \mathbf{z})_{s,t}$; we can conclude $\mathbf{Q}_{s,t}^{n;W} \equiv \mathbf{Q}_{s,t}^W$ for $s, t \in D^n$. A similar argument shows $\mathbf{P}_{s,t}^{n;V} \equiv \mathbf{P}_{s,t}^V$ for $s, t \in D^n$. We finish the argument in the same way as in the previous example: solutions of $\pi_{V,(W)}(s, x; (\xi^n, \mathbf{z}^n))$ converge uniformly to $\pi_{V,(W)}(s, x; (\xi, \mathbf{z}))$. On neighbouring points $s, t \in D^n$, $\pi_{V,(W)}(s, x; (\xi^n, \mathbf{z}^n))_{s,t}$ can be identified as

$$\left[\mathbf{Q}_{s,t}^{n,W} \circ \mathbf{P}_{s,t}^{n,V} \right] (x) = \left[\mathbf{Q}_{s,t}^W \circ \mathbf{P}_{s,t}^V \right] (x).$$

Hence, for every $t \in [0, T]$

$$y_t^{n;\text{Split}} := \prod_{k=0}^{\lfloor t/n \rfloor - 1} \left[\mathbf{Q}_{k/n, (k+1)/n}^W \circ \mathbf{P}_{k/n, (k+1)/n}^V \right] (y_0) \rightarrow \pi_{V, (W)} (0, y_0; (\xi, \mathbf{z}))_t \text{ as } n \rightarrow \infty \text{ a.s.}$$

Moreover, $\sup_n \|y^{n;\text{Split}}\|_{p\text{-var}; [0, T]} < \infty$ which implies by interpolation convergence in $(p + \epsilon)$ -variation norm of $y^{n;\text{Split}}$ for every $\epsilon > 0$.

Remark 2 *Similarly, one shows convergence of a splitting scheme, running the semigroups in the different order $\mathbf{P}^V \circ \mathbf{Q}^W$. Note also, that we restrict ourselves in this article to Lie splitting schemes but the methods can be easily modified to include Strang splitting (see [MQ02] for the difference between Lie- and Strang splitting schemes) by using an appropriate modification of the time change. Further, we just deal with equidistant partitions. Numerous variations of all this are possible (as long as one can show convergence in a rough path topology of the approximating sequence) and such modifications are of great importance for rates of convergence - a topic which we hope to address in future work.*

3 Rough partial differential equations

One could hope for similar results for SPDEs in Stratonovich form,

$$\begin{aligned} du &= F(t, x, u, Du, D^2u) dt + \sum_{k=1}^d \Lambda_k(t, x, u, Du) \circ dB^k \\ u(0, x) &= u_0(x) \text{ on } \mathbb{R}^e \end{aligned} \quad (13)$$

Indeed, splitting for equation (13) has been treated in [KG03] for the case of F being a linear operator. To give meaning to (13) for nonlinear F one can introduce the concept of (rough) viscosity solutions (cf. [LS98a, LS98b, LS98b, LS00a, LS00b] and [CFO09] or [FO10]). Let us informally discuss the idea before we give the precise definition in section 4: a real-valued, bounded and continuous function u on $[0, T] \times \mathbb{R}^e$ is called a solution if it is the uniform limit (locally on compacts) of (standard) viscosity solutions (u^n) of the equations

$$\begin{aligned} du^n &= F(t, x, u^n, Du^n, D^2u^n) d\xi^n + \sum_{i=1}^d \Lambda_k(t, x, u, Du) dz^{n;i}, \\ u^n(0, x) &= u_0(x) \text{ on } (0, T] \times \mathbb{R}^e, \end{aligned}$$

where $(z^n) \subset C^\infty([0, T], \mathbb{R}^d)$ and $(\xi^n) \subset C^\infty([0, T], \mathbb{R})$ are sequences of smooth driving signals, converging to a weak geometric rough path (ξ, \mathbf{z}) (see (8)). Formally we write

$$du = F(t, x, u, Du, D^2u) d\xi + \Lambda(t, x, u, Du) dz.$$

(a solutions is then also a solution in the sense of [LS98b] when \mathbf{z} is an enhanced Brownian motion). In the case when the rough path \mathbf{z} is an enhanced Brownian motion and $\xi(t) = t$, this gives a natural concept of Stratonovich solutions to (13). However, to apply the methods outlined in the sections above to derive a splitting method, care has to be taken: firstly, in the RDE case “any

sequence" (ξ^n, z^n) of smooth paths converging to (ξ, \mathbf{z}) gives rise to a solution, but in the RPDE case, if $\dot{\xi}_t^n < 0$, one can not expect to treat even the simple second order equation

$$du^n = D^2 u^n d\xi^n \quad (14)$$

since it is no longer degenerate elliptic ($\dot{\xi}_t^n < 0$ amounts to running the heat equation backwards in time). Secondly, assume we want to use Lie-splitting on dyadic partitions, i.e. approximate $\xi(t) = t$ with $\xi^n(t) = a(n^{-1}, t)$. This introduces a discontinuous time-dependence ($\dot{\xi}^n$ does not exist on points of the partition)⁷ in equation (14). Such time-discontinuities are in general difficult to handle in a viscosity setting. Thirdly, one has to show continuous dependence of the solution⁸ of (13), not only on \mathbf{z} but continuous dependence on (ξ, \mathbf{z}) in a rough path sense.

The first point is dealt with by characterizing the class of legit approximations ξ^n , leading to the path space $C_0^{1-\text{var};+}([0, T], \mathbb{R})$, described in section 3.1. Section 3.2 deals with nonlinear PDEs with a discontinuity-in-time introduced by ξ^n and section 4 gives the precise definitions of rough viscosity solutions and stability, contains the main theorem and examples of stable RPDEs.

3.1 The space $C_0^{1-\text{var},+}([0, T], \mathbb{R})$

As pointed out above, we have to avoid to fall outside the scope of (degenerate) elliptic PDEs. Using the notation $C^{0,1-\text{var}}([0, T], \mathbb{R})$ for the closure of the space of smooth paths in variation norm ($\overline{C^\infty}^{|\cdot|_{1-\text{var}}}([0, T], \mathbb{R})$) we recall that

$$\begin{aligned} W_0^{1,1}([0, T], \mathbb{R}) &\equiv \left\{ x : [0, T] \rightarrow \mathbb{R}, \exists y \in L^1([0, T], \mathbb{R}) \text{ s.t. } x(t) = \int_0^t y(u) du \right\} \\ &= \{ x : [0, T] \rightarrow \mathbb{R}, x \text{ absolutely continuous, } x(0) = 0 \} \\ &= \overline{\{ x : [0, T] \rightarrow \mathbb{R}, x \in C^\infty, x(0) = 0 \}}^{|\cdot|_{1-\text{var}}} \\ &\equiv C_0^{0,1-\text{var}}([0, T], \mathbb{R}) \subsetneq C_0^{1-\text{var}}([0, T], \mathbb{R}). \end{aligned}$$

Definition 3 $C_0^{1,+}([0, T], \mathbb{R}) = \left\{ \xi \in C_0^1([0, T], \mathbb{R}) : \xi_T = T, \dot{\xi} > 0 \right\}$.

Note that the paths $a(\Delta, \cdot)$ and $b(\Delta, \cdot)$ are not elements of $C_0^{1,+}([0, T], \mathbb{R})$, but elements of its closure, $\overline{C_0^{1,+}}^{|\cdot|_\infty}([0, T], \mathbb{R})$, in sup-norm. Working with $C_0^{1,+}$ enables us in section below to give a short proof of existence, uniqueness and stability of a solution to PDEs of the type $\partial_t u = F\xi_t$ for paths $\xi \in \overline{C_0^{1,+}}^{|\cdot|_\infty}([0, T], \mathbb{R})$.

Proposition 4 Denote $C_0^{1-\text{var},+}([0, T], \mathbb{R}) = \overline{C_0^{1,+}}^{|\cdot|_\infty}([0, T], \mathbb{R})$. Then

$$\begin{aligned} C_0^{1-\text{var},+}([0, T], \mathbb{R}) &= \left\{ \xi_t \in C_0([0, T], \mathbb{R}) : \xi_T = T \text{ and } \exists \dot{\xi} \in L^1([0, T], \mathbb{R}_{\geq 0}), \right. \\ &\quad \left. \exists a \in C_0^{1-\text{var}}([0, T], \mathbb{R}_{\geq 0}), a \text{ increasing, } \dot{a} = 0 \text{ a.s. and } \xi_t = a_t + \int_0^t \dot{\xi}_u du \right\} \end{aligned}$$

⁷One could avoid discontinuous time-dependence by restricting the class of splitting schemes (i.e. the class with $(\dot{\xi}^n) \subset C^1$). However, nearly all popular schemes (Strang, Lie, etc.) would then not be covered.

⁸The results in [CFO09] and [FO10] do not cover this due the time-discontinuity of the approximating sequence $(d\xi^n)$.

and $C_0^{1-var,+}([0, T], \mathbb{R}) \subsetneq C_0^{1-var}([0, T], \mathbb{R})$.

Proof. \subset : Let (ξ^ε) be a Cauchy sequence wrt. $|\cdot|_\infty$. Since $(C_0([0, T], \mathbb{R}), |\cdot|_\infty)$ is complete, ξ^ε converges uniformly to some $\xi \in C_0([0, T], \mathbb{R})$. This ξ is monotone (not necessarily strict) increasing and hence $|\xi|_{1-var;[0, T]} < \infty$ (recall that $\xi_T = T$). Every function of finite 1-variation is Lebesgue-a.e. differentiable and has a representation of the form

$$\xi_t = a_t + \int_{[0, t]} \dot{\xi}_u du$$

where a is a function of 1-variation with $\dot{a} = 0$ Lebesgue-a.e. Now $\xi_{s, s+h} \geq 0$, for every $h > 0$; $s \in [0, 1)$. Hence we have $a_{s, s+h} \geq -\int_s^{s+h} \dot{\xi}_u du$ and sending $h \rightarrow 0$ shows together with $\dot{a} = 0$ a.s. that a is monotone increasing and this implies $\dot{\xi}_u \geq 0$ Lebesgue-a.e.

\supset : $F(t) := \xi_t$ defines a continuous distribution function on $[0, T]$ and let X be a random variable with distribution F . For $\varepsilon > 0$ denote by F^ε the distribution function of the random variable $X + \varepsilon N$ where N is a standard normal, independent of X . Clearly, $X + \varepsilon N \rightarrow X$ a.s. as $\varepsilon \rightarrow 0$ and so the F^ε converge pointwise. By the lemma below this implies uniform convergence of F^ε to F . It remains to show that $\xi_t^\varepsilon := F^\varepsilon(t)$ is C^1 but this follows from

$$F^\varepsilon(t) = \int_0^t F(t-u) dF_{\varepsilon N}(u)$$

where $F_{\varepsilon N}$ is the distribution function of εN . ■

Lemma 5 *Let $(f^\eta)_{\eta>0} \subset C_0([0, T], \mathbb{R})$, $f^\eta(1) = 1$, each f^η increasing (not necessarily strictly) and assume $f^\eta \rightarrow f \in C_0([0, T], \mathbb{R})$ pointwise as $\eta \rightarrow 0$. Then, $|f^\eta - f|_{\infty;[0, T]} \rightarrow 0$ as $\eta \rightarrow 0$.*

Proof. Given $\varepsilon > 0$ we can choose a $n \in \mathbb{N}$ big enough s.t.

$$\left| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right) \right| < \frac{\varepsilon}{2}$$

for every $i \in \{0, 1, \dots, n\}$. Now choose η small enough such that

$$\left| f^\eta\left(\frac{i}{n}\right) - f\left(\frac{i}{n}\right) \right| < \frac{\varepsilon}{2}$$

for all $i \in \{0, 1, \dots, n\}$. This implies $|f^\eta(x) - f(x)| < \varepsilon$ since every x is an element of (at least one) interval $[\frac{i-1}{n}, \frac{i}{n}]$ and by monotonicity and using above estimates

$$f^\eta(x) \leq f^\eta\left(\frac{i}{n}\right) \leq f\left(\frac{i-1}{n}\right) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq f(x) + \varepsilon.$$

Similarly

$$f^\eta(x) > f(x) - \varepsilon$$

and so $|f^\eta(x) - f(x)| < \varepsilon$ for all $x \in [0, T]$. ■

Remark 6 *Concerning the choice of notation $C_0^{1-var,+}$, note that the space of paths of finite 1-variation C_0^{1-var} is given as the closure of C^1 -paths with uniformly bounded 1-variation. Since paths in $C_0^{1,+}$ have 1-variation bounded by T , the notation $C_0^{1-var,+}([0, T], \mathbb{R})$ seems natural.*

Remark 7 The paths $a(\Delta, \cdot)$ and $b(\Delta, \cdot)$ converge in $(1 + \varepsilon)$ -variation to the path $id_1 : t \mapsto t$ for every $\varepsilon > 0$ and therefore also uniformly (but not in 1-variation!).

Remark 8 $C_0^{1-var,+}([0, T], \mathbb{R})$ is not a linear space but a convex subset of $C_0^{1-var}([0, T], \mathbb{R})$.

Remark 9 Despite the restriction of $C_0^{1-var,+}([0, T], \mathbb{R})$ to paths with $\xi(T) = T$ which is convenient in the proofs, one can handle PDEs with general increasing processes by rescaling; e.g. replace ξ by $\tilde{\xi}(t) := \xi(t) \frac{T}{\xi(T)} \in C_0^{1-var,+}([0, T], \mathbb{R})$ and write $du = F(t, x, u, Du, D^2u) d\xi = \tilde{F}(t, x, u, Du, D^2u) d\tilde{\xi}$ with $\tilde{F} := F \frac{\xi(T)}{T}$.

3.2 PDEs with discontinuous time-dependence

This section extends the notion of viscosity solutions to equations of the form

$$du = F(t, x, u, Du, D^2u) d\xi(t), \quad u(0, x) = u_0(x),$$

with F a continuous function and $\xi \in C_0^{1-var,+}([0, T], \mathbb{R})$. In the appendix we show that this solution concept coincides with the notion of generalized viscosity solutions (going back to [Ish85]) whenever the latter exists. In view of applications in sections section 4 and 5 and to keep technicalities down, we focus on time-independent F . A proof for time-dependent F is given in the appendix.

Proposition 10 Let $(\xi^\varepsilon)_\varepsilon \subset C_0^{1,+}([0, T], \mathbb{R})$ converge uniformly to some $\xi \in C_0^{1-var,+}([0, T], \mathbb{R})$ as $\varepsilon \rightarrow 0$. Assume $(v^\varepsilon)_\varepsilon \subset BUC([0, T] \times \mathbb{R}^n, \mathbb{R})$ are locally uniformly bounded viscosity solutions of

$$\partial_t v^\varepsilon = F^\varepsilon(x, v^\varepsilon, Dv^\varepsilon, D^2v^\varepsilon) \dot{\xi}_t^\varepsilon, \quad v^\varepsilon(0, x) = v_0(x). \quad (15)$$

with $F^\varepsilon : \mathbb{R}^e \times \mathbb{R} \times \mathbb{R}^e \times \mathbb{S}^e \rightarrow \mathbb{R}$, \mathbb{S}^e denoting the space of symmetric $(e \times e)$ -matrices, a continuous and degenerate elliptic function. Further, assume that F^ε converges locally uniformly to a continuous, degenerate elliptic function F and that a comparison result holds for $\partial_t - F^\varepsilon = 0$ and $\partial_t - F = 0$. Then there exists a v such that

$$v^\varepsilon \rightarrow v \text{ locally uniformly as } \varepsilon \rightarrow 0.$$

Further, v does not depend on the choice of the sequence approximating ξ and we also write $v \equiv v^\xi$ to emphasize the dependence on ξ and say that v solves

$$dv = F(x, v, Dv, D^2v) d\xi_t, \quad v(0, x) = v_0(x).$$

Prepare the proof with

Lemma 11 Let $\xi \in C_0^{1,+}([0, T], \mathbb{R})$,

$$F : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \rightarrow \mathbb{R}$$

and let $\tilde{F}(t, x, r, p, X) = F(\xi^{-1}(t), r, x, p, X)$ for $(t, x, r, p, X) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$. Then

1. if $u \in BUC([0, T] \times \mathbb{R}^n)$ is a sub- (resp. super) solution of $\partial_t - F\dot{\xi} = 0$, $u(0, \cdot) = u_0(\cdot)$ then $w(t, x) := u(\xi_t^{-1}, x)$ is a sub- (resp. super) solution of $\partial_t - \tilde{F} = 0$, $w(0, \cdot) = u_0(\cdot)$.

2. if $w \in BUC([0, T] \times \mathbb{R}^n)$ is a sub- (resp. super) solution of $\partial_t - \tilde{F} = 0$, $w(0, \cdot) = w_0(\cdot)$ then $u(t, x) := w(\xi_t, x)$ is a sub- (resp. super) solution of $\partial_t - F\dot{\xi}$, $u(0, \cdot) = w_0(\cdot)$.

Proof. 1. Let $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^n)$ and assume $w(t, x) - \varphi(t, x)$ attains a local maximum at $(\hat{t}, \hat{x}) \in [0, T] \times \mathbb{R}^n$. Then

$$w(\hat{t}, \hat{x}) - \varphi(\hat{t}, \hat{x}) = u(\xi^{-1}(\hat{t}), \hat{x}) - \varphi(\hat{t}, \hat{x}) = u(\xi^{-1}(\hat{t}), \hat{x}) - \tilde{\varphi}(\xi^{-1}(\hat{t}), \hat{x})$$

where $\tilde{\varphi}(\hat{t}, \hat{x}) := \varphi(\xi_{\hat{t}}, \hat{x})$. Using that u is a subsolution gives

$$\begin{aligned} \partial_t \tilde{\varphi}|_{\xi_{\hat{t}}^{-1}, \hat{x}} &\leq F(\hat{x}, \xi^{-1}(\hat{t}), u(\xi^{-1}(\hat{t}), \hat{x}), D\tilde{\varphi}|_{\xi_{\hat{t}}^{-1}, \hat{x}}, D^2\tilde{\varphi}|_{\xi_{\hat{t}}^{-1}, \hat{x}}) \dot{\xi}_{\xi_{\hat{t}}^{-1}} \\ &= F(\hat{x}, \xi^{-1}(\hat{t}), w(\hat{t}, \hat{x}), D\varphi|_{\hat{t}, \hat{x}}, D^2\varphi|_{\hat{t}, \hat{x}}) \dot{\xi}_{\xi_{\hat{t}}^{-1}} \\ &= \tilde{F}(\hat{x}, \hat{t}, w(\hat{t}, \hat{x}), D\varphi|_{\hat{t}, \hat{x}}, D^2\varphi|_{\hat{t}, \hat{x}}) \dot{\xi}_{\xi_{\hat{t}}^{-1}} \end{aligned}$$

where we used that $D\tilde{\varphi}|_{\xi_{\hat{t}}^{-1}, \hat{x}} = D\varphi|_{\hat{t}, \hat{x}}$ and $D^2\tilde{\varphi}|_{\xi_{\hat{t}}^{-1}, \hat{x}} = D^2\varphi|_{\hat{t}, \hat{x}}$. Since $\partial_t \tilde{\varphi}|_{\xi_{\hat{t}}^{-1}, \hat{x}} = \partial_t \varphi|_{\hat{t}, \hat{x}} \dot{\xi}|_{\xi_{\hat{t}}^{-1}}$ and $\dot{\xi} > 0$ it follows that

$$\partial_t \varphi|_{\hat{t}, \hat{x}} \leq \tilde{F}(\hat{x}, \hat{t}, w(\hat{t}, \hat{x}), D\varphi|_{\hat{t}, \hat{x}}, D^2\varphi|_{\hat{t}, \hat{x}}).$$

Now the same argument as above when u is a supersolution.

2. Let $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^n)$ and assume $u(t, x) - \varphi(t, x)$ attains a local maximum at $(\hat{t}, \hat{x}) \in [0, T] \times \mathbb{R}^n$. Then

$$u(\hat{t}, \hat{x}) - \varphi(\hat{t}, \hat{x}) = w(\xi(\hat{t}), \hat{x}) - \varphi(\hat{t}, \hat{x}) = w(\xi(\hat{t}), \hat{x}) - \tilde{\varphi}(\xi(\hat{t}), \hat{x})$$

where $\tilde{\varphi}(t, x) := \varphi(\xi^{-1}(t), x)$. Using that w is a subsolution gives

$$\begin{aligned} \partial_t \tilde{\varphi}|_{\xi_{\hat{t}}, \hat{x}} &\leq \tilde{F}(\hat{x}, \xi(\hat{t}), w(\xi(\hat{t}), \hat{x}), D\tilde{\varphi}|_{\xi_{\hat{t}}, \hat{x}}, D^2\tilde{\varphi}|_{\xi_{\hat{t}}, \hat{x}}) \\ &= F(\hat{x}, \hat{t}, u(\hat{t}, \hat{x}), D\varphi|_{\hat{t}, \hat{x}}, D^2\varphi|_{\hat{t}, \hat{x}}) \end{aligned}$$

where we used that $D\tilde{\varphi}|_{\xi_{\hat{t}}, \hat{x}} = D\varphi|_{\hat{t}, \hat{x}}$ and $D^2\tilde{\varphi}|_{\xi_{\hat{t}}, \hat{x}} = D^2\varphi|_{\hat{t}, \hat{x}}$. Since $\dot{\xi} > 0$ and $\partial_t \tilde{\varphi}|_{\xi_{\hat{t}}, \hat{x}} = \partial_t \varphi|_{\hat{t}, \hat{x}} (\xi^{-1})'|_{\xi_{\hat{t}}} = \partial_t \varphi|_{\hat{t}, \hat{x}} (\dot{\xi}(\hat{t}))^{-1}$ it follows that

$$\partial_t \varphi|_{\hat{t}, \hat{x}} \leq F(\hat{x}, \hat{t}, u(\hat{t}, \hat{x}), D\varphi|_{\hat{t}, \hat{x}}, D^2\varphi|_{\hat{t}, \hat{x}}) \dot{\xi}(\hat{t}).$$

Now the same argument as above when w is a supersolution. ■

Proof of Proposition 10. Set $w^\varepsilon(t, x) := v^\varepsilon((\xi^\varepsilon)^{-1}(t), x)$, by lemma 11,

$$v^\varepsilon \text{ is a solution of } \partial_t - F^\varepsilon \dot{\xi}^\varepsilon = 0 \text{ iff } w^\varepsilon \text{ is a solution of } \partial_t - F^\varepsilon = 0.$$

Let

$$\overline{w} := \limsup_\varepsilon * w^\varepsilon \text{ and } \underline{w} := \liminf_\varepsilon * w^\varepsilon$$

and note that

$$F^\varepsilon(x, r, p, X) \rightarrow F(x, r, p, X) \text{ locally uniformly.}$$

Standard viscosity theory tells us that \bar{w} and \underline{w} are sub- resp. supersolutions of $\partial_t - F = 0$. Using the method of semi-relaxed limits (by definition, $\bar{w} \geq \underline{w}$ and the reversed inequality follows from comparison which holds by assumption) conclude that $w(t, x) := \bar{w}(t, x) = \underline{w}(t, x)$. Further, using a Dini-type argument, for every compact set $K \subset \mathbb{R}^n$,

$$|w^\varepsilon - w|_{\infty; [0, T] \times K} = \sup_{t \in [0, T], x \in K} |w^\varepsilon(t, x) - w(t, x)| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Now define

$$v(t, x) := w(\xi(t), x).$$

and we get the claimed convergence $v^\varepsilon \rightarrow v$. ■

4 Splitting RPDEs

4.1 Rough viscosity solutions and stability

Solutions of RDEs can be defined as limit points of ODEs. Similarly one can define solutions of rough partial differential equations as limit points of PDE solutions.

Definition 12 *Let $\mathbf{z} \in C_0^{0, p\text{-var}}([0, T], G^{[p]}(\mathbb{R}^d))$, $\xi \in C_0^{1\text{-var}; +}([0, T], \mathbb{R})$ and denote $(\xi, \mathbf{z}) \in C_0^{0, p\text{-var}}([0, T], G^{[p]}(\mathbb{R}^{d+1}))$ the Young pairing (given canonically via Young integration). Further, let $(\xi^\varepsilon, z^\varepsilon) \in C_0^{1, +}([0, T], \mathbb{R}) \times C_0^{1\text{-var}}([0, T], \mathbb{R}^d)$, converge to (ξ, \mathbf{z}) in the sense of (1) and assume the PDE*

$$\begin{aligned} du^\varepsilon &= F(x, u^\varepsilon, Du^\varepsilon, D^2u^\varepsilon) d\xi^\varepsilon + \sum_{k=1}^d \Lambda_k(t, x, u^\varepsilon, Du^\varepsilon) dz^{\varepsilon; k} \text{ on } (0, T] \times \mathbb{R}^n, \\ u(0, x) &= u_0(x) \in BUC(\mathbb{R}^e, \mathbb{R}) \end{aligned}$$

has a unique solution $u^\varepsilon \in BUC([0, T] \times \mathbb{R}^e; \mathbb{R})$ for every $\varepsilon > 0$. We call every limit point u of (u^ε) (in BUC topology) a solution of the RPDE

$$du = F(x, u, Du, D^2u) d\xi + \Lambda(t, x, u, Du) d\mathbf{z} \text{ on } (0, T] \times \mathbb{R}^n, \quad u(0, x) = u_0(x). \quad (16)$$

If additionally, the limit is unique, does not depend on the choice of the approximating sequence $(\xi^\varepsilon, z^\varepsilon)$ and the map

$$(\xi, \mathbf{z}) \in C_0^{1\text{-var}; +}([0, T], \mathbb{R}) \times C_0^{0, p\text{-var}}([0, T], G^{[p]}(\mathbb{R}^d)) \mapsto u \in BUC([0, T] \times \mathbb{R}^e, \mathbb{R})$$

is continuous then we say that the RPDE (16) is stable in a rough path sense and we also write $u = u^{\xi, \mathbf{z}}$ (or $u = u^{\mathbf{z}}$ when $\xi(t) = t$) to emphasize dependence on the rough path (ξ, \mathbf{z}) .

4.2 The main theorem

We are now able to formulate our main theorem. The proof is an easy consequence of the results in the previous sections. In section 4 we show that the assumptions are satisfied for a large class of RPDEs.

Theorem 13 Let $\mathbf{z} \in C_0^{0,p\text{-var}}([0, T], G^{[p]}(\mathbb{R}^d))$, $\xi \in C_0^{+,1\text{-var}}([0, T], \mathbb{R})$ and assume $u \in BUC$ is the unique solution of the stable (in the sense of definition 12) RPDE,

$$du = F(x, u, Du, D^2u) d\xi + \Lambda(t, x, u, Du) dz \text{ on } (0, T] \times \mathbb{R}^e, u(0, x) = u_0(x) \in BUC. \quad (17)$$

Assume further that also the two (R)PDEs given by setting either $F \equiv 0$ or $\Lambda \equiv 0$ in (17) are stable. Denote $\{\mathbf{P}_{s,t}, 0 \leq u \leq T\}$ and $\{\mathbf{Q}_{s,t}, 0 \leq s \leq t \leq T\}$ the solution operators

$$\begin{aligned} \mathbf{P}_{s,t} &: BUC(\mathbb{R}^e, \mathbb{R}) \rightarrow BUC([0, T] \times \mathbb{R}^e, \mathbb{R}), \varphi \mapsto v, \\ \mathbf{Q}_{s,t} &: BUC(\mathbb{R}^e, \mathbb{R}) \rightarrow BUC([0, T] \times \mathbb{R}^e, \mathbb{R}), \phi \mapsto w, \end{aligned}$$

with

$$\begin{aligned} dv &= F(x, v, Dv, D^2v) d\xi, v(0, x) = \varphi(x), \\ dw &= \Lambda(t, x, w, Dw) dz, w(s, x) = \phi(x). \end{aligned}$$

and set

$$u^{n;\text{Split}}(t, x) := \prod_{i=0}^{\lfloor tn^{-1} \rfloor - 1} [\mathbf{Q}_{in^{-1}, (i+1)n^{-1}} \circ \mathbf{P}_{in^{-1}, (i+1)n^{-1}}](u_0(x)).$$

Then

$$u^{n;\text{Split}} \rightarrow u \text{ locally uniformly as } n \rightarrow \infty.$$

Proof. Define $\mathbf{z}^n = \mathbf{z}(b(n^{-1}, t))$, $\xi^n(t) = \xi(a(n^{-1}, t))$. By lemma 1, $(\xi^n, \mathbf{z}^n) \rightarrow_n (\xi, \mathbf{z})$ in the sense of (1) and by stability, the solutions u^n of

$$du^n = F(x, u^n, Du^n, D^2u^n) d\xi^n + \Lambda(t, x, u^n, Du^n) dz^n \text{ on } (0, T] \times \mathbb{R}^n, u(0, x) = u_0(x)$$

converge to u , the solution of (16). Now for each given n one can identify on points of the dissection $\{k \frac{T}{n}, i = 0, \dots, n\}$ the solutions of $u^{n;\text{Split}}$ with u^n and by assumptions u^n converges locally uniformly to u . ■

4.3 Examples of stable RPDEs

This section shows stability in a rough path sense for a large class of RPDEs. Splitting results then follow readily by theorem 13. Throughout this section \mathbf{z} is a geometric p -rough path, i.e. $\mathbf{z} \in C_0^{0,p\text{-var}}([0, T], G^p(\mathbb{R}^d))$, $p \geq 1$.

Proposition 14 Let

$$\begin{aligned} L(x, r, p, X) &= \text{Tr} \left[A(x)^T X \right] + b(x) \cdot p + f(x, r) \\ \Lambda_k(t, x, r, p) &= (p \cdot \sigma_k(t, x)) + r \nu_k(t, x) + g_k(t, x) \end{aligned}$$

with $A(x) = \bar{\sigma}(x) \bar{\sigma}^T(x) \in \mathbb{S}^e$ and $\bar{\sigma} : \mathbb{R}^e \rightarrow \mathbb{R}^{e \times e'}$, $b(x) : \mathbb{R}^e \rightarrow \mathbb{R}^e$ bounded, Lipschitz continuous in x . Also assume that $f : \mathbb{R}^e \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, bounded whenever r remains bounded, and with a lower Lipschitz bound, i.e. $\exists C < 0$ s.t.

$$f(x, r) - f(x, s) \geq C(r - s) \text{ for all } r \geq s, x \in \mathbb{R}^e.$$

and that the coefficients of $\Lambda = (\Lambda_1, \dots, \Lambda_d)$, that is σ, ν and g , have Lip^γ -regularity for $\gamma > p + 2$. Then the RPDE

$$du = L(x, u, Du, D^2u) dt + \Lambda(t, x, u, Du) dz, \quad u(0, \cdot) \equiv u_0 \quad (18)$$

is stable in a rough path sense and has a unique solution $u^z \in BUC([0, T] \times \mathbb{R}^e, \mathbb{R})$.

Proof. We use the same technique of “rough semi-relaxed limits” as in [FO10]; for details of the transform of u^ε to \tilde{v}^ε we refer to [FO10]: the key remark being that u^ε is a solution of

$$du^\varepsilon = L(x, u^\varepsilon, Du^\varepsilon, D^2u^\varepsilon) d\xi^\varepsilon + \Lambda(t, x, u^\varepsilon, Du^\varepsilon) dz^\varepsilon, \quad u(0, \cdot) \equiv u_0(\cdot) \quad (19)$$

iff

$$\tilde{v}^\varepsilon(t, x) := (\phi^\varepsilon)^{-1}(t, u^\varepsilon(t, \psi^\varepsilon(t, x)), x) + \alpha^\varepsilon(t, x);$$

is a solution of

$$d\tilde{v}^\varepsilon = \tilde{L}^\varepsilon(x, \tilde{v}^\varepsilon, D\tilde{v}^\varepsilon, D^2\tilde{v}^\varepsilon) d\xi^\varepsilon, \quad \tilde{v}^\varepsilon(0, \cdot) \equiv u_0(\cdot).$$

Here \tilde{L}^ε is a linear operator with coefficients determined by the characteristics of the PDE $\partial w = \Lambda(t, x, w^\varepsilon, Dw^\varepsilon) dz^\varepsilon$ and $\psi^\varepsilon, \phi^\varepsilon, \alpha^\varepsilon$ are ODE flows converging to RDE flows ψ^z, ϕ^z, α^z which depend on z (but not the approximating sequence of (z^ε)). Further a comparison principle applies to $\partial_t - \tilde{L}^\varepsilon = 0$. The assumptions of proposition 10 are then fulfilled, $\tilde{L}^\varepsilon \rightarrow \tilde{L}$ locally uniformly and using the method of semirelaxed limits,

$$\tilde{v}^\varepsilon \rightarrow \tilde{v} \text{ locally uniformly.}$$

Unwrapping the transformation, that is, setting

$$u^z(t, x) := \phi^z\left(t, \tilde{v}\left(t, (\psi^z)^{-1}(t, x)\right) - \alpha^z\left(t, (\psi^z)^{-1}(t, x)\right)\right), \quad (20)$$

finishes the proof since stability of the RPDE follows directly from the representation (20). ■

Proposition 15 *Let*

$$F(x, r, p, X) = \inf_{\alpha \in \mathcal{A}} \left\{ \text{Tr} \left[A(x; \alpha)^T X \right] + b(x; \alpha) \cdot p + f(x, r; \alpha) \right\}$$

where A, σ and b, f satisfy the assumption of proposition 14 uniformly with respect to $\alpha \in \mathcal{A}$ and $\nu = (\nu_k)_{k=1}^d \subset Lip^\gamma(\mathbb{R}^n, \mathbb{R}^n)$ $\gamma > p + 2$. Then the RPDE

$$du = F(x, u, Du, D^2u) dt + Du \cdot \nu(x) dz, \quad u(0, \cdot) \equiv u_0$$

has a unique solution $u^z \in BUC([0, T] \times \mathbb{R}^n)$ and is stable in a rough path sense.

Proof. Similar to the proof above. ■

5 Applications to stochastic PDEs

The typical applications to SPDEs are path-by-path, i.e. by taking \mathbf{z} to be a realization of a continuous semi-martingale Y and its stochastic area, say $\mathbf{Y}(\omega) = (Y, A)$; the most prominent example being Brownian motion and Lévy's area. Taking the linear case as an example, the stability result of proposition 14 allows to identify

$$du = L(t, x, u, Du, D^2u) dt + \Lambda(t, x, u, Du) d\mathbf{z}, \quad u(0, \cdot) \equiv u_0$$

with $\mathbf{z} = \mathbf{Y}(\omega)$ as *Stratonovich solution* to the SPDE

$$du = L(t, x, u, Du, D^2u) dt + \Lambda(t, x, u, Du) \circ dY, \quad u(0, \cdot) = u_0.$$

Indeed, under the stated assumptions, the Wong-Zakai approximations, in which Y is replaced by its piecewise linear approximation, based on some mesh $\{0, \frac{T}{n}, \frac{2T}{n} \dots, T\}$, the approximate solution will converge (locally uniformly on $[0, T] \times \mathbb{R}^n$ and in probability, say) to the solution of

$$du = L(t, x, u, Du, D^2u) dt + \Lambda(t, x, u, Du) d\mathbf{Y}, \quad u(0, \cdot) = u_0,$$

as constructed in proposition 14. In view of well-known Wong-Zakai approximation results for SPDEs, ranging from [BF95, Twa95] to [GM04, GS06], the rough PDE solution is then identified as Stratonovich solution. (At least for L uniformly elliptic: the (Stratonovich) integral interpretations can break down in degenerate situations; as example, consider non-differentiable initial data u_0 and the (one-dimensional) random transport equation $du = u_x \circ dB$ with explicit "Stratonovich" solution $u_0(x + B_t)$. A similar situation occurs for the classical transport equation $\dot{u} = u_x$, of course.) Motivated by this, if $u^{\mathbf{z}}$ is a RPDE solution of

$$du = F(t, x, u, Du, D^2u) dt + \Lambda(t, x, u, Du) d\mathbf{z}, \quad u(0, \cdot) = u_0,$$

then we call $u^{\mathbf{z}}$ with $\mathbf{z} = \mathbf{Y}(\omega)$ as *Stratonovich solution* and write

$$du = F(t, x, u, Du, D^2u) dt + \Lambda(t, x, u, Du) \circ dY, \quad u(0, \cdot) = u_0.$$

The following example was suggested in [LS98c] and carefully worked out in [BM02, BM07].

Example 16 (Pathwise stochastic control) *Consider*

$$dX = b(X; \alpha) dt + W(X; \alpha) \circ d\tilde{B} + V(X) \circ dB,$$

where b, W, V are (collections of) sufficiently nice vector fields (with b, W dependent on a suitable control $\alpha = \alpha(t) \in \mathcal{A}$, applied at time t) and \tilde{B}, B are multi-dimensional (independent) Brownian motions. Define⁹

$$v(x, t; B) = \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[\left(g(X_T^{x,t}) + \int_t^T f(X_s^{x,t}, \alpha_s) ds \right) \middle| B \right]$$

⁹Remark that any optimal control $\alpha(\cdot)$ here will depend on knowledge of the entire path of B . Such anticipative control problems and their link to classical stochastic control problems were discussed early on by Davis and Burnstein [DB92].

where $X^{x,t}$ denotes the solution process to the above SDE started at $X(t) = x$. Then, at least by a formal computation,

$$dv + \inf_{\alpha \in \mathcal{A}} [b(x, \alpha) Dv + L_\alpha v + f(x, \alpha)] dt + Dv \cdot \nu(x) \circ dB = 0$$

with terminal data $v(\cdot, T) \equiv g$, and $L_\alpha = \sum W_i^2$ in Hörmander form. Setting $u(x, t) = v(x, T - t)$ turns this into the initial value (Cauchy) problem,

$$du = \inf_{\alpha \in \mathcal{A}} [b(x, \alpha) Du + L_\alpha u + f(x, \alpha)] dt + Du \cdot V(x) \circ dB_{T-}.$$

with initial data $u(\cdot, 0) \equiv g$; and hence of a form which is covered by theorem 15. (Moreover, the rough driving signal in proposition 15 is taken as $\mathbf{z}_t := \mathbf{B}_{T-t}(\omega)$ where $\mathbf{B}(\omega)$ is a fixed Brownian motion).

Using theorem 13 we immediately get a splitting result:

Example 17 (Splitting HJB-equations) Let B be standard d -dimensional Brownian motion. Then the SPDE

$$\begin{aligned} du &= \inf_{\alpha \in \mathcal{A}} \left\{ \text{Tr} \left[\sigma(x; \alpha) \sigma(x; \alpha)^T D^2 u \right] + b(x; \alpha) \cdot Du + f(x; \alpha) \right\} dt + (Du \cdot \nu(x)) \circ dB, \\ u(\omega; 0, x) &= u_0(x), \end{aligned} \quad (21)$$

has a unique solution u if $\sigma(x, \alpha) : \mathbb{R}^e \times \mathcal{A} \rightarrow \mathbb{R}^{e \times e'}$ and $b(x, \alpha) : \mathbb{R}^e \times \mathcal{A} \rightarrow \mathbb{R}^e$ are Lipschitz continuous in x , uniformly in $\alpha \in \mathcal{A}$, $\nu = (\nu_1, \dots, \nu_d) \subset \text{Lip}^\gamma(\mathbb{R}^e; \mathbb{R}^e)$ with $\gamma > 4$. Denote $\{\mathbf{P}_u, u \geq 0\}$ the solution operator¹⁰ of

$$du = \inf_{\alpha \in \mathcal{A}} \left\{ \text{Tr} \left[\sigma(x; \alpha) \sigma(x; \alpha)^T D^2 u \right] + b(x; \alpha) \cdot Du + f(x; \alpha) \right\} dt, \quad (22)$$

i.e. $\mathbf{P}_t u_0(\cdot) = u(t, \cdot)$, and $\{\mathbf{Q}_{s,t}, 0 \leq s \leq t \leq T\}$ the solution operator given by $\mathbf{Q}_{s,t} \varphi(\cdot) = \varphi(\pi_{-V}(s, \cdot; B)_t)$, with $\pi_{-V}(s, x; B)_t$ the SDE solution of

$$dy = -V(y) \circ dB, \quad y_s = x \in \mathbb{R}^e. \quad (23)$$

Then¹¹,

$$u^{n; \text{Split}}(t, x) := \prod_{i=0}^{\lfloor t/n \rfloor - 1} [\mathbf{Q}_{i/n, i/n+1/n} \circ \mathbf{P}_{1/n}](u_0(x)) \rightarrow u(t, x) \text{ as } n \rightarrow \infty$$

and the convergence also holds locally uniformly.

Thus, equation (21) can be approximated by solutions of a standard HJB equation (21) and by solutions of the RDE (23) (for numerical schemes for HJB, see [FS06a]).

¹⁰Note that in the case $\xi = t$ one can use a one-parameter semigroup since $\xi^n = 2$ on the time interval on which the approximation evolve.

¹¹A priori the leftmost two terms would have to be $\mathbf{Q}_{\lfloor t/n \rfloor, t} \circ \mathbf{P}_{t - \lfloor t/n \rfloor}$. However, the claimed convergence follows from Lyons' limit theorem.

Example 18 (Linear SPDEs, Filtering) Let L and Λ be as in proposition 14. Then there exists a unique solution to

$$du = L(x, u, Du, D^2u) dt + \sum_{k=1}^d \Lambda_k(t, x, u, Du) \circ dB^k, u(0, \cdot) = u_0(\cdot)$$

Denote by $\{\mathbf{P}_u, 0 \leq u \leq T\}$ the solution operator

$$\varphi \mapsto v \text{ with } v \text{ solution of } \partial v = L(x, v, Dv, D^2v) dt, v(0, \cdot) = \varphi(\cdot)$$

and by $\{\mathbf{Q}_{s,t}, 0 \leq s \leq t\}$ the solution operator

$$\varphi \mapsto y \text{ with } y \text{ solution of } dy = \Lambda(t, x, u, Du) \circ dB, v(0, \cdot) = \varphi(\cdot)$$

Then,

$$u^{n; Split}(t, x) := \prod_{i=0}^{\lfloor t/n \rfloor - 1} [\mathbf{Q}_{i/n, i/n+1/n} \circ \mathbf{P}_{1/n}](u_0(x)) \rightarrow u(t, x) \text{ as } n \rightarrow \infty$$

and the convergence also holds locally uniformly.

A Appendix: Time-dependent F

To deal with time-dependent F we need the additional assumption of uniform bounds on the derivatives of the approximating sequence ξ^ε .

Proposition 19 Let $(\xi^\varepsilon)_\varepsilon \subset C_0^{1,+}([0, T], \mathbb{R})$, $\sup_\varepsilon \left| \dot{\xi}^\varepsilon \right|_{\infty; [0, T]} < \infty$, converge uniformly to some $\xi \in C_0^{1-var;+}([0, T], \mathbb{R})$ as $\varepsilon \rightarrow 0$. Assume $(v^\varepsilon)_\varepsilon \subset BUC([0, T] \times \mathbb{R}^e, \mathbb{R})$ are locally uniformly bounded viscosity solutions of

$$\partial_t v^\varepsilon = F^\varepsilon(t, x, v^\varepsilon, Dv^\varepsilon, D^2v^\varepsilon) \dot{\xi}_t^\varepsilon, v^\varepsilon(0, x) = v_0(x).$$

with $F^\varepsilon : [0, T] \times \mathbb{R}^e \times \mathbb{R} \times \mathbb{R}^e \times \mathbb{S}^e \rightarrow \mathbb{R}$ a continuous and degenerate elliptic function. Further, assume that F^ε converges locally uniformly to a continuous, degenerate elliptic function F and that a comparison result holds for $\partial_t - F^\varepsilon = 0$ and $\partial_t - F = 0$. Then there exists a v such that

$$v^\varepsilon \rightarrow v \text{ locally uniformly as } \varepsilon \rightarrow 0.$$

Further, v does not depend on the choice of the sequence approximating ξ and we also write $v \equiv v^\xi$ to emphasize the dependence on ξ and say that v solves

$$dv = F(t, x, v, Dv, D^2v) d\xi_t, v(0, x) = v_0(x).$$

Proof. Set $w^\varepsilon(t, x) := v^\varepsilon((\xi^\varepsilon)^{-1}(t), x)$ and divide $[0, T]$ into intervals on which ξ is strictly increasing resp. constant, i.e. $0 = s_1 \leq t_1 \leq s_2 \leq \dots \leq t_n = T$, ξ strictly increasing on $[s_i, t_i]$, constant on $[t_i, s_{i+1}]$. By lemma 11 on intervals $[s_i, t_i]$

$$v^\varepsilon \text{ is a solution of } \partial_t - F\dot{\xi}^\varepsilon = 0 \text{ iff } w^\varepsilon \text{ is a solution of } \partial_t - \tilde{F}^\varepsilon = 0$$

where $\tilde{F}^\varepsilon(t, x, r, p, X) = F((\xi^\varepsilon)^{-1}(t), x, r, p, X)$. Let

$$\bar{w} := \limsup_\varepsilon {}^*w^\varepsilon \text{ and } \underline{w} := \liminf_\varepsilon {}^*w^\varepsilon$$

and note that on intervals $[s_i, t_i]$

$$\tilde{F}^\varepsilon(t, x, r, p, X) \rightarrow F(\xi^{-1}(t), x, r, p, X) =: \tilde{F}(t, x, r, p, X) \text{ locally uniformly.}$$

Standard viscosity theory tells us that on intervals $[s_i, t_i]$, \bar{w} and \underline{w} are sub- resp. supersolutions of $\partial_t - \tilde{F} = 0$. Using the method of semi-relaxed limits (by definition, $\bar{w} \geq \underline{w}$ and the reversed inequality follows from comparison) conclude that $w := \bar{w} = \underline{w}$ and that for every compact set $K \subset \mathbb{R}^n$ (by using a Dini-type argument),

$$|w^\varepsilon - w|_{\infty; [s_i, t_i] \times K} = \sup_{t \in [s_i, t_i], x \in K} |w^\varepsilon(t, x) - w(t, x)| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Now define

$$v(t, x) := \begin{cases} w(\xi(t), x), & \xi^{-1}(s_i) \leq t \leq \xi^{-1}(t_i) \\ w(\xi(t_i), x), & \xi^{-1}(t_i) < t < \xi^{-1}(s_{i+1}) \end{cases}$$

We get the claimed convergence $v^\varepsilon \rightarrow v$ on intervals $[\xi^{-1}(s_i), \xi^{-1}(t_i)]$. However, on $[\xi^{-1}(t_i), \xi^{-1}(s_{i+1})]$, v^ε is by definition viscosity solution of $\partial_t - F\dot{\xi}^\varepsilon = 0$ with initial condition $v^\varepsilon(\xi^{-1}(t_i), \cdot)$ and $F\dot{\xi}^\varepsilon \rightarrow_\varepsilon 0$ locally uniformly since $\sup_t |\dot{\xi}^\varepsilon(t)|$ is uniformly bounded in ε by assumption. Hence the standard stability result of viscosity theory applies and v^ε converges locally uniformly on $[\xi^{-1}(t_i), \xi^{-1}(s_{i+1})]$ against the constant-in-time function $v^\varepsilon(\xi^{-1}(t_i), \cdot)$, the only solution to $\partial_t = 0$ with initial condition $v^\varepsilon(\xi^{-1}(t_i), \cdot)$. This proves the claimed convergence. Further, note that v is given as the unique viscosity solution of $\partial_t - \tilde{F} = 0$, hence every other sequence approximating ξ will lead to the same limit. ■

The proof of the main theorem (theorem 13) and applications to examples adapt now in a straightforward way to time-dependent F .

B Appendix: Generalized viscosity solutions

Section 3.2 and appendix A extend the notion of viscosity solutions to equations of the form

$$du = F(t, x, u, Du, D^2u) d\xi(t), \quad u(0, x) = u_0(x). \quad (24)$$

with $\xi \in C^{1-\text{var};+}([0, T], \mathbb{R})$. Generalizations of viscosity solutions go back to [Ish85],[LP87] and for the parabolic case [Nun92]. Let us recall the definition given in [Nun92].

Condition 20 $F(\cdot, x, r, p, X) \in L^1((0, T), \mathbb{R})$ for all $(x, r, p, X) \in \mathbb{R}^e \times \mathbb{R} \times \mathbb{R}^e \times \mathbb{S}^e$ and F is continuous on $\mathbb{R}^e \times \mathbb{R} \times \mathbb{R}^e \times \mathbb{S}^e$ for almost all $t \in (0, T)$

Condition 21 $F(t, x, \cdot, p, X)$ is nondecreasing on \mathbb{R} for all $t \in (0, T)$ and for all $(x, p, X) \in \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{S}^e$.

Definition 22 Let F satisfy conditions 20 and 21. A locally bounded uniformly upper semicontinuous function $u \in BUC([0, T] \times \mathbb{R}^e)$ is called a generalized subsolution of

$$du = F(t, x, u, Du, D^2u), \quad u(0, x) = u_0(x) \quad (25)$$

if for any $(\hat{t}, \hat{x}) \in [0, T] \times \mathbb{R}^e$, $b \in L^1([0, T], \mathbb{R})$, $\phi \in C^2(\mathbb{R}^e, \mathbb{R})$, $G : [0, T] \times \mathbb{R}^e \times \mathbb{R} \times \mathbb{R}^e \times \mathbb{S}^e \rightarrow \mathbb{R}$ continuous and degenerate parabolic, such that

$$u(t, x) + \int_0^t b(r) dr - \phi(t, x) \text{ attains a local maximum at } (\hat{t}, \hat{x}),$$

and

$$\begin{aligned} b(\hat{t}) + G(\hat{t}, x, r, p, X) &\leq F(\hat{t}, x, r, p, X) \text{ for a.e. } t \in B_\delta(\hat{t}) \text{ and} \\ \text{for all } (x, r, p, X) &\in B_\delta(\hat{x}, u(\hat{t}, \hat{x}), D\phi|_{\hat{t}, \hat{x}}, D^2\phi|_{\hat{t}, \hat{x}}) \text{ for some } \delta > 0 \end{aligned}$$

it follows that

$$b(\hat{t}) + G(\hat{x}, u(\hat{t}, \hat{x}), D\phi|_{\hat{t}, \hat{x}}, D^2\phi|_{\hat{t}, \hat{x}}) \leq 0.$$

A locally bounded uniformly lower semicontinuous function is called a supersolution if the above estimates hold when one replaces maximum by minimum and reverses the inequality sign.

Note that equation (25) is covered by this definition. However, it is quite cumbersome to derive existence, comparison and stability results in this very general setting and in the case of interest to us, the time-discontinuity only appears multiplicatively.

Proposition 23 Under the assumptions of proposition 19 and additionally $\xi \in W^{1,1}$ the function $u = u^\xi$ is a viscosity solution of

$$du = F(t, x, u, Du, D^2u) d\xi_t, \quad u(0, x) = u_0(x)$$

in the sense of definition 22.

Proof. We partition $[0, T]$ into $0 \leq s_1 \leq t_1 \leq \dots \leq s_n \leq t_n \leq T$ such that ξ is increasing on $[s_i, t_i]$, constant on $[t_i, s_{i+1}]$. Say $u(t, x) + \int_0^t b(r) dr - \phi(t, x)$ attains a local maximum at (\hat{t}, \hat{x}) . If $\hat{t} \in [s_i, t_i]$ by construction $u(t, x) \equiv w(t, \xi_t)$ with w a viscosity subsolution of $\partial_t - \tilde{F} = 0$, $\tilde{F}(t, r, x, p, X) = F(\xi^{-1}(t), r, x, p, X)$, hence also a generalized subsolution and using that ξ is invertible on $[s_i, t_i]$ one sees by a change of variable that also u is a generalized subsolution on $[s_i, t_i]$ of $\partial_t - \tilde{F} = 0$.

If $\hat{t} \in [t_i, s_{i+1}]$, then ξ is constant, hence $\dot{\xi} = 0$ a.s. and so $F(t, x, r, p, X) \dot{\xi}(t) = 0$ for a.e. $t \in B_\delta(\hat{t})$ and u is a generalized subsolution on that interval. This shows that u is a generalized subsolution and the same argument shows that u is a generalized supersolution. ■

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