Special Lagrangian submanifolds with isolated conical singularities. IV. Desingularization, obstructions and families

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1 Introduction

Special Lagrangian m-folds (SL m-folds) are a distinguished class of real m-dimensional minimal submanifolds which may be defined in $\mathbb{C}^m$, or in *Calabi–Yau m-folds*, or more generally in *almost Calabi–Yau m-folds* (compact Kähler m-folds with trivial canonical bundle). We write an almost Calabi–Yau m-fold as M or (M,J,ω,Ω), where the manifold M has complex structure J, Kähler form ω and holomorphic volume form Ω.

This is the fourth in a series of five papers [7, 8, 9, 10] studying SL m-folds with *isolated conical singularities*. That is, we consider an SL m-fold X in an almost Calabi–Yau m-fold M for $m>2$ with singularities at $x_1,\ldots,x_n$ in M, such that for some special Lagrangian cones C_i in $T_{x_i}M\cong\mathbb{C}^m$ with $C_i\setminus\{0\}$ nonsingular, X approaches C_i near x_i in an asymptotic C^1 sense. Readers are advised to begin with the final paper [10], which surveys the series, and applies the results to prove some conjectures.

The first paper [7] laid the foundations for the series, and studied the *regularity* of SL m-folds with conical singularities near their singular points. The second paper [8] discussed the *deformation theory* of compact SL m-folds X with conical singularities in an almost Calabi–Yau m-fold M.

The third paper [9] and this one study *desingularizations* of compact SL m-folds X with conical singularities. That is, we construct a family of compact, *nonsingular* SL m-folds $\tilde N^t$ in M for $t\in(0,\epsilon]$ with $\tilde N^t\rightarrow X$ as $t\rightarrow 0$, in the sense of currents. In [9] we did this for simple situations, working in a single almost Calabi–Yau m-fold (M,J,ω,Ω), and making topological assumptions to avoid problems with obstructions to the existence of $\tilde N^t$.

This paper extends the results of [9] to more complicated situations, in which there are *topological obstructions* to the existence of desingularizations $\tilde N^t$ of X, and to desingularizations in *families* of almost Calabi–Yau m-folds $(M,J^s,\omega^s,\Omega^s)$ for $s\in\mathcal{F}$.

Having a good understanding of the singularities of special Lagrangian submanifolds will be essential in clarifying the Strominger–Yau–Zaslow conjecture on the Mirror Symmetry of Calabi–Yau 3-folds [16], and also in resolving conjectures made by the author [5] on defining new invariants of Calabi–Yau 3-folds by counting special Lagrangian homology 3-spheres with weights. The series aims to develop such an understanding for simple singularities of SL m-folds.

Here is the basic idea of [9]. Let X be a compact SL m-fold with conical singularities $x_1, \ldots, x_n$ in an almost Calabi–Yau m-fold (M, J, ω, Ω). Choose an isomorphism $\upsilon_i : \mathbb{C}^m \rightarrow T_{x_i}M$ for $i = 1, \ldots, n$. Then there is a unique *SL cone* C_i in $\mathbb{C}^m$ with X asymptotic to $\upsilon_i(C_i)$ at x_i. Let L_i be an *Asymptotically Conical SL m-fold* (*AC SL m-fold*) in $\mathbb{C}^m$, asymptotic to C_i at infinity.

Now $tL_i = \{t\,\mathbf{x} : \mathbf{x} \in L_i\}$ is also an AC SL m-fold asymptotic to C_i for $t > 0$. We construct a 1-parameter family of compact, nonsingular *Lagrangian* m-folds N^t in (M, ω) for $t \in (0, \delta)$ by gluing tL_i into X at x_i. When t is small, N^t is close to being special Lagrangian, but also close to being singular. We prove using analysis that for small t we can deform N^t to a *special* Lagrangian m-fold $\tilde{N}^t$ in M, using a small Hamiltonian deformation.

In this paper we shall study the following issues, not tackled in [9]. The AC SL m-folds L_i have topological invariants $Y(L_i), Z(L_i)$ defined in §4.1, which measure the relative de Rham cohomology classes of ω and $\operatorname{Im}\Omega$ in $H^k(\mathbb{C}^m, L_i; \mathbb{R})$. In [9] we assumed that $Y(L_i) = 0$ for $i = 1, \ldots, n$. Section 6 shows how to extend the results of [9] to the case $Y(L_i) \neq 0$, so that they are applicable to a much larger class of AC SL m-folds.

Doing this is a problem in symplectic geometry. If $Y(L_i) = 0$ then we can choose the Lagrangian m-folds N^t to coincide with X away from x_i, and work locally near x_i. But if $Y(L_i) \neq 0$ we cannot do this. Instead, we must define N^t away from x_i as the graph of a closed 1-form on $X' = X \setminus \{x_1, \ldots, x_n\}$ with nonzero cohomology class, and there can be *topological obstructions* to the existence of N^t as a Lagrangian m-fold.

In §7 and §8 we extend the results to *smooth families* of almost Calabi–Yau m-folds $(M, J^s, \omega^s, \Omega^s)$, for $s \in \mathcal{F} \subset \mathbb{R}^d$ with $0 \in \mathcal{F}$ and $(M, J, \omega, \Omega) = (M, J^0, \omega^0, \Omega^0)$. It turns out that the cohomology classes $[\omega^s]$ and $[\operatorname{Im}\Omega^s]$ contribute to the obstruction equations involving $Y(L_i)$ and $Z(L_i)$ for the existence of desingularizations $\tilde{N}^t$.

Because of this, it can happen that a singular SL m-fold X in (M, J, ω, Ω) admits no desingularizations $\tilde{N}^t$ in (M, J, ω, Ω), but does admit desingularizations $\tilde{N}^{s,t}$ in $(M, J^s, \omega^s, \Omega^s)$ for small $s \neq 0$. Thus we can overcome obstructions to the existence of desingularizations by varying the underlying almost Calabi–Yau m-fold (M, J, ω, Ω).

We begin in §2 with an introduction to special Lagrangian geometry. Sections 3 and 4 define SL m-folds with conical singularities and Asymptotically Conical SL m-folds, and review results we need from [7]. Section 5 recalls and discusses the major definitions and theorems from the previous paper [9].

The new material of the paper is §6–§8. Section 6 generalizes the results of [9] to the case when $Y(L_i) \neq 0$, and §7 to families of almost Calabi–Yau m-folds $(M, J^s, \omega^s, \Omega^s)$ when $Y(L_i) = 0$. Finally §8 considers the most complicated case, in families $(M, J^s, \omega^s, \Omega^s)$ when $Y(L_i) \neq 0$.

For simplicity we generally take all submanifolds to be *embedded*. However, all our results generalize immediately to *immersed* submanifolds, with only cosmetic changes.

Other authors have also desingularized SL m-folds using gluing techniques. Those known to me are Salur [14, 15], Butscher [2] and Lee [4], which were discussed in [9, §1]. They all involve connect sum constructions in Calabi–Yau m-folds or $\mathbb{C}^m$, rather than the more general kinds of singularities we consider.

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2 Special Lagrangian geometry

We introduce special Lagrangian submanifolds (SL m-folds) in two different geometric contexts. First, in §2.1, we define SL m-folds in $\mathbb{C}^m$. Then §2.2 discusses SL m-folds in *almost Calabi–Yau m-folds*, compact Kähler manifolds equipped with a holomorphic volume form, which generalize Calabi–Yau manifolds. Some references for this section are Harvey and Lawson [3] and the author [6].

2.1 Special Lagrangian submanifolds in $\mathbb{C}^m$

We begin by defining *calibrations* and *calibrated submanifolds*, following [3].

Definition 2.1 Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane* V on M is a vector subspace V of some tangent space T_xM to M with $\dim V = k$, equipped with an orientation. If V is an oriented tangent k-plane on M then $g|_V$ is a Euclidean metric on V, so combining $g|_V$ with the orientation on V gives a natural *volume form* vol_V on V, which is a k-form on V.

Now let φ be a closed k-form on M. We say that φ is a *calibration* on M if for every oriented k-plane V on M we have $\varphi|_V \leqslant \mathrm{vol}_V$. Here $\varphi|_V = \alpha \cdot \mathrm{vol}_V$ for some $\alpha \in \mathbb{R}$, and $\varphi|_V \leqslant \mathrm{vol}_V$ if $\alpha \leqslant 1$. Let N be an oriented submanifold of M with dimension k. Then each tangent space T_xN for $x \in N$ is an oriented tangent k-plane. We say that N is a *calibrated submanifold* if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

It is easy to show that calibrated submanifolds are automatically *minimal submanifolds* [3, Th. II.4.2]. Here is the definition of special Lagrangian submanifolds in $\mathbb{C}^m$, taken from [3, §III].

Definition 2.2 Let $\mathbb{C}^m$ have complex coordinates $(z_1, \ldots, z_m)$, and define a metric g', a real 2-form ω' and a complex m-form Ω' on $\mathbb{C}^m$ by

$$\begin{aligned} g' = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \quad \omega' &= \tfrac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar z_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar z_m), \\ \text{and} \quad \Omega' &= \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m. \end{aligned} \tag{1}$$

Then $\mathrm{Re}\,\Omega'$ and $\mathrm{Im}\,\Omega'$ are real m-forms on $\mathbb{C}^m$. Let L be an oriented real submanifold of $\mathbb{C}^m$ of real dimension m. We say that L is a *special Lagrangian* *submanifold* of $\mathbb{C}^m$, or *SL m-fold* for short, if L is calibrated with respect to $\operatorname{Re}\Omega'$, in the sense of Definition 2.1.

Harvey and Lawson [3, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds:

Proposition 2.3 *Let L be a real m-dimensional submanifold of $\mathbb{C}^m$. Then L admits an orientation making it into an SL submanifold of $\mathbb{C}^m$ if and only if $\omega'|_L \equiv 0$ and $\operatorname{Im}\Omega'|_L \equiv 0$.*

An m-dimensional submanifold L in $\mathbb{C}^m$ is called *Lagrangian* if $\omega'|_L \equiv 0$. Thus special Lagrangian submanifolds are Lagrangian submanifolds satisfying the extra condition that $\operatorname{Im}\Omega'|_L \equiv 0$, which is how they get their name.

2.2 Almost Calabi–Yau m-folds and SL m-folds

We shall define special Lagrangian submanifolds not just in Calabi–Yau manifolds, as usual, but in the much larger class of *almost Calabi–Yau manifolds*.

Definition 2.4 Let $m \geqslant 2$. An *almost Calabi–Yau m-fold* is a quadruple (M, J, ω, Ω) such that (M, J) is a compact m-dimensional complex manifold, ω is the Kähler form of a Kähler metric g on M, and Ω is a non-vanishing holomorphic $(m, 0)$-form on M.

We call (M, J, ω, Ω) a *Calabi–Yau m-fold* if in addition ω and Ω satisfy

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega}. \tag{2}$$

Then for each $x \in M$ there exists an isomorphism $T_x M \cong \mathbb{C}^m$ that identifies g_x, ω_x and Ω_x with the flat versions g', ω', Ω' on $\mathbb{C}^m$ in (1). Furthermore, g is Ricci-flat and its holonomy group is a subgroup of $\mathrm{SU}(m)$.

This is not the usual definition of a Calabi–Yau manifold, but is essentially equivalent to it.

Definition 2.5 Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold, and N a real m-dimensional submanifold of M. We call N a *special Lagrangian submanifold*, or *SL m-fold* for short, if $\omega|_N \equiv \operatorname{Im}\Omega|_N \equiv 0$. It easily follows that $\operatorname{Re}\Omega|_N$ is a nonvanishing m-form on N. Thus N is orientable, with a unique orientation in which $\operatorname{Re}\Omega|_N$ is positive.

Again, this is not the usual definition of SL m-fold, but is essentially equivalent to it. Suppose (M, J, ω, Ω) is an almost Calabi–Yau m-fold, with metric g. Let $\psi : M \to (0, \infty)$ be the unique smooth function such that

$$\psi^{2m}\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega}, \tag{3}$$

and define $\tilde{g}$ to be the conformally equivalent metric $\psi^2 g$ on M. Then $\operatorname{Re}\Omega$ is a *calibration* on the Riemannian manifold $(M, \tilde{g})$, and SL m-folds N in (M, J, ω, Ω) are calibrated with respect to it, so that they are minimal with respect to $\tilde{g}$.

If M is a Calabi–Yau m-fold then $\psi \equiv 1$ by (2), so $\tilde{g} = g$, and an m-submanifold N in M is special Lagrangian if and only if it is calibrated w.r.t. $\mathrm{Re}\,\Omega$ on (M, g), as in Definition 2.2. This recovers the usual definition of special Lagrangian m-folds in Calabi–Yau m-folds.

The *deformation theory* of special Lagrangian submanifolds was studied by McLean [13, §3], who proved the following result in the Calabi–Yau case. The extension to the almost Calabi–Yau case is described in [6, §9.5].

Theorem 2.6 *Let N be a compact SL m-fold in an almost Calabi–Yau m-fold (M, J, ω, Ω). Then the moduli space $\mathcal{M}_N$ of special Lagrangian deformations of N is a smooth manifold of dimension $b^1(N)$, the first Betti number of N.*

We shall often consider *families* of almost Calabi–Yau m-folds.

Definition 2.7 Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold. A *smooth family of deformations of* (M, J, ω, Ω) is a connected open set $\mathcal{F} \subset \mathbb{R}^d$ for $d \geqslant 0$ with $0 \in \mathcal{F}$ called the *base space*, and a smooth family $\big\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\big\}$ of almost Calabi–Yau structures on M with $(J^0, \omega^0, \Omega^0) = (J, \omega, \Omega)$.

If N is an SL m-fold in (M, J, ω, Ω), the moduli spaces of deformations of N in each $(M, J^s, \omega^s, \Omega^s)$ for $s \in \mathcal{F}$ fit together into a big moduli space $\mathcal{M}_N^{\mathcal{F}}$.

Definition 2.8 Let $\big\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\big\}$ be a smooth family of deformations of an almost Calabi–Yau m-fold (M, J, ω, Ω), and N be a compact SL m-fold in (M, J, ω, Ω). Define the *moduli space $\mathcal{M}_N^{\mathcal{F}}$ of deformations of N in the family $\mathcal{F}$* to be the set of pairs $(s, \hat{N})$ for which $s \in \mathcal{F}$ and $\hat{N}$ is a compact SL m-fold in $(M, J^s, \omega^s, \Omega^s)$ which is diffeomorphic to N and isotopic to N in M. Define a *projection* $\pi^{\mathcal{F}} : \mathcal{M}_N^{\mathcal{F}} \to \mathcal{F}$ by $\pi^{\mathcal{F}}(s, \hat{N}) = s$. Then $\mathcal{M}_N^{\mathcal{F}}$ has a natural topology, and $\pi^{\mathcal{F}}$ is continuous.

The following result is proved by Marshall [11, Th. 3.2.9], using similar methods to Theorem 2.6.

Theorem 2.9 *Let $\big\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\big\}$ be a smooth family of deformations of an almost Calabi–Yau m-fold (M, J, ω, Ω), with base space $\mathcal{F} \subset \mathbb{R}^d$. Suppose N is a compact SL m-fold in (M, J, ω, Ω) with $[\omega^s|_N] = 0$ in $H^2(N, \mathbb{R})$ and $[\mathrm{Im}\,\Omega^s|_N] = 0$ in $H^m(N, \mathbb{R})$ for all $s \in \mathcal{F}$. Let $\mathcal{M}_N^{\mathcal{F}}$ be the moduli space of deformations of N in $\mathcal{F}$, and $\pi^{\mathcal{F}} : \mathcal{M}_N^{\mathcal{F}} \to \mathcal{F}$ the natural projection.*

Then $\mathcal{M}_N^{\mathcal{F}}$ is a smooth manifold of dimension $d + b^1(N)$, and $\pi^{\mathcal{F}} : \mathcal{M}_N^{\mathcal{F}} \to \mathcal{F}$ a smooth submersion. For small $s \in \mathcal{F}$ the moduli space $\mathcal{M}_N^s = (\pi^{\mathcal{F}})^{-1}(s)$ of deformations of N in $(M, J^s, \omega^s, \Omega^s)$ is a nonempty smooth manifold of dimension $b^1(N)$, with $\mathcal{M}_N^0 = \mathcal{M}_N$.

This describes the *obstructions* to the existence of SL m-folds when we deform the underlying almost Calabi–Yau m-fold.

3 SL m-folds with conical singularities

The preceding papers [7, 8, 9] defined and studied *compact SL m-folds X with conical singularities* in an almost Calabi–Yau m-fold (M, J, ω, Ω). We now recall the definitions and results from [7] that we will need later. For brevity we shall keep explanations to a minimum, and readers are referred to [7] for further discussion and motivation.

3.1 Preliminaries on special Lagrangian cones

Following [7, §2.1] we give definitions and results on *special Lagrangian cones*.

Definition 3.1 A (singular) SL m-fold C in $\mathbb{C}^m$ is called a *cone* if $C = tC$ for all $t > 0$, where $tC = \{t\,\mathbf{x} : \mathbf{x} \in C\}$. Let C be a closed SL cone in $\mathbb{C}^m$ with an isolated singularity at 0. Then $\Sigma = C \cap \mathcal{S}^{2m-1}$ is a compact, nonsingular $(m-1)$-submanifold of $\mathcal{S}^{2m-1}$, not necessarily connected. Let g_Σ be the restriction of g' to Σ, where g' is as in (1).

Set $C' = C \setminus \{0\}$. Define $\iota : \Sigma \times (0,\infty) \to \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Then ι has image C'. By an abuse of notation, *identify* C' with $\Sigma \times (0,\infty)$ using ι. The *cone metric* on $C' \cong \Sigma \times (0,\infty)$ is $g' = \iota^*(g') = \mathrm{d}r^2 + r^2 g_\Sigma$.

For $\alpha \in \mathbb{R}$, we say that a function $u : C' \to \mathbb{R}$ is *homogeneous of order* α if $u \circ t \equiv t^\alpha u$ for all $t > 0$. Equivalently, u is homogeneous of order α if $u(\sigma, r) \equiv r^\alpha v(\sigma)$ for some function $v : \Sigma \to \mathbb{R}$.

In [7, Lem. 2.3] we study *homogeneous harmonic functions* on C'.

Lemma 3.2 *In the situation of Definition 3.1, let $u(\sigma, r) \equiv r^\alpha v(\sigma)$ be a homogeneous function of order α on $C' = \Sigma \times (0,\infty)$, for $v \in C^2(\Sigma)$. Then*

$$\Delta u(\sigma, r) = r^{\alpha-2}\bigl(\Delta_\Sigma v - \alpha(\alpha + m - 2)v\bigr),$$

where Δ, Δ_Σ are the Laplacians on (C', g') and (Σ, g_Σ). Hence, u is harmonic on C' if and only if v is an eigenfunction of Δ_Σ with eigenvalue $\alpha(\alpha + m - 2)$.

Following [7, Def. 2.5], we define:

Definition 3.3 In the situation of Definition 3.1, suppose $m > 2$ and define

$$\mathcal{D}_\Sigma = \bigl\{\alpha \in \mathbb{R} : \alpha(\alpha + m - 2) \text{ is an eigenvalue of } \Delta_\Sigma\bigr\}. \tag{4}$$

Then $\mathcal{D}_\Sigma$ is a countable, discrete subset of $\mathbb{R}$. By Lemma 3.2, an equivalent definition is that $\mathcal{D}_\Sigma$ is the set of $\alpha \in \mathbb{R}$ for which there exists a nonzero homogeneous harmonic function u of order α on C'.

Define $m_\Sigma : \mathcal{D}_\Sigma \to \mathbb{N}$ by taking $m_\Sigma(\alpha)$ to be the multiplicity of the eigenvalue $\alpha(\alpha + m - 2)$ of Δ_Σ, or equivalently the dimension of the vector space of homogeneous harmonic functions u of order α on C'. Define $N_\Sigma : \mathbb{R} \to \mathbb{Z}$ by

$$N_\Sigma(\delta) = -\sum_{\alpha \in \mathcal{D}_\Sigma \cap (\delta, 0)} m_\Sigma(\alpha) \text{ if } \delta < 0, \text{ and } N_\Sigma(\delta) = \sum_{\alpha \in \mathcal{D}_\Sigma \cap [0, \delta]} m_\Sigma(\alpha) \text{ if } \delta \geqslant 0.$$

Then N_Σ is monotone increasing and upper semicontinuous, and is discontinuous exactly on $\mathcal{D}_\Sigma$, increasing by $m_\Sigma(\alpha)$ at each $\alpha \in \mathcal{D}_\Sigma$. As the eigenvalues of Δ_Σ are nonnegative, we see that $\mathcal{D}_\Sigma \cap (2-m, 0) = \emptyset$ and $N_\Sigma \equiv 0$ on $(2-m, 0)$.

3.2 The definition of SL m-folds with conical singularities

Now we can define *conical singularities* of SL m-folds, following [7, Def. 3.6].

Definition 3.4 Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold for $m > 2$, and define $\psi : M \to (0, \infty)$ as in (3). Suppose X is a compact singular SL m-fold in M with singularities at distinct points $x_1, \ldots, x_n \in X$, and no other singularities.

Fix isomorphisms $\upsilon_i : \mathbb{C}^m \to T_{x_i}M$ for $i = 1, \ldots, n$ such that $\upsilon_i^*(\omega) = \omega'$ and $\upsilon_i^*(\Omega) = \psi(x_i)^m \Omega'$, where ω', Ω' are as in (1). Let $C_1, \ldots, C_n$ be SL cones in $\mathbb{C}^m$ with isolated singularities at 0. For $i = 1, \ldots, n$ let $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$, and let $\mu_i \in (2, 3)$ with $(2, \mu_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset$, where $\mathcal{D}_{\Sigma_i}$ is defined in (4). Then we say that X has a *conical singularity* at x_i, with *rate* μ_i and *cone* C_i for $i = 1, \ldots, n$, if the following holds.

By Darboux' Theorem [12, Th. 3.15] there exist embeddings $\Upsilon_i : B_R \to M$ for $i = 1, \ldots, n$ satisfying $\Upsilon_i(0) = x_i$, $\mathrm{d}\Upsilon_i|_0 = \upsilon_i$ and $\Upsilon_i^*(\omega) = \omega'$, where B_R is the open ball of radius R about 0 in $\mathbb{C}^m$ for some small $R > 0$. Define $\iota_i : \Sigma_i \times (0, R) \to B_R$ by $\iota_i(\sigma, r) = r\sigma$ for $i = 1, \ldots, n$.

Define $X' = X \setminus \{x_1, \ldots, x_n\}$. Then there should exist a compact subset $K \subset X'$ such that $X' \setminus K$ is a union of open sets $S_1, \ldots, S_n$ with $S_i \subset \Upsilon_i(B_R)$, whose closures $\bar{S}_1, \ldots, \bar{S}_n$ are disjoint in X. For $i = 1, \ldots, n$ and some $R' \in (0, R]$ there should exist a smooth $\phi_i : \Sigma_i \times (0, R') \to B_R$ such that $\Upsilon_i \circ \phi_i : \Sigma_i \times (0, R') \to M$ is a diffeomorphism $\Sigma_i \times (0, R') \to S_i$, and

$$\left|\nabla^k(\phi_i - \iota_i)\right| = O(r^{\mu_i - 1 - k}) \quad \text{as } r \to 0 \text{ for } k = 0, 1. \tag{5}$$

Here $\nabla, |\,.\,|$ are computed using the cone metric $\iota_i^*(g')$ on $\Sigma_i \times (0, R')$.

The reasoning behind this definition was discussed in [7, §3.3]. We suppose $m > 2$ for two reasons. Firstly, the only SL cones in $\mathbb{C}^2$ are finite unions of SL planes $\mathbb{R}^2$ in $\mathbb{C}^2$ intersecting only at 0. Thus any SL 2-fold with conical singularities is actually *nonsingular* as an immersed 2-fold, so there is really no point in studying them.

Secondly, $m = 2$ is a special case in the analysis of [7, §2], and it is simpler to exclude it. Therefore we will suppose $m > 2$ throughout the paper. We will need the following tool [7, Def. 2.6], a smoothed out version of the distance from the singular set $\{x_1, \ldots, x_n\}$ in X.

Definition 3.5 Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$, and use the notation of Definition 3.4. Define a *radius function* ρ on X' to be a smooth function $\rho : X' \to (0, 1]$ such that $\rho \equiv 1$ on K and $\rho(y) = d(x_i, y)$ for $y \in S_i$ close to x_i, where d is the metric on X. Radius functions always exist.

3.3 Homology, cohomology and Hodge theory

Next we discuss *homology* and *cohomology* of SL m-folds with conical singularities, following [7, §2.4]. For a general reference, see for instance Bredon [1]. When Y is a manifold, write $H^k(Y,\mathbb{R})$ for the k^{th} *de Rham cohomology group* and $H^k_{\text{cs}}(Y,\mathbb{R})$ for the k^{th} *compactly-supported de Rham cohomology group* of Y. If Y is compact then $H^k(Y,\mathbb{R}) = H^k_{\text{cs}}(Y,\mathbb{R})$. The *Betti numbers* of Y are $b^k(Y) = \dim H^k(Y,\mathbb{R})$ and $b^k_{\text{cs}}(Y) = \dim H^k_{\text{cs}}(Y,\mathbb{R})$.

Let Y be a topological space, and $Z \subset Y$ a subspace. Write $H_k(Y,\mathbb{R})$ for the k^{th} *real singular homology group* of Y, and $H_k(Y;Z,\mathbb{R})$ for the k^{th} *real singular relative homology group* of $(Y;Z)$. When Y is a manifold and Z a submanifold we define $H_k(Y,\mathbb{R})$ and $H_k(Y;Z,\mathbb{R})$ using *smooth* simplices, as in [1, §V.5]. Then the pairing between (singular) homology and (de Rham) cohomology is defined at the chain level by integrating k-forms over k-simplices.

Suppose X is a compact SL m-fold in M with conical singularities $x_1,\ldots,x_n$ and cones $C_1,\ldots,C_n$, and set $X' = X \setminus \{x_1,\ldots,x_n\}$ and $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$ as in §3.2. Then by [7, §2.4] there is a natural long exact sequence

$$\cdots \to H^k_{\text{cs}}(X',\mathbb{R}) \to H^k(X',\mathbb{R}) \to \bigoplus_{i=1}^n H^k(\Sigma_i,\mathbb{R}) \to H^{k+1}_{\text{cs}}(X',\mathbb{R}) \to \cdots, \quad (6)$$

and natural isomorphisms

$$H_k\bigl(X;\{x_1,\ldots,x_n\},\mathbb{R}\bigr)^* \cong H^k_{\text{cs}}(X',\mathbb{R}) \cong H_{m-k}(X',\mathbb{R}) \cong H^{m-k}(X',\mathbb{R})^* \quad (7)$$

$$\text{and} \quad H^k_{\text{cs}}(X',\mathbb{R}) \cong H_k(X,\mathbb{R})^* \quad \text{for all } k > 1. \quad (8)$$

The inclusion $\iota : X \to M$ induces homomorphisms $\iota_* : H_k(X,\mathbb{R}) \to H_k(M,\mathbb{R})$.

If (Y,g) is a compact Riemannian manifold, then *Hodge theory* shows that each class in $H^k(Y,\mathbb{R})$ is represented by a unique k-form α with $\mathrm{d}\alpha = \mathrm{d}^*\alpha = 0$. Here is an analogue of this on X' when $k = 1$, given in [7, Th. 5.4].

Theorem 3.6 *Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold, and define $\psi : M \to (0,\infty)$ as in (3). Suppose X is a compact SL m-fold in M with conical singularities at $x_1,\ldots,x_n$, and let $X', K, R', \Sigma_i, \Upsilon_i, \phi_i, S_i$ and μ_i be as in Definition 3.4, $\mathcal{D}_{\Sigma_i}$ as in Definition 3.3, and ρ as in Definition 3.5. Define*

$$Y_{X'} = \bigl\{\alpha \in C^\infty(T^*X') : \mathrm{d}\alpha = 0, \quad \mathrm{d}^*(\psi^m\alpha) = 0, \\ |\nabla^k\alpha| = O(\rho^{-1-k}) \text{ for } k \geqslant 0\bigr\}. \quad (9)$$

Then $\pi : Y_{X'} \to H^1(X',\mathbb{R})$ given by $\pi : \alpha \mapsto [\alpha]$ is an isomorphism. Furthermore:

(a) *Fix $\alpha \in Y_{X'}$. By Hodge theory there exists a unique $\gamma_i \in C^\infty(T^*\Sigma_i)$ with $\mathrm{d}\gamma_i = \mathrm{d}^*\gamma_i = 0$ for $i = 1,\ldots,n$, such that the image of $\pi(\alpha)$ under the map $H^1(X',\mathbb{R}) \to \bigoplus_{i=1}^n H^1(\Sigma_i,\mathbb{R})$ of (6) is $\bigl([\gamma_1],\ldots,[\gamma_n]\bigr)$. There exist unique $T_i \in C^\infty\bigl(\Sigma_i \times (0,R')\bigr)$ for $i = 1,\ldots,n$ such that*

$$(\Upsilon_i \circ \phi_i)^*(\alpha) = \pi_i^*(\gamma_i) + \mathrm{d}T_i \quad \text{on } \Sigma_i \times (0,R') \text{ for } i=1,\ldots,n, \text{ and} \tag{10}$$

$$\nabla^k T_i(\sigma,r) = O(r^{\nu_i - k}) \qquad \begin{array}{l} \text{as } r \to 0, \text{ for all } k \geqslant 0 \text{ and} \\ \nu_i \in (0,\mu_i - 2) \text{ with } (0,\nu_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset. \end{array} \tag{11}$$

(b) *Suppose* $\gamma_i \in C^\infty(T^*\Sigma_i)$ *with* $\mathrm{d}\gamma_i = \mathrm{d}^*\gamma_i = 0$ *for* $i = 1,\ldots,n$, *and the image of* $\bigl([\gamma_1],\ldots,[\gamma_n]\bigr)$ *under* $\bigoplus_{i=1}^n H^1(\Sigma_i,\mathbb{R}) \to H^2_{\mathrm{cs}}(X',\mathbb{R})$ *in* (6) *is* $[\beta]$ *for some exact 2-form* β *on* X' *supported on* K. *Then there exists* $\alpha \in C^\infty(T^*X')$ *with* $\mathrm{d}\alpha = \beta$, $\mathrm{d}^*(\psi^m\alpha) = 0$ *and* $|\nabla^k\alpha| = O(\rho^{-1-k})$ *for* $k \geqslant 0$, *such that* (10) *and* (11) *hold for* $T_i \in C^\infty\bigl(\Sigma_i \times (0,R')\bigr)$.

(c) *Let* $f \in C^\infty(X')$ *with* $|\nabla^k f| = O(\rho^{-1-k})$ *for* $k \geqslant 0$ *and* $\int_{X'} f\,\mathrm{d}V = 0$. *Then there exists a unique exact 1-form* α *on* X' *with* $\mathrm{d}^*(\psi^m\alpha) = f$ *and* $|\nabla^k\alpha| = O(\rho^{-1-k})$ *for* $k \geqslant 0$, *such that* (10) *and* (11) *hold for* $\gamma_i = 0$ *and* $T_i \in C^\infty\bigl(\Sigma_i \times (0,R')\bigr)$.

3.4 Lagrangian Neighbourhood Theorems and regularity

We recall some symplectic geometry, which can be found in McDuff and Salamon [12]. Let N be a real m-manifold. Then its tangent bundle T^*N has a *canonical symplectic form* $\hat\omega$, defined as follows. Let $(x_1,\ldots,x_m)$ be local coordinates on N. Extend them to local coordinates $(x_1,\ldots,x_m,y_1,\ldots,y_m)$ on T^*N such that $(x_1,\ldots,y_m)$ represents the 1-form $y_1\mathrm{d}x_1 + \cdots + y_m\mathrm{d}x_m$ in $T^*_{(x_1,\ldots,x_m)}N$. Then $\hat\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \cdots + \mathrm{d}x_m \wedge \mathrm{d}y_m$.

Identify N with the zero section in T^*N. Then N is a *Lagrangian submanifold* of T^*N. The *Lagrangian Neighbourhood Theorem* [12, Th. 3.33] shows that any compact Lagrangian submanifold N in a symplectic manifold looks locally like the zero section in T^*N.

Theorem 3.7 *Let* (M,ω) *be a symplectic manifold and* $N \subset M$ *a compact Lagrangian submanifold. Then there exists an open tubular neighbourhood* U *of the zero section* N *in* T^*N, *and an embedding* $\Phi : U \to M$ *with* $\Phi|_N = \mathrm{id} : N \to N$ *and* $\Phi^*(\omega) = \hat\omega$, *where* $\hat\omega$ *is the canonical symplectic structure on* T^*N.

In [7, §4] we extend Theorem 3.7 to situations involving conical singularities, first to *SL cones*, [7, Th. 4.3].

Theorem 3.8 *Let* C *be an SL cone in* $\mathbb{C}^m$ *with isolated singularity at* 0, *and set* $\Sigma = C \cap \mathcal{S}^{2m-1}$. *Define* $\iota : \Sigma \times (0,\infty) \to \mathbb{C}^m$ *by* $\iota(\sigma,r) = r\sigma$, *with image* $C \setminus \{0\}$. *For* $\sigma \in \Sigma$, $\tau \in T^*_\sigma\Sigma$, $r \in (0,\infty)$ *and* $u \in \mathbb{R}$, *let* (σ,r,τ,u) *represent the point* $\tau + u\,\mathrm{d}r$ *in* $T^*_{(\sigma,r)}\bigl(\Sigma \times (0,\infty)\bigr)$. *Identify* $\Sigma \times (0,\infty)$ *with the zero section* $\tau = u = 0$ *in* $T^*\bigl(\Sigma \times (0,\infty)\bigr)$. *Define an action of* $(0,\infty)$ *on* $T^*\bigl(\Sigma \times (0,\infty)\bigr)$ *by*

$$t : (\sigma,r,\tau,u) \longmapsto (\sigma,tr,t^2\tau,tu) \quad \text{for } t \in (0,\infty), \tag{12}$$

so that $t^*(\hat\omega) = t^2\hat\omega$, *for* $\hat\omega$ *the canonical symplectic structure on* $T^*\bigl(\Sigma \times (0,\infty)\bigr)$.

Then there exists an open neighbourhood U_C of $\Sigma\times(0,\infty)$ in $T^*\bigl(\Sigma\times(0,\infty)\bigr)$ invariant under (12) *given by*

$$U_C = \bigl\{(\sigma,r,\tau,u)\in T^*\bigl(\Sigma\times(0,\infty)\bigr) : \bigl|(\tau,u)\bigr| < 2\zeta r\bigr\} \quad \textit{for some } \zeta>0, \qquad (13)$$

where $|\,.\,|$ is calculated using the cone metric $\iota^*(g')$ on $\Sigma\times(0,\infty)$, and an embedding $\Phi_C : U_C \to \mathbb{C}^m$ with $\Phi_C|_{\Sigma\times(0,\infty)}=\iota$, $\Phi_C^*(\omega')=\hat\omega$ and $\Phi_C\circ t = t\,\Phi_C$ for all $t>0$, where t acts on U_C as in (12) *and on $\mathbb{C}^m$ by multiplication.*

In [7, Th. 4.4] we construct a particular choice of ϕ_i in Definition 3.4.

Theorem 3.9 *Let (M,J,ω,Ω), $\psi,X,n,x_i,\upsilon_i,C_i,\Sigma_i,\mu_i,R,\Upsilon_i$ and ι_i be as in Definition 3.4. Theorem 3.8 gives $\zeta>0$, neighbourhoods U_{C_i} of $\Sigma_i\times(0,\infty)$ in $T^*\bigl(\Sigma_i\times(0,\infty)\bigr)$ and embeddings $\Phi_{C_i}:U_{C_i}\to\mathbb{C}^m$ for $i=1,\ldots,n$.*

Then for sufficiently small $R'\in(0,R]$ there exist unique closed 1-forms η_i on $\Sigma_i\times(0,R')$ for $i=1,\ldots,n$ written $\eta_i(\sigma,r)=\eta_i^1(\sigma,r)+\eta_i^2(\sigma,r)\mathrm{d}r$ for $\eta_i^1(\sigma,r)\in T^*_\sigma\Sigma_i$ and $\eta_i^2(\sigma,r)\in\mathbb{R}$, and satisfying $|\eta_i(\sigma,r)|<\zeta r$ and $\bigl|\nabla^k\eta_i\bigr| = O(r^{\mu_i-1-k})$ as $r\to0$ for $k=0,1$, computing $\nabla,|\,.\,|$ using the cone metric $\iota_i^*(g')$, such that the following holds.

Define $\phi_i:\Sigma_i\times(0,R')\to B_R$ by $\phi_i(\sigma,r)=\Phi_{C_i}\bigl(\sigma,r,\eta_i^1(\sigma,r),\eta_i^2(\sigma,r)\bigr)$. Then $\Upsilon_i\circ\phi_i:\Sigma_i\times(0,R')\to M$ is a diffeomorphism $\Sigma_i\times(0,R')\to S_i$ for open sets $S_1,\ldots,S_n$ in X' with $\bar S_1,\ldots,\bar S_n$ disjoint, and $K=X'\setminus(S_1\cup\cdots\cup S_n)$ is compact. Also ϕ_i satisfies (5)*, so that R',ϕ_i,S_i,K satisfy Definition 3.4.*

In [7, §5] we study the asymptotic behaviour of the maps ϕ_i of Theorem 3.9, using the elliptic regularity of the special Lagrangian condition. Combining [7, Th. 5.1], [7, Lem. 4.5] and [7, Th. 5.5] proves:

Theorem 3.10 *In the situation of Theorem 3.9 we have $\eta_i=\mathrm{d}A_i$ for $i=1,\ldots,n$, where $A_i:\Sigma_i\times(0,R')\to\mathbb{R}$ is given by $A_i(\sigma,r)=\int_0^r\eta_i^2(\sigma,s)\mathrm{d}s$. Suppose $\mu_i'\in(2,3)$ with $(2,\mu_i']\cap\mathcal{D}_{\Sigma_i}=\emptyset$ for $i=1,\ldots,n$. Then*

$$\begin{gathered}\bigl|\nabla^k(\phi_i-\iota_i)\bigr| = O(r^{\mu_i'-1-k}),\quad \bigl|\nabla^k\eta_i\bigr| = O(r^{\mu_i'-1-k}) \quad \textit{and}\\ \bigl|\nabla^k A_i\bigr| = O(r^{\mu_i'-k}) \quad \textit{as } r\to0 \textit{ for all } k\geqslant0 \textit{ and } i=1,\ldots,n.\end{gathered} \qquad (14)$$

Hence X has conical singularities at x_i with cone C_i and rate μ_i', for all possible rates μ_i' allowed by Definition 3.4. Therefore, the definition of conical singularities is essentially independent of the choice of rate μ_i.

Next we extend Theorem 3.7 to SL m-folds with conical singularities [7, Th. 4.6], in a way compatible with Theorems 3.8 and 3.9.

Theorem 3.11 *Suppose (M,J,ω,Ω) is an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1,\ldots,x_n$. Let the notation $\psi,\upsilon_i,C_i,\Sigma_i,\mu_i,R,\Upsilon_i$ and ι_i be as in Definition 3.4, and let ζ,U_{C_i}, $\Phi_{C_i},R',\eta_i,\eta_i^1,\eta_i^2,\phi_i,S_i$ and K be as in Theorem 3.9.*

Then making R' smaller if necessary, there exists an open tubular neighbourhood $U_{X'} \subset T^*X'$ of the zero section X' in T^*X', such that under $\mathrm{d}(\Upsilon_i \circ \phi_i) : T^*\bigl(\Sigma_i \times (0,R')\bigr) \to T^*X'$ for $i = 1,\ldots,n$ we have

$$\bigl(\mathrm{d}(\Upsilon_i \circ \phi_i)\bigr)^*(U_{X'}) = \bigl\{(\sigma,r,\tau,u) \in T^*\bigl(\Sigma_i \times (0,R')\bigr) : \bigl|(\tau,u)\bigr| < \zeta r\bigr\}, \qquad (15)$$

and there exists an embedding $\Phi_{X'} : U_{X'} \to M$ with $\Phi_{X'}|_{X'} = \mathrm{id} : X' \to X'$ and $\Phi_{X'}^*(\omega) = \hat{\omega}$, where $\hat{\omega}$ is the canonical symplectic structure on T^*X', such that

$$\Phi_{X'} \circ \mathrm{d}(\Upsilon_i \circ \phi_i)(\sigma,r,\tau,u) \equiv \Upsilon_i \circ \Phi_{C_i}\bigl(\sigma,r,\tau+\eta_i^1(\sigma,r),u+\eta_i^2(\sigma,r)\bigr) \qquad (16)$$

for all $i = 1,\ldots,n$ and $(\sigma,r,\tau,u) \in T^*\bigl(\Sigma_i \times (0,R')\bigr)$ with $\bigl|(\tau,u)\bigr| < \zeta r$. Here $|(\tau,u)|$ is computed using the cone metric $\iota_i^*(g')$ on $\Sigma_i \times (0,R')$.

In [7, Th. 4.8] we extend Theorem 3.11 to *families* of almost Calabi–Yau m-folds $(M,J^s,\omega^s,\Omega^s)$ for $s \in \mathcal{F}$. If $\omega^s|_{X'}$ is not exact then we cannot deform X' to a Lagrangian m-fold in (M,ω^s). Therefore we replace the condition $\Phi_{X'}^*(\omega) = \hat{\omega}$ in Theorem 3.11 by $(\Phi_{X'}^s)^*(\omega^s) = \hat{\omega} + \pi^*(\nu^s)$, where ν^s is a compactly-supported closed 2-form on X'.

Theorem 3.12 *Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1,\ldots,x_n$. Let the notation $R,\Upsilon_i,\zeta,\Phi_{C_i},R',\eta_i,\eta_i^1,\eta_i^2,\phi_i,S_i,K$ be as in Theorem 3.9, and let $U_{X'},\Phi_{X'}$ be as in Theorem 3.11. Suppose $\bigl\{(M,J^s,\omega^s,\Omega^s) : s \in \mathcal{F}\bigr\}$ is a smooth family of deformations of (M,J,ω,Ω) with base space $\mathcal{F} \subset \mathbb{R}^d$. Define $\psi^s : M \to (0,\infty)$ for $s \in \mathcal{F}$ as in (3), but using ω^s,Ω^s.*

Then making R,R' and $U_{X'}$ smaller if necessary, for some connected open $\mathcal{F}' \subseteq \mathcal{F}$ with $0 \in \mathcal{F}'$ and all $s \in \mathcal{F}'$ there exist

- (a) *isomorphisms $\upsilon_i^s : \mathbb{C}^m \to T_{x_i}M$ for $i = 1,\ldots,n$ with $\upsilon_i^0 = \upsilon_i$, $(\upsilon_i^s)^*(\omega^s) = \omega'$ and $(\upsilon_i^s)^*(\Omega) = \psi^s(x_i)^m\Omega'$,*
- (b) *embeddings $\Upsilon_i^s : B_R \to M$ for $i = 1,\ldots,n$ with $\Upsilon_i^0 = \Upsilon_i$, $\Upsilon_i^s(0) = x_i$, $\mathrm{d}\Upsilon_i^s|_0 = \upsilon_i^s$ and $(\Upsilon_i^s)^*(\omega^s) = \omega'$,*
- (c) *a closed 2-form $\nu^s \in C^\infty(\Lambda^2T^*X')$ supported in $K \subset X'$ with $\nu^0 = 0$, and*
- (d) *an embedding $\Phi_{X'}^s : U_{X'} \to M$ with $\Phi_{X'}^0 = \Phi_{X'}$ and $(\Phi_{X'}^s)^*(\omega^s) = \hat{\omega} + \pi^*(\nu^s)$,*

all depending smoothly on $s \in \mathcal{F}'$ with

$$\Phi_{X'}^s \circ \mathrm{d}(\Upsilon_i \circ \phi_i)(\sigma,r,\tau,u) \equiv \Upsilon_i^s \circ \Phi_{C_i}\bigl(\sigma,r,\tau+\eta_i^1(\sigma,r),u+\eta_i^2(\sigma,r)\bigr) \qquad (17)$$

for all $s \in \mathcal{F}'$, $i = 1,\ldots,n$ and $(\sigma,r,\tau,u) \in T^*\bigl(\Sigma_i \times (0,R')\bigr)$ with $\bigl|(\tau,u)\bigr| < \zeta r$.

The 2-forms ν^s in Theorem 3.12 define classes $[\nu^s]$ in $H^2_{\mathrm{cs}}(X',\mathbb{R})$. In [7, Th. 4.9] we investigate these classes, and the freedom to choose ν^s.

Theorem 3.13 *In the situation of Theorem 3.12, under the isomorphism* (8)*, the class* $[\nu^s] \in H^2_{\rm cs}(X',\mathbb{R})$ *is identified with the map* $H_2(X,\mathbb{R}) \rightarrow \mathbb{R}$ *given by* $\gamma \mapsto \iota_*(\gamma)\cdot[\omega^s]$*, where* $\iota : X \rightarrow M$ *is the inclusion,* $\iota_* : H_2(X,\mathbb{R}) \rightarrow H_2(M,\mathbb{R})$ *the induced map, and* $[\omega^s] \in H^2(M,\mathbb{R})$*. Thus* $[\nu^s]$ *depends only on* X, M *and* $[\omega^s] \in H^2(M,\mathbb{R})$*.*

Let $V \cong H^2_{\rm cs}(X',\mathbb{R})$ *be a vector space of smooth closed* 2*-forms on* X' *supported in* K *representing* $H^2_{\rm cs}(X',\mathbb{R})$*. Then making* $\mathcal{F}'$ *smaller if necessary, we can choose* Υ^s_i, ν^s *and* $\Phi^s_{X'}$ *in Theorem 3.12 so that* $\nu^s \in V$ *for all* $s \in \mathcal{F}'$*. In particular, if* $[\nu^s]=0$ *in* $H^2_{\rm cs}(X',\mathbb{R})$ *then we can choose* $\nu^s = 0$*.*

4 Asymptotically Conical SL m-folds

Let C be an SL cone in $\mathbb{C}^m$ with an isolated singularity at 0. Section 3 considered SL m-folds with conical singularities, which are asymptotic to C at 0. We now discuss *Asymptotically Conical* SL m-folds L in $\mathbb{C}^m$, which are asymptotic to C at infinity. Here is the definition.

Definition 4.1 Let C be an SL cone in $\mathbb{C}^m$ with isolated singularity at 0 for $m > 2$, as in Definition 3.1, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$, so that Σ is a compact, nonsingular $(m-1)$-manifold, not necessarily connected. Let g_Σ be the metric on Σ induced by the metric g' on $\mathbb{C}^m$ in (1), and r the radius function on $\mathbb{C}^m$. Define $\iota : \Sigma \times (0,\infty) \rightarrow \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Then the image of ι is $C \setminus \{0\}$, and $\iota^*(g') = r^2 g_\Sigma + {\rm d}r^2$ is the cone metric on $C \setminus \{0\}$.

Let L be a closed, nonsingular SL m-fold in $\mathbb{C}^m$ and $\lambda < 2$. We call L *Asymptotically Conical (AC)* with *rate* λ and *cone* C if there exists a compact subset $K \subset L$ and a diffeomorphism $\varphi : \Sigma \times (T,\infty) \rightarrow L \setminus K$ for some $T > 0$, such that

$$\bigl|\nabla^k(\varphi - \iota)\bigr| = O(r^{\lambda-1-k}) \quad \text{as } r \rightarrow \infty \text{ for } k = 0, 1. \tag{18}$$

Here $\nabla, |\,.\,|$ are computed using the cone metric $\iota^*(g')$ on $\Sigma \times (T,\infty)$.

This is very similar to Definition 3.4, and in fact there are strong parallels between the theories of SL m-folds with conical singularities and of Asymptotically Conical SL m-folds. We recall some results from [7, §7], including versions of the material in §3.4. We continue to assume $m > 2$ throughout.

4.1 Cohomological invariants of AC SL m-folds

Let L be an AC SL m-fold in $\mathbb{C}^m$ with cone C, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$. Using the notation of §3.3, as in (6) there is a long exact sequence

$$\cdots \rightarrow H^k_{\rm cs}(L,\mathbb{R}) \rightarrow H^k(L,\mathbb{R}) \rightarrow H^k(\Sigma,\mathbb{R}) \rightarrow H^{k+1}_{\rm cs}(L,\mathbb{R}) \rightarrow \cdots. \tag{19}$$

Following [7, Def. 7.2] we define *cohomological invariants* $Y(L), Z(L)$ of L.

Definition 4.2 Let L be an AC SL m-fold in $\mathbb{C}^m$ with cone C, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$. As $\omega', {\rm Im}\,\Omega'$ in (1) are closed forms with $\omega'|_L \equiv {\rm Im}\,\Omega'|_L = 0$, they define classes in the relative de Rham cohomology groups $H^k(\mathbb{C}^m;L,\mathbb{R})$ for $k=2,m$. For $k>1$ we have the exact sequence

$$0=H^{k-1}(\mathbb{C}^m,\mathbb{R})\rightarrow H^{k-1}(L,\mathbb{R})\xrightarrow{\cong}H^k(\mathbb{C}^m;L,\mathbb{R})\rightarrow H^k(\mathbb{C}^m,\mathbb{R})=0.$$

Let $Y(L)\in H^1(\Sigma,\mathbb{R})$ be the image of $[\omega']$ in $H^2(\mathbb{C}^m;L,\mathbb{R})\cong H^1(L,\mathbb{R})$ under $H^1(L,\mathbb{R})\rightarrow H^1(\Sigma,R)$ in (19), and $Z(L)\in H^{m-1}(\Sigma,\mathbb{R})$ be the image of $[\operatorname{Im}\Omega']$ in $H^m(\mathbb{C}^m;L,\mathbb{R})\cong H^{m-1}(L,\mathbb{R})$ under $H^{m-1}(L,\mathbb{R})\rightarrow H^{m-1}(\Sigma,R)$ in (19).

Here are some conditions for $Y(L)$ or $Z(L)$ to be zero, [7, Prop. 7.3].

Proposition 4.3 *Let L be an AC SL m-fold in $\mathbb{C}^m$ with cone C and rate λ, and let $\Sigma=C\cap\mathcal{S}^{2m-1}$. If $\lambda<0$ or $b^1(L)=0$ then $Y(L)=0$. If $\lambda<2-m$ or $b^0(\Sigma)=1$ then $Z(L)=0$.*

Here is a (trivial) lemma on *dilations* of AC SL m-folds.

Lemma 4.4 *Let L be an AC SL m-fold in $\mathbb{C}^m$ with rate λ and cone C, and let $t>0$. Then $tL=\{t\,\mathbf{x}:\mathbf{x}\in L\}$ is also an AC SL m-fold in $\mathbb{C}^m$ with rate λ and cone C, satisfying $Y(tL)=t^2Y(L)$ and $Z(tL)=t^mZ(L)$.*

4.2 Lagrangian Neighbourhood Theorems and regularity

Next we give versions of §3.4 for AC SL m-folds rather than SL m-folds with conical singularities. Here [7, Th. 7.4] is the analogue of Theorem 3.9.

Theorem 4.5 *Let C be an SL cone in $\mathbb{C}^m$ with isolated singularity at 0, and set $\Sigma=C\cap\mathcal{S}^{2m-1}$. Define $\iota:\Sigma\times(0,\infty)\rightarrow\mathbb{C}^m$ by $\iota(\sigma,r)=r\sigma$. Let ζ, $U_C\subset T^*\bigl(\Sigma\times(0,\infty)\bigr)$ and $\Phi_C:U_C\rightarrow\mathbb{C}^m$ be as in Theorem 3.8.*

Suppose L is an AC SL m-fold in $\mathbb{C}^m$ with cone C and rate $\lambda<2$. Then there exists a compact $K\subset L$ and a diffeomorphism $\varphi:\Sigma\times(T,\infty)\rightarrow L\setminus K$ for some $T>0$ satisfying (18), and a closed 1-form χ on $\Sigma\times(T,\infty)$ written $\chi(\sigma,r)=\chi^1(\sigma,r)+\chi^2(\sigma,r)\mathrm{d}r$ for $\chi^1(\sigma,r)\in T^*_\sigma\Sigma$ and $\chi^2(\sigma,r)\in\mathbb{R}$, satisfying

$$\begin{aligned}&\bigl|\chi(\sigma,r)\bigr|<\zeta r,\quad \varphi(\sigma,r)\equiv\Phi_C\bigl(\sigma,r,\chi^1(\sigma,r),\chi^2(\sigma,r)\bigr)\\ &\text{and}\quad \bigl|\nabla^k\chi\bigr|=O(r^{\lambda-1-k})\quad \text{as } r\rightarrow\infty \text{ for } k=0,1,\end{aligned}\tag{20}$$

computing $\nabla,|\,.\,|$ using the cone metric $\iota^*(g')$.

The next two theorems are analogous to Theorem 3.10. In [7, Prop. 7.6] and [7, Th. 7.7] we decompose χ in Theorem 4.5.

Theorem 4.6 *In Theorem 4.5 we have $[\chi]=Y(L)$ in $H^1\bigl(\Sigma\times(T,\infty),\mathbb{R}\bigr)\cong H^1(\Sigma,\mathbb{R})$, where $Y(L)$ is as in Definition 4.2. Let γ be the unique 1-form on Σ with $\mathrm{d}\gamma=\mathrm{d}^*\gamma=0$ and $[\gamma]=Y(L)\in H^1(\Sigma,\mathbb{R})$, which exists by Hodge* *theory. Then* $\chi = \pi^*(\gamma) + \mathrm{d}E$, *where* $\pi : \Sigma \times (T,\infty) \to \Sigma$ *is the projection and* $E \in C^\infty\bigl(\Sigma \times (T,\infty)\bigr)$ *satisfies*

$$\begin{gathered} \bigl|\nabla^k(\varphi-\iota)\bigr| = O(r^{\lambda-1-k}), \;\; \bigl|\nabla^k\chi\bigr| = O(r^{\lambda-1-k}), \;\; \bigl|\nabla^{k+1}E\bigr| = O(r^{\lambda-1-k}) \\ \text{for all } k \geqslant 0, \text{ and } \quad |E| = \begin{cases} O(r^\lambda), & \lambda \neq 0, \\ O\bigl(|\log r|\bigr), & \lambda = 0. \end{cases} \end{gathered} \tag{21}$$

Here $\nabla, |\,.\,|$ *are computed using the cone metric* $\iota^*(g')$ *on* $\Sigma \times (T,\infty)$.

Then [7, Th. 7.11] we improve the rate of convergence λ.

Theorem 4.7 *Let* L *be an AC SL* m*-fold in* $\mathbb{C}^m$ *with cone* C *and rate* λ. *Set* $\Sigma = C \cap \mathcal{S}^{2m-1}$, *and let* $\mathcal{D}_\Sigma, N_\Sigma$ *be as in Definition 3.3. Let* ι, T, φ, χ *be as in Theorem 4.5, and* $Y(L), \gamma, E$ *as in Theorem 4.6. Then*

(a) *Suppose* λ, λ' *lie in the same connected component of* $\mathbb{R} \setminus \mathcal{D}_\Sigma$. *Then*

$$\begin{gathered} \bigl|\nabla^k(\varphi-\iota)\bigr| = O(r^{\lambda'-1-k}), \quad \bigl|\nabla^k\chi\bigr| = O(r^{\lambda'-1-k}) \quad \text{and} \\ \bigl|\nabla^k E\bigr| = O(r^{\lambda'-k}) \quad \text{as } r \to \infty \text{ for all } k \geqslant 0. \end{gathered} \tag{22}$$

Hence L *is an AC SL* m*-fold with rate* λ'. *In particular, if* $\lambda \in (2-m, 0)$ *then* L *is an AC SL* m*-fold with rate* λ' *for all* $\lambda' \in (2-m, 0)$.

(b) *Suppose* $0 \leqslant \lambda < \min\bigl(\mathcal{D}_\Sigma \cap (0,\infty)\bigr)$. *Then adding a constant to* E *if necessary, for all* $\lambda' \in \bigl(\max(-2, 2-m), 0\bigr)$ *we have*

$$\bigl|\nabla^k E\bigr| = O(r^{\lambda'-k}) \quad \text{as } r \to \infty \text{ for all } k \geqslant 0. \tag{23}$$

Thus if $Y(L) = 0 = \gamma$ *then* L *is an AC SL* m*-fold with rate* λ', *and if* $Y(L) \neq 0 \neq \gamma$ *then* L *is an AC SL* m*-fold with rate* 0.

Here is the analogue of Theorem 3.11, proved in [7, Th. 7.5].

Theorem 4.8 *Suppose* L *is an AC SL* m*-fold in* $\mathbb{C}^m$ *with cone* C. *Let* $\Sigma, \iota, \zeta, U_C, \Phi_C, K, T, \varphi, \chi, \chi^1, \chi^2$ *be as in Theorem 4.5. Then making* T, K *larger if necessary, there exists an open tubular neighbourhood* $U_L \subset T^*L$ *of the zero section* L *in* T^*L, *such that under* $\mathrm{d}\varphi : T^*\bigl(\Sigma \times (T,\infty)\bigr) \to T^*L$ *we have*

$$(\mathrm{d}\varphi)^*(U_L) = \bigl\{(\sigma, r, \tau, u) \in T^*\bigl(\Sigma \times (T,\infty)\bigr) : \bigl|(\tau, u)\bigr| < \zeta r\bigr\}, \tag{24}$$

and there exists an embedding $\Phi_L : U_L \to \mathbb{C}^m$ *with* $\Phi_L|_L = \mathrm{id} : L \to L$ *and* $\Phi_L^*(\omega') = \hat\omega$, *where* $\hat\omega$ *is the canonical symplectic structure on* T^*L, *such that*

$$\Phi_L \circ \mathrm{d}\varphi(\sigma, r, \tau, u) \equiv \Phi_C\bigl(\sigma, r, \tau + \chi^1(\sigma, r), u + \chi^2(\sigma, r)\bigr) \tag{25}$$

for all $(\sigma, r, \tau, u) \in T^*\bigl(\Sigma \times (T,\infty)\bigr)$ *with* $|(\tau, u)| < \zeta r$, *computing* $|\,.\,|$ *using* $\iota^*(g')$.

In [7, Th. 7.10] we study the *bounded harmonic functions* on L.

Theorem 4.9 *Suppose L is an AC SL m-fold in $\mathbb{C}^m$, with cone C. Let Σ, T and φ be as in Theorem 4.5. Let $l = b^0(\Sigma)$, and $\Sigma^1, \ldots, \Sigma^l$ be the connected components of Σ. Let V be the vector space of bounded harmonic functions on L. Then $\dim V = l$, and for each $\mathbf{c} = (c^1, \ldots, c^l) \in \mathbb{R}^l$ there exists a unique $v^{\mathbf{c}} \in V$ such that for all $j = 1, \ldots, l$, $k \geqslant 0$ and $\beta \in (2-m, 0)$ we have*

$$\nabla^k\bigl(\varphi^*(v^{\mathbf{c}}) - c^j\bigr) = O\bigl(|\mathbf{c}|r^{\beta-k}\bigr) \quad \text{on } \Sigma^j \times (T, \infty) \text{ as } r \to \infty. \tag{26}$$

Note also that $V = \{v^{\mathbf{c}} : \mathbf{c} \in \mathbb{R}^l\}$ and $v^{(1,\ldots,1)} \equiv 1$.

5 Review of the main results of [9]

Our goal is to generalize the results of [9] to more complicated situations. This gave me a problem in writing this paper, as I want it to make sense on its own without constant reference to [9], but to control the length I don't want to reproduce large parts of [9] as introductory material here.

The solution I have adopted is to reproduce only the three major theorems from [9] in this section, with some supporting definitions and explanations. However, much of §6–§8 (for instance, Definitions 6.1–6.3 below) has in effect been copied from [9, §6–§7] and then modified. I hope this is more economical and readable than reproducing long definitions from [9] unchanged, and then explaining later how to change them.

The other way I save space is that if the proof of a result in [9] requires only superficial changes for the new situations in this paper, then I give only the result but not the proof here, or else make only brief comments on how to adapt the proof in [9].

5.1 An analytic existence result for SL m-folds

The results of [9] hinged upon an existence result [9, Th. 5.3] for compact SL m-folds proved using analysis. Here is some notation we will need.

Definition 5.1 Let (N, g) be a Riemannian manifold with Levi-Civita connection ∇. For each integer $k \geqslant 0$, define $C^k(N)$ to be the Banach space of functions f on N with k continuous derivatives, for which the norm $\|f\|_{C^k} = \sum_{j=0}^k \sup_N |\nabla^j f|$ is finite. Let $C^\infty(N) = \bigcap_{k\geqslant 0} C^k(N)$.

For $q \geqslant 1$, define the *Lebesgue space* $L^q(N)$ to be the Banach space of locally integrable functions f on N for which the norm $\|f\|_{L^q} = \bigl(\int_N |f|^q \,\mathrm{d}V_g\bigr)^{1/q}$ is finite. For $q \geqslant 1$ and $k \geqslant 0$ an integer, define the *Sobolev space* $L^q_k(N)$ to be the set of $f \in L^q(N)$ such that f is k times weakly differentiable and $|\nabla^j f| \in L^q(N)$ for $j \leqslant k$, with norm $\|f\|_{L^q_k} = \bigl(\sum_{j=0}^k \int_N |\nabla^j f|^q \,\mathrm{d}V_g\bigr)^{1/q}$.

The following definition [9, Def. 5.2] sets up the situation we shall consider.

Definition 5.2 Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold, with metric g. Let N be a compact, oriented, immersed, Lagrangian m-submanifold in M, with immersion $\iota : N \to M$, so that $\iota^*(\omega) \equiv 0$. Define $h = \iota^*(g)$, so that (N, h) is a Riemannian manifold. Let $\mathrm{d}V$ be the volume form on N induced by the metric h and orientation.

Let $\psi : M \to (0, \infty)$ be the smooth function given in (3). Then $\Omega|_N$ is a complex m-form on N, and using (3) and the Lagrangian condition we find that $\bigl|\Omega|_N\bigr| = \psi^m$, calculating $|\,.\,|$ using h on N. Therefore we may write

$$\Omega|_N = \psi^m \mathrm{e}^{i\theta}\,\mathrm{d}V \quad \text{on } N, \tag{27}$$

for some phase function $\mathrm{e}^{i\theta}$ on N. Suppose that $\cos\theta \geqslant \frac{1}{2}$ on N. Then we can choose θ to be a smooth function $\theta : N \to (-\frac{\pi}{3}, \frac{\pi}{3})$. Suppose that $[\iota^*(\mathrm{Im}\,\Omega)] = 0$ in $H^m(N, \mathbb{R})$. Then $\int_N \psi^m \sin\theta\,\mathrm{d}V = 0$, by (27).

Suppose we are given a finite-dimensional vector subspace $W \subset C^\infty(N)$ with $1 \in W$. Define $\pi_W : L^2(N) \to W$ to be the projection onto W using the L^2-inner product.

For $r > 0$, define $\mathcal{B}_r \subset T^*N$ to be the bundle of 1-forms α on N with $|\alpha| < r$. Regard $\mathcal{B}_r$ as a noncompact $2m$-manifold with natural projection $\pi : \mathcal{B}_r \to N$, whose fibre at $x \in N$ is the ball of radius r about 0 in T_x^*N. We will sometimes identify N with the zero section of $\mathcal{B}_r$, and write $N \subset \mathcal{B}_r$.

At each $y \in \mathcal{B}_r$ with $\pi(y) = x \in N$, the Levi-Civita connection ∇ of h on T^*N defines a splitting $T_y\mathcal{B}_r = H \oplus V$ into horizontal and vertical subspaces H, V, with $H \cong T_xN$ and $V \cong T_x^*N$. Write $\hat\omega$ for the natural symplectic structure on $\mathcal{B}_r \subset T^*N$, defined using $T\mathcal{B}_r \cong H \oplus V$ and $H \cong V^*$. Define a natural Riemannian metric $\hat h$ on $\mathcal{B}_r$ such that the subbundles H, V are orthogonal, and $\hat h|_H = \pi^*(h)$, $\hat h|_V = \pi^*(h^{-1})$.

Let $\hat\nabla$ be the connection on $T\mathcal{B}_r \cong H \oplus V$ given by the lift of the Levi-Civita connection ∇ of h on N in the horizontal directions H, and by partial differentiation in the vertical directions V, which is well-defined as $T\mathcal{B}_r$ is naturally trivial along each fibre. Then $\hat\nabla$ preserves $\hat h, \hat\omega$ and the splitting $T\mathcal{B}_r \cong H \oplus V$. It is *not* torsion-free in general, but has torsion $T(\hat\nabla)$ depending linearly on the Riemann curvature $R(h)$.

As N is a Lagrangian submanifold of M, by Theorem 3.7 the symplectic manifold (M, ω) is locally isomorphic near N to T^*N with its canonical symplectic structure. That is, for some small $r > 0$ there exists an immersion $\Phi : \mathcal{B}_r \to M$ such that $\Phi^*(\omega) = \hat\omega$ and $\Phi|_N = \iota$. Define an m-form β on $\mathcal{B}_r$ by $\beta = \Phi^*(\mathrm{Im}\,\Omega)$.

If $\alpha \in C^\infty(T^*N)$ with $|\alpha| < r$, write $\Gamma(\alpha)$ for the *graph* of α in $\mathcal{B}_r$. Then $\Phi_*(\Gamma(\alpha))$ is a compact, immersed submanifold in M diffeomorphic to N.

With this notation, here is the existence result [9, Th. 5.3].

Theorem 5.3 *Let $\kappa > 1$ and $A_1, \ldots, A_8 > 0$ be real, and $m \geqslant 3$ an integer. Then there exist $\epsilon, K > 0$ depending only on $\kappa, A_1, \ldots, A_8$ and m such that the following holds.*

Suppose $0 < t \leqslant \epsilon$ and Definition 5.2 holds with $r = A_1 t$, and

(i) $\|\psi^m \sin\theta\|_{L^{2m/(m+2)}} \leqslant A_2 t^{\kappa+m/2}$, $\|\psi^m \sin\theta\|_{C^0} \leqslant A_2 t^{\kappa-1}$, $\|\mathrm{d}(\psi^m \sin\theta)\|_{L^{2m}} \leqslant A_2 t^{\kappa-3/2}$ *and* $\|\pi_W(\psi^m \sin\theta)\|_{L^1} \leqslant A_2 t^{\kappa+m-1}$.

(ii) *$\psi \geqslant A_3$ on N.*

(iii) *$\|\hat\nabla^k\beta\|_{C^0} \leqslant A_4 t^{-k}$ for $k = 0, 1, 2$ and 3.*

(iv) *The injectivity radius $\delta(h)$ satisfies $\delta(h) \geqslant A_5 t$.*

(v) *The Riemann curvature $R(h)$ satisfies $\|R(h)\|_{C^0} \leqslant A_6 t^{-2}$.*

(vi) *If $v \in L^2_1(N)$ with $\pi_W(v) = 0$, then $v \in L^{2m/(m-2)}(N)$ by the Sobolev Embedding Theorem, and $\|v\|_{L^{2m/(m-2)}} \leqslant A_7\|\mathrm{d}v\|_{L^2}$.*

(vii) *For all $w \in W$ we have $\|\mathrm{d}^*\mathrm{d}w\|_{L^{2m/(m+2)}} \leqslant \frac{1}{2}A_7^{-1}\|\mathrm{d}w\|_{L^2}$.*
For all $w \in W$ with $\int_N w\,\mathrm{d}V = 0$ we have $\|w\|_{C^0} \leqslant A_8 t^{1-m/2}\|\mathrm{d}w\|_{L^2}$.

Here norms are computed using the metric h on N in (i), (v), (vi) *and* (vii), *and the metric $\hat h$ on $\mathcal{B}_{A_1 t}$ in* (iii). *Then there exists $f \in C^\infty(N)$ with $\int_N f\,\mathrm{d}V = 0$, such that $\|\mathrm{d}f\|_{C^0} \leqslant Kt^\kappa < A_1 t$ and $\tilde N = \Phi_*\bigl(\Gamma(\mathrm{d}f)\bigr)$ is an immersed special Lagrangian m-fold in (M, J, ω, Ω).*

Its proof in [9, §5] is long and technical. The basic idea is to write the equation for $\tilde N = \Phi_*\bigl(\Gamma(\mathrm{d}f)\bigr)$ to be special Lagrangian as a second-order nonlinear elliptic p.d.e. on f. Conditions (i)–(vii) ensure that this p.d.e. is close to being linear, in a certain sense. We then solve the p.d.e. for f by a series method, using facts about the solutions of second-order linear elliptic p.d.e.s.

On a first reading, Definition 5.2 and Theorem 5.3 may look like formidably technical abstract nonsense. We now try to explain (informally, and oversimplifying a bit) what the theorem does, and the reasons behind its design.

- The theorem says that given a Lagrangian m-fold N in (M, J, ω, Ω) which is *close to special Lagrangian* in a certain sense, we can deform N to a nearby SL m-fold $\tilde N$ in M by a small Hamiltonian deformation.
- A Lagrangian m-fold N has a *phase function* $\mathrm{e}^{i\theta}$, and N is special Lagrangian with some orientation if and only if $\sin\theta \equiv 0$. Part (i) of Theorem 5.3 requires four norms of $\psi^m \sin\theta$ to be small, so it forces N to be *close to special Lagrangian*.
- We shall apply Theorem 5.3 when N is an explicitly constructed desingularization of a singular SL m-fold. The construction depends on $t \in (0, \delta)$, the *length scale* at which the singularities are resolved. Thus, we actually construct a 1-*parameter family* of Lagrangian m-folds N^t for $t \in (0, \delta)$.

  When t is small, N^t is *close to special Lagrangian*, but it is also *close to singular*, in that the metric $h^t = g|_{N^t}$ on N^t has large Riemann curvature $R(h^t) = O(t^{-2})$, and small injectivity radius $\delta(h^t) = O(t)$.

  So the theorem is set up using a real parameter $t > 0$. Parts (iv), (v) say that (N, h) is *not too close to singular*, and part (iii) that the geometry of M near N is *not too close to singular*, in terms of t.

- When t is small N is close to special Lagrangian by (i), which is an *advantage*, but (iii)–(v) allow N to be close to singular, which is a *disadvantage*. The proof is a delicate balancing act between these two influences. The advantages win, and for small $t \leqslant \epsilon$ we can deform N to an SL m-fold $\tilde{N}$.
- Roughly speaking, to solve the p.d.e. on f we need an *inverse* Δ^{-1} *of the Laplacian* Δ on N, which should be *bounded independent of* t.

  However $\Delta 1 = 0$, so Δ is not invertible. Also, in our applications Δ has a finite number of *small positive eigenvalues* of size $O(t^{m-2})$, so that if it existed Δ^{-1} would be $O(t^{2-m})$, and not bounded independent of t.

  To get round this we introduce a vector space $W \subset C^\infty(N)$ with $1 \in W$, which approximates the eigenspaces of Δ with small eigenvalues. Part (vi) roughly says that $\Delta^{-1} = O(1)$ on $W^\perp$. Part (vii) roughly says that $\Delta^{-1} = O(t^{2-m})$ on the subspace of W orthogonal to 1.

  To control W-components of functions in the proof although Δ^{-1} is large on W, we assume in (i) that the W-component $\pi_W(\psi^m \sin\theta)$ is very small.

5.2 Theorems on desingularizing SL m-folds

In [9, §6–§7] we apply Theorem 5.3 to construct desingularizations of compact SL m-folds X in (M, J, ω, Ω) with conical singularities. The first main result is [9, Th. 6.13], which deals with the simplest case in which there are no obstructions to desingularization.

Theorem 5.4 *Suppose (M, J, ω, Ω) is an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Let $L_1, \ldots, L_n$ be Asymptotically Conical SL m-folds in $\mathbb{C}^m$ with cones $C_1, \ldots, C_n$ and rates $\lambda_1, \ldots, \lambda_n$. Suppose $\lambda_i < 0$ for $i = 1, \ldots, n$, and $X' = X \setminus \{x_1, \ldots, x_n\}$ is connected.*

Then there exists $\epsilon > 0$ and a smooth family $\{\tilde{N}^t : t \in (0, \epsilon]\}$ of compact, nonsingular SL m-folds in (M, J, ω, Ω), such that $\tilde{N}^t$ is constructed by gluing tL_i into X at x_i for $i = 1, \ldots, n$. In the sense of currents, $\tilde{N}^t \to X$ as $t \to 0$.

The second main result is [9, Th. 7.10]. It strengthens Theorem 5.4 by dropping the assumption that X' is connected. In doing this we encounter *topological obstructions*, so that desingularizations $\tilde{N}^t$ exist only if an equation (28) holds on topological invariants of X and $L_1, \ldots, L_n$.

Theorem 5.5 *Suppose (M, J, ω, Ω) is an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Define $\psi : M \to (0, \infty)$ as in (3). Let $L_1, \ldots, L_n$ be Asymptotically Conical SL m-folds in $\mathbb{C}^m$ with cones $C_1, \ldots, C_n$ and rates $\lambda_1, \ldots, \lambda_n$. Suppose $\lambda_i < 0$ for $i = 1, \ldots, n$. Write $X' = X \setminus \{x_1, \ldots, x_n\}$ and $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$.*

Set $q = b^0(X')$, and let $X'_1, \ldots, X'_q$ be the connected components of X'. For $i = 1, \ldots, n$ let $l_i = b^0(\Sigma_i)$, and let $\Sigma_i^1, \ldots, \Sigma_i^{l_i}$ be the connected components of

Σ_i. *Define* $k(i,j) = 1,\ldots,q$ *by* $\Upsilon_i \circ \varphi_i\big(\Sigma_i^j \times (0,R')\big) \subset X'_{k(i,j)}$ *for* $i = 1,\ldots,n$ *and* $j = 1,\ldots,l_i$. *Suppose that*

$$\sum_{\substack{1\leqslant i\leqslant n,\ 1\leqslant j\leqslant l_i:\\ k(i,j)=k}} \psi(x_i)^m Z(L_i)\cdot[\Sigma_i^j] = 0 \quad \textit{for all } k = 1,\ldots,q. \tag{28}$$

Suppose also that the compact m*-manifold* N *obtained by gluing* L_i *into* X' *at* x_i *for* $i = 1,\ldots,n$ *is connected. A sufficient condition for this to hold is that* X *and* L_i *for* $i = 1,\ldots,n$ *are connected.*

Then there exists $\epsilon > 0$ *and a smooth family* $\big\{\tilde N^t : t \in (0,\epsilon]\big\}$ *of compact, nonsingular SL* m*-folds in* (M,J,ω,Ω) *diffeomorphic to* N*, such that* $\tilde N^t$ *is constructed by gluing* tL_i *into* X *at* x_i *for* $i = 1,\ldots,n$*. In the sense of currents in Geometric Measure Theory,* $\tilde N^t \to X$ *as* $t \to 0$.

When X' is connected, so that $q = 1$, it turns out that (28) holds automatically and Theorem 5.5 reduces to Theorem 5.4. Theorems 5.4 and 5.5 are proved by the following method, which will also be used in §6–§8 below.

We shrink L_i by a small factor $t > 0$ to get tL_i, which is also an AC SL m-fold in $\mathbb{C}^m$. Using the Lagrangian neighbourhood results of §3.4 and §4.2 we glue tL_i into X at x_i using a partition of unity, to get a Lagrangian m-fold N^t in (M,ω) for $t \in (0,\delta)$. We also glue the Lagrangian neighbourhoods of X' and L_i together to get a Lagrangian neighbourhood Φ_{N^t} for N^t.

We define vector spaces $W^t \subset C^\infty(N^t)$, using spaces of bounded harmonic functions on L_i. We then show that N^t, W^t and Φ^t satisfy Definition 5.2 and parts (i)–(vii) of Theorem 5.3 for all $t \in (0,\delta)$, for some $\kappa > 1$ and $A_1,\ldots,A_8 > 0$ independent of t. Theorem 5.3 then gives $\epsilon > 0$ depending on $\kappa, A_1,\ldots,A_8$ such that N^t can be deformed to an SL m-fold $\tilde N^t$ if $t \leqslant \epsilon$.

Here is how the obstruction equation (28) arises. For Theorem 5.4 we take $W^t = \langle 1\rangle$, and then $\pi_{W^t}(\psi^m \sin\theta^t) = 0$ for topological reasons. But for Theorem 5.5 we have $W^t \cong \mathbb{R}^q$, and $\pi_{W^t}(\psi^m \sin\theta^t)$ need not be zero. Calculation shows that the leading contribution to $\pi_{W^t}(\psi^m \sin\theta^t)$ is $O(t^m)$, and is proportional to the left hand side of (28). For part (i) of Theorem 5.4 we need $\|\pi_{W^t}(\psi^m \sin\theta^t)\|_{L^1} \leqslant A_2 t^{\kappa+m-1}$, and this holds if and only if (28) does.

6 Desingularizing when $Y(L_i) \neq 0$

In Theorems 5.4 and 5.5 we desingularized an SL m-fold X with conical singularities using AC SL m-folds L_i with rates λ_i for $i = 1,\ldots,n$, where we assumed that $\lambda_i < 0$, so that $Y(L_i) = 0$ by Proposition 4.3. We now explain how to relax this to allow $\lambda_i \leqslant 0$ and $Y(L_i) \neq 0$. As in [10, §6.4] there are many examples of AC SL m-folds L with rate 0 and $Y(L) \neq 0$, so this significantly increases the scope of the main result.

Allowing $\lambda_i = 0$ complicates the proofs in two main ways. Firstly, gluing tL_i into X at x_i to make N^t is no longer a local matter. Instead, there is a *global* condition for N^t to exist as a *Lagrangian* m-fold, that $\big(Y(L_1),\ldots,Y(L_n)\big)$ should lie in the image of the map $H^1(X',\mathbb{R}) \to \bigoplus_{i=1}^n H^1(\Sigma_i,\mathbb{R})$ in (6).

Secondly, the modifications to the definition of N^t introduce extra error terms in $\operatorname{Im}\Omega|_{N^t}$, which contribute $O(t^4)$ to $\|\psi^m \sin\theta^t\|_{L^{2m/(m+2)}}$. But for part (i) of Theorem 5.3 to hold we need $\|\psi^m \sin\theta^t\|_{L^{2m/(m+2)}} = O(t^{\kappa+m/2})$ for $\kappa > 1$, so we must assume $m < 6$. Therefore our main result, Theorem 6.12, holds only in dimensions $m = 3, 4, 5$, rather than all $m \geqslant 3$ as in Theorem 5.4.

6.1 Setting up the problem

We shall consider the following situation, the analogue of [9, Def. 6.1].

Definition 6.1 Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold with metric g, and define $\psi : M \to (0,\infty)$ as in (3). Let X be a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$ with identifications υ_i, cones C_i and rates $\mu_i \in (2,3)$, as in Definition 3.4. Let $L_1, \ldots, L_n$ be AC SL m-folds in $\mathbb{C}^m$ as in Definition 4.1, where L_i has cone C_i. Define $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$ and $Y(L_i) \in H^1(\Sigma_i, \mathbb{R})$ and $Z(L_i) \in H^{m-1}(\Sigma_i, \mathbb{R})$ for $i = 1, \ldots, n$ as in Definition 4.2.

Set $q = b^0(X')$, so that X' has q connected components, and number them $X'_1, \ldots, X'_q$. For $i = 1, \ldots, n$ let $l_i = b^0(\Sigma_i)$, so that Σ_i has l_i connected components, and number them $\Sigma_i^1, \ldots, \Sigma_i^{l_i}$. If $\Upsilon_i, \varphi_i, S_i$ are as in Definition 3.4, then $\Upsilon_i \circ \varphi_i$ is a diffeomorphism $\Sigma_i \times (0, R') \to S_i \subset X'$. For each $j = 1, \ldots, l_i$, $\Upsilon_i \circ \varphi_i\bigl(\Sigma_i^j \times (0,R')\bigr)$ is a connected subset of X', and so lies in exactly one of the X'_k for $k = 1, \ldots, q$. Define numbers $k(i,j) = 1, \ldots, q$ for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$. by $\Upsilon_i \circ \varphi_i(\Sigma_i^j \times (0,R')) \subset X'_{k(i,j)}$. Suppose that:

(i) The dimension m satisfies $2 < m < 6$,

(ii) The AC SL m-fold L_i has rate $\lambda_i \leqslant 0$ for $i = 1, \ldots, n$, and

(iii) There exists $\varrho \in H^1(X', \mathbb{R})$ such that $\bigl(Y(L_1), \ldots, Y(L_n)\bigr)$ is the image of ϱ under the map $H^1(X', \mathbb{R}) \to \bigoplus_{i=1}^n H^1(\Sigma_i, \mathbb{R})$ in (6).

(iv) $\displaystyle\sum_{\substack{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l_i: \\ k(i,j) = k}} \psi(x_i)^m Z(L_i) \cdot [\Sigma_i^j] = 0$ for all $k = 1, \ldots, q$.

(v) Let N be the compact m-manifold obtained by gluing L_i into X' at x_i for $i = 1, \ldots, n$ in the obvious way. Suppose N is connected. A sufficient condition for this to hold is that X and L_i for $i = 1, \ldots, n$ are connected.

We use the following notation:

- Let R, B_R, X' and ι_i, Υ_i for $i = 1, \ldots, n$ be as in Definition 3.4.
- Let ζ and U_{C_i}, Φ_{C_i} for $i = 1, \ldots, n$ be as in Theorem 3.8.
- Let R', K and $\phi_i, \eta_i, \eta_i^1, \eta_i^2, S_i$ for $i = 1, \ldots, n$ be as in Theorem 3.9.
- Let $U_{X'}, \Phi_{X'}$ be as in Theorem 3.11.
- Let A_i be as in Theorem 3.10 for $i = 1, \ldots, n$, so that $\eta_i = \mathrm{d}A_i$.
- For $i = 1, \ldots, n$ let γ_i be the unique 1-form on Σ_i with $\mathrm{d}\gamma_i = \mathrm{d}^*\gamma_i = 0$ and $[\gamma_i] = Y(L_i)$ in $H^1(\Sigma_i, \mathbb{R})$, which exists by Hodge theory.

Let $\pi_i : \Sigma_i \times (0,\infty) \to \Sigma_i$ be the projection, so that $\pi_i^*(\gamma_i)$ is a 1-form on $\Sigma_i \times (0,\infty)$ with $\bigl|\pi_i^*(\gamma_i)\bigr| = O(r^{-1})$ and $\mathrm{d}\bigl(\pi_i^*(\gamma_i)\bigr) = \mathrm{d}^*\bigl(\pi_i^*(\gamma_i)\bigr) = 0$, computing $|\,.\,|$ and d^* using the cone metric $\iota_i^*(g')$ on $\Sigma_i \times (0,\infty)$.

- Let $Y_{X'}$ be as in (9), and $\alpha \in Y_{X'}$ the unique element with $\pi(\alpha) = \varrho$. Then $\mathrm{d}\alpha = \mathrm{d}^*(\psi^m \alpha) = 0$. For $i = 1,\ldots,n$ let $T_i \in C^\infty\bigl(\Sigma_i \times (0,R')\bigr)$ be as in part (a) of Theorem 3.6. Choose $\nu_i \in (0, \mu_i - 2)$ with $(0,\nu_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset$. Then part (a) of Theorem 3.6 shows that

$$(\Upsilon_i \circ \phi_i)^*(\alpha) = \pi_i^*(\gamma_i) + \mathrm{d}T_i \quad \text{on } \Sigma_i \times (0,R') \text{ for } i = 1,\ldots,n, \tag{29}$$

$$\text{and} \quad \nabla^k T_i(\sigma,r) = O(r^{\nu_i - k}) \quad \text{as } r \to 0, \text{ for all } k \geqslant 0. \tag{30}$$

- Apply Theorem 4.5 to L_i with $\zeta, U_{C_i}, \Phi_{C_i}$ as above, for $i = 1,\ldots,n$. Let $T > 0$ be as in the theorem, the same for all i. Let the subset $K_i \subset L_i$, the diffeomorphism $\varphi_i : \Sigma_i \times (T,\infty) \to L_i \setminus K_i$ and the 1-form χ_i on $\Sigma_i \times (T,\infty)$ with components χ_i^1, χ_i^2 be as in Theorem 4.5.

- Let U_{L_i}, Φ_{L_i} be as in Theorem 4.8 for $i = 1,\ldots,n$.

- Let $E_i \in C^\infty\bigl(\Sigma_i \times (T,\infty)\bigr)$ be as in Theorem 4.6 for $i = 1,\ldots,n$. Fix $\lambda \in \bigl(\max(-2, 2-m), \frac{1}{2}(2-m)\bigr)$. (This interval is nonempty as $2 < m < 6$.) Then Theorem 4.6 and part (b) of Theorem 4.7 show that

$$\chi_i = \pi_i^*(\gamma_i) + \mathrm{d}E_i \quad \text{on } \Sigma_i \times (T,\infty) \text{ for } i = 1,\ldots,n \tag{31}$$

$$\text{and} \quad \nabla^k E_i(\sigma,r) = O(r^{\lambda - k}) \quad \text{as } r \to \infty \text{ for all } k \geqslant 0. \tag{32}$$

By (29) and (31) we see that the 1-forms α on X' and χ_i on L_i both have leading term $\pi_i^*(\gamma_i)$ in their asymptotic expansion on the cone C_i. The construction below will work by matching up these terms $\pi_i^*(\gamma_i)$, so that we do not have to taper them off to zero.

Here are some remarks on conditions (i)–(v) above.

(i) It will turn out in §6.2 that defining the Lagrangian m-folds N^t when $Y(L_i) \neq 0$ introduces $O(t^4)$ error terms in $\|\psi^m \sin\theta^t\|_{L^{2m/(m+2)}}$. However, for part (i) of Theorem 5.3 to hold we need $\|\psi^m \sin\theta^t\|_{L^{2m/(m+2)}} = O(t^{\kappa + m/2})$ for $\kappa > 1$.

Thus we need $\kappa + m/2 \leqslant 4$ and $\kappa > 1$, giving $m < 6$, which is why we suppose $m < 6$ in (i) above. We also use $2 < m < 6$ in choosing $\lambda \in \bigl(\max(-2, 2-m), \frac{1}{2}(2-m)\bigr)$. With some more work on the definition of N^t, the result can probably be extended to the case $m = 6$.

(ii) The point of this section is to relax the assumption $\lambda_i < 0$ in Theorems 5.4 and 5.5, and so allow $Y(L_i) \neq 0$. We suppose $\lambda_i \leqslant 0$ so that part (b) of Theorem 4.7 applies. Although going from $\lambda_i < 0$ to $\lambda_i \leqslant 0$ may not seem like much of an improvement, in fact as in [10, §6.4] there are many examples of AC SL m-folds L with rate exactly $\lambda = 0$, so allowing $\lambda = 0$ will make our results much more useful.

(iii) The condition that $\big(Y(L_1),\ldots,Y(L_n)\big)$ lies in the image of $H^1(X',\mathbb{R})\rightarrow\bigoplus_{i=1}^n H^1(\Sigma_i,\mathbb{R})$ is necessary for the existence of any *Lagrangian* m-fold N^t made by gluing tL_i into X at x_i for $i=1,\ldots,n$. So it is clearly also necessary for the existence of an SL m-fold $\tilde N^t$ made in the same way.

(iv) This condition was introduced in [9, Def. 7.1], to deal with analytic problems in desingularizing X when X' is not connected. As in [9, §7.4], when X' is connected $q=1$ and (iv) holds automatically since $Z(L_i)\cdot[\Sigma_i]=0$.

(v) We assume this for simplicity. If N is not connected then we can apply Theorem 6.12 below to each component of N separately.

Here is the analogue of [9, Def.s 6.2 & 6.3], constructing N^t for $t\in(0,\delta)$.

Definition 6.2 Choose a smooth, increasing $F:(0,\infty)\rightarrow[0,1]$ with $F(r)\equiv 0$ for $r\in(0,1)$ and $F(r)\equiv 1$ for $r>2$, and write $F'=\mathrm{d}F/\mathrm{d}r$. Let τ satisfy

$$0<\max\Big\{\frac{m}{m+1},\frac{m+2}{2\mu_i+m-2},\frac{m-2}{2\nu_i+m-2}:i=1,\ldots,n\Big\}<\tau<1, \tag{33}$$

which is possible as $\mu_i>2$, $\nu_i>0$ and $m>2$. For $i=1,\ldots,n$ and small enough $t>0$, define $P_i^t=\Upsilon_i(tK_i)$. This is well-defined, Lagrangian in (M,ω), and diffeomorphic to K_i.

For $i=1,\ldots,n$ and $t>0$ with $tT<t^\tau<2t^\tau<R'$ define a 1-form ξ_i^t on $\Sigma_i\times(tT,R')$ by

$$\begin{aligned}\xi_i^t(\sigma,r)&=\mathrm{d}\big[F(t^{-\tau}r)A_i(\sigma,r)+t^2(1-F(t^{-\tau}r))E_i(\sigma,t^{-1}r)\big]\\&\quad+t^2\pi_i^*(\gamma_i)+t^2\mathrm{d}\big[F(t^{-\tau}r)T_i(\sigma,r)\big]\\&=F(t^{-\tau}r)\eta_i(\sigma,r)+t^{-\tau}F'(t^{-\tau}r)A_i(\sigma,r)\mathrm{d}r\\&\quad+t^2(1-F(t^{-\tau}r))\chi_i(\sigma,t^{-1}r)-t^{2-\tau}F'(t^{-\tau}r)E_i(\sigma,t^{-1}r)\mathrm{d}r\\&\quad+t^2F(t^{-\tau}r)(\Upsilon_i\circ\phi_i)^*(\alpha)(\sigma,r)+t^{2-\tau}F'(t^{-\tau}r)T_i(\sigma,r)\mathrm{d}r,\end{aligned}\tag{34}$$

by (29) and (31). Let $\xi_i^{1,t},\xi_i^{2,t}$ be the components of ξ_i^t in $T^*\Sigma$ and $\mathbb{R}$. Then when $r\geqslant 2t^\tau$ we have $\xi_i^t\equiv\eta_i+t^2(\Upsilon_i\circ\phi_i)^*(\alpha)$, and when $r\leqslant t^\tau$ we have $\xi_i^t(\sigma,r)=t^2\chi_i(\sigma,t^{-1}r)$. Thus ξ_i^t is a closed 1-form which interpolates between $\eta_i+t^2(\Upsilon_i\circ\phi_i)^*(\alpha)$ near $r=R'$ and $t^2\chi_i(\sigma,t^{-1}r)$ near $r=tT$.

Choose $\delta\in(0,1]$ with $\delta T\leqslant\delta^\tau<2\delta^\tau\leqslant R'$, $\delta K_i\subset B_R\subset\mathbb{C}^m$ and $\big|\xi_i^t(\sigma,r)\big|<\zeta r$ on $\Sigma_i\times(tT,R')$ for all $i=1,\ldots,n$ and $t\in(0,\delta)$. As in [9, Def. 6.2], this is possible. Define $\Xi_i^t:\Sigma_i\times(tT,R')\rightarrow M$ by

$$\Xi_i^t(\sigma,r)=\Upsilon_i\circ\Phi_{C_i}\big(\sigma,r,\xi_i^{1,t}(\sigma,r),\xi_i^{2,t}(\sigma,r)\big)$$

for $i=1,\ldots,n$ and $t\in(0,\delta)$. Making R' smaller if necessary, this is well-defined, and an embedding. Define $Q_i^t=\Xi_i^t\big(\Sigma_i\times(tT,R')\big)$ for $i=1,\ldots,n$ and $t\in(0,\delta)$. As $\Upsilon_i^*(\omega)=\omega'$, $\Phi_{C_i}^*(\omega')=\hat\omega$ and ξ_i^t is a closed 1-form we see that $(\Xi_i^t)^*(\omega)\equiv 0$. Thus Q_i^t is *Lagrangian* in (M,ω), and is a noncompact embedded submanifold diffeomorphic to $\Sigma_i\times(tT,R')$.

Let $\Gamma(t^2\alpha)$ be the graph of the 1-form $t^2\alpha$ in T^*X'. Then $\Gamma(t^2\alpha)\cap\pi^*(K)\subset T^*K$ is the graph of $t^2\alpha|_K$. Recall that $U_{X'}$ is an open neighbourhood of the zero section in T^*X'. By compactness of K, making δ smaller if necessary, we can suppose that $\Gamma(t^2\alpha)\cap\pi^*(K)\subset U_{X'}$ for all $t\in(0,\delta)$. Define

$$K^t=\Phi_{X'}\bigl(\Gamma(t^2\alpha)\cap\pi^*(K)\bigr)\quad\text{for } t\in(0,\delta). \tag{35}$$

Then K^t is a submanifold of M with boundary, diffeomorphic to K. As α is closed $\Gamma(t^2\alpha)$ is Lagrangian in $(T^*X',\hat\omega)$, and $\Phi_{X'}^*(\omega)=\hat\omega$, so K^t is *Lagrangian* in (M,ω). For $t\in(0,\delta)$, define N^t to be the disjoint union of K^t, $P_1^t,\ldots,P_n^t$ and $Q_1^t,\ldots,Q_n^t$.

Then N^t is *Lagrangian* in (M,ω), as K^t,P_i^t and Q_i^t are. Moreover, N^t is a compact, smooth submanifold of M *without boundary*. To see this, note that $\xi_i^t(\sigma,r)=t^2\chi_i(\sigma,t^{-1}r)$ on $\Sigma_i\times(tT,t^\tau]$, and so

$$\begin{aligned}\Xi_i^t(\sigma,r)&=\Upsilon_i\circ\Phi_{C_i}\bigl(\sigma,r,t^2\chi_i^1(\sigma,t^{-1}r),t\chi_i^2(\sigma,t^{-1}r)\bigr)\\&=\Upsilon_i\bigl(t\,\Phi_{C_i}\bigl(\sigma,t^{-1}r,\chi_i^1(\sigma,t^{-1}r),\chi_i^2(\sigma,t^{-1}r)\bigr)\bigr)=\Upsilon_i\bigl(t\,\varphi_i(\sigma,t^{-1}r)\bigr)\end{aligned}$$

on $\Sigma_i\times(tT,t^\tau]$, using (20) and the dilation equivariance of Φ_{C_i} in Theorem 3.8.

Thus the end $\Xi_i^t\bigl(\Sigma_i\times(tT,t^\tau]\bigr)$ of Q_i^t is $\Upsilon_i\bigl(t\,\varphi_i(\Sigma_i\times(T,t^{\tau-1}])\bigr)\subset\Upsilon_i(tL_i)$, and as $\varphi_i\bigl(\Sigma_i\times(T,t^{\tau-1}]\bigr)\subset L_i$ joins smoothly onto $K_i\subset L_i$ we see that $\Xi_i^t\bigl(\Sigma_i\times(tT,t^\tau]\bigr)\subset Q_i^t$ joins smoothly onto $P_i^t=\Upsilon_i(tK_i)$. Similarly, the boundary ∂K^t is the disjoint union of pieces Σ_i for $i=1,\ldots,n$ which join smoothly onto $Q_i^t\cong\Sigma_i\times(tT,R')$ at the $\Sigma_i\times\{R'\}$ end, as K^t is the graph of $t^2\alpha$ over K and $\xi_i^t\equiv\eta_i+t^2(\Upsilon_i\circ\phi_i)^*(\alpha)$ on $\Sigma_i\times[2t^\tau,R')$.

As X',L_i are SL m-folds they are oriented, and N^t is made by gluing $X',L_1,\ldots,L_n$ together in an orientation-preserving way, so N^t is also oriented. Let h^t be the restriction of g to N^t for $t\in(0,\delta)$, so that (N^t,h^t) is a compact Riemannian manifold, and let $\mathrm{d}V^t$ be the induced volume form on N^t. Then $\Omega|_{N^t}=\psi^m e^{i\theta^t}\,\mathrm{d}V^t$ for some phase function $e^{i\theta^t}$ on N^t. Write $\varepsilon^t=\psi^m\sin\theta^t$, so that $\operatorname{Im}\Omega|_{N^t}=\varepsilon^t\,\mathrm{d}V^t$ for $t\in(0,\delta)$.

From (34) we can explain the reason for condition (iii) in Definition 6.1. The problem is that we need to define a *closed* 1-form ξ_i^t on $\Sigma_i\times(tT,R')$ which interpolates between $t^2\chi_i(\sigma,t^{-1}r)$ near $r=tT$ and $\eta_i+t^2(\Upsilon_i\circ\phi_i)^*(\alpha)$ near $r=R'$. This is possible if and only if $t^2\chi_i(\sigma,t^{-1}r)$ and $\eta_i+t^2(\Upsilon_i\circ\phi_i)^*(\alpha)$ have the *same cohomology class* in $H^1\bigl(\Sigma_i\times(tT,R'),\mathbb{R}\bigr)\cong H^1(\Sigma_i,\mathbb{R})$.

The cohomology class of $\chi_i(\sigma,t^{-1}r)$ is $Y(L_i)\in H^1(\Sigma_i,\mathbb{R})$ by Theorem 4.6, and $\eta_i=\mathrm{d}A_i$ is exact. Hence the cohomology class of $(\Upsilon_i\circ\phi_i)^*(\alpha)$ in $H^1(\Sigma_i,\mathbb{R})$ must be $Y(L_i)$. That is, the cohomology class $[\alpha]=\varrho\in H^1(X',\mathbb{R})$ must have image $Y(L_i)$ in $H^1(\Sigma_i,\mathbb{R})$ for all $i=1,\ldots,n$ under the map $H^1(X',\mathbb{R})\rightarrow\bigoplus_{i=1}^n H^1(\Sigma_i,\mathbb{R})$ in (6), which gives (iii). This is a *necessary condition* for N^t to exist as a Lagrangian m-fold.

Following [9, Def. 7.2] we define a vector subspace $W^t\subset C^\infty(N^t)$. This will be W in Definition 5.2. It is an approximation to the eigenspaces of the Laplacian Δ on N^t with small eigenvalues, and is a tool to deal with some analytic problems when X' is not connected.

Definition 6.3 We work in the situation of Definitions 6.1 and 6.2. For $i=1,\ldots,n$ apply Theorem 4.9 to the AC SL m-fold L_i in $\mathbb{C}^m$, using the numbering Σ_i^j chosen in Definition 6.1 for the connected components of Σ_i. This gives a vector space V_i of bounded harmonic functions on L_i with $\dim V_i = l_i$. For each $\mathbf{c}_i = (c_i^1, \ldots, c_i^{l_i}) \in \mathbb{R}^{l_i}$ there exists a unique $v_i^{\mathbf{c}_i} \in V_i$ with

$$\nabla^k\bigl(\varphi_i^*(v_i^{\mathbf{c}_i}) - c_i^j\bigr) = O\bigl(|\mathbf{c}_i| r^{\beta-k}\bigr) \quad \text{on } \Sigma_i^j \times (T,\infty) \text{ as } r \to \infty, \tag{36}$$

for all $i = 1, \ldots, n$, $j = 1, \ldots, l_i$, $k \geqslant 0$ and $\beta \in (2-m, 0)$.

We shall define a vector subspace $W^t \subset C^\infty(N^t)$ for $t \in (0,\delta)$, with an isomorphism $W^t \cong \mathbb{R}^q$. Fix $\mathbf{d} = (d_1, \ldots, d_q) \in \mathbb{R}^q$, and set $c_i^j = d_{k(i,j)}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$. Let $\mathbf{c}_i = (c_i^1, \ldots, c_i^{l_i})$. This defines vectors $\mathbf{c}_i \in \mathbb{R}^{l_i}$ for $i = 1, \ldots, n$, which depend linearly on $\mathbf{d}$. Hence we have harmonic functions $v_i^{\mathbf{c}_i} \in V_i \subset C^\infty(L_i)$, which also depend linearly on $\mathbf{d}$.

Let $F : (0,\infty) \to [0,1]$ and $\tau \in (0,1)$ be as in Definition 6.2. Make $\delta > 0$ smaller if necessary so that $tT < \frac{1}{2}t^\tau$ for all $t \in (0,\delta)$. For $t \in (0,\delta)$, define a function $w_{\mathbf{d}}^t \in C^\infty(N^t)$ as follows:

(i) The subset $K^t \subset N^t$ is diffeomorphic to K, and so has q connected components K_k^t diffeomorphic to $K \cap X_k'$ for $k = 1, \ldots, q$. Define $w_{\mathbf{d}}^t$ on K^t by $w_{\mathbf{d}}^t \equiv d_k$ on K_k^t for $k = 1, \ldots, q$.

(ii) Define $w_{\mathbf{d}}^t$ on $P_i^t \subset N^t$ by $(\Upsilon_i \circ t \circ \varphi_i)^*(w_{\mathbf{d}}^t) \equiv v_i^{\mathbf{c}_i}$ on K_i for $i = 1, \ldots, n$.

(iii) Define $w_{\mathbf{d}}^t$ on $Q_i^t \subset N^t$ by

$$(\Xi_i^t)^*(w_{\mathbf{d}}^t)(\sigma, r) = \bigl(1 - F(2t^{-\tau}r)\bigr)\varphi_i^*(v_i^{\mathbf{c}_i})(\sigma, t^{-1}r) + F(2t^{-\tau}r)c_i^j \tag{37}$$

on $\Sigma_i^j \times (tT, R')$, for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$.

It is easy to see that $w_{\mathbf{d}}^t$ is smooth over the joins between P_i^t, Q_i^t and K^t, so $w_{\mathbf{d}}^t \in C^\infty(N^t)$. Also $w_{\mathbf{d}}^t$ is linear in $\mathbf{d}$, as $v_i^{\mathbf{c}_i}$ is. Thus $W^t = \{w_{\mathbf{d}}^t : \mathbf{d} \in \mathbb{R}^q\}$ is a vector subspace of $C^\infty(N^t)$ isomorphic to $\mathbb{R}^q$, for all $t \in (0,\delta)$.

If $\mathbf{d} = (1, \ldots, 1)$ then $c_i^j \equiv 1$, so $\mathbf{c}_i = (1, \ldots, 1)$ for $i = 1, \ldots, n$, and thus $v_i^{\mathbf{c}_i} \equiv 1$ for $i = 1, \ldots, n$ by Theorem 4.9. Therefore $w_{(1,\ldots,1)}^t \equiv 1$ by (i)–(iii) above, and $1 \in W^t$ for all $t \in (0,\delta)$. This corresponds to the condition $1 \in W$ in Definition 5.2. Define $\pi_{W^t} : L^2(N^t) \to W^t$ to be the projection onto W^t using the L^2-inner product, as for π_W in Definition 5.2.

Note that if X' is connected, so that $q = 1$, then $W^t = \langle 1 \rangle$.

6.2 Estimating $\operatorname{Im}\Omega|_{N^t}$

We now prove estimates for $\operatorname{Im}\Omega|_{N^t} = \varepsilon^t \, \mathrm{d}V^t$, following [9, §6.2]. First we compute bounds for ε^t at each point, as in [9, Prop. 6.4].

Proposition 6.4 *In the situation above, making $\delta > 0$ smaller if necessary,* *there exists* $C>0$ *such that for all* $t\in(0,\delta)$ *we have*

$$\bigl|(\Xi_i^t)^*(\varepsilon^t)\bigr|(\sigma,r)\leqslant\begin{cases}Cr, & r\in(tT,t^\tau],\\ Ct^{4-4\tau}+Ct^{\tau(\mu_i-2)}+\\ Ct^{(1-\tau)(2-\lambda)}+Ct^{2+\tau(\nu_i-2)}, & r\in(t^\tau,2t^\tau),\\ Ct^4r^{-4}, & r\in[2t^\tau,R'),\end{cases}\tag{38}$$

$$\bigl|(\Xi_i^t)^*(\mathrm{d}\varepsilon^t)\bigr|(\sigma,r)\leqslant\begin{cases}C, & r\in(tT,t^\tau],\\ Ct^{4-5\tau}+Ct^{\tau(\mu_i-3)}+\\ Ct^{(1-\tau)(2-\lambda)-\tau}+Ct^{2+\tau(\nu_i-3)}, & r\in(t^\tau,2t^\tau),\\ Ct^4r^{-5}, & r\in[2t^\tau,R'),\end{cases}\tag{39}$$

$$|\varepsilon^t|\leqslant Ct^4,\quad |\mathrm{d}\varepsilon^t|\leqslant Ct^4\quad \textit{on } K^t,\tag{40}$$

$$\textit{and}\quad |\varepsilon^t|\leqslant Ct,\quad |\mathrm{d}\varepsilon^t|\leqslant C\quad \textit{on } P_i^t \textit{ for all } i=1,\dots,n.\tag{41}$$

Here $|\,.\,|$ *is computed using* $(\Xi_i^t)^*(h^t)$ *in* (39) *and* h^t *in* (40) *and* (41).

Proof. As $\Upsilon_i^*(\operatorname{Im}\Omega)$ is a smooth m-form on B_R and $\Upsilon_i^*(\operatorname{Im}\Omega)|_0=\upsilon_i^*(\operatorname{Im}\Omega)=\psi(x_i)^m\operatorname{Im}\Omega'$ by Definition 3.4, we see that $\Upsilon_i^*(\operatorname{Im}\Omega)=\psi(x_i)^m\operatorname{Im}\Omega'+O(r)$ on B_R, by Taylor's Theorem. Since tL_i is special Lagrangian in $\mathbb{C}^m$ we have $\operatorname{Im}\Omega'|_{tL_i}\equiv 0$. Thus

$$\bigl|\Upsilon_i^*(\operatorname{Im}\Omega)|_{tL_i}\bigr|=O(r)\quad\text{on } tL_i\cap B_R,\tag{42}$$

computing $|\,.\,|$ using the metric $\Upsilon_i^*(g)$ on B_R, restricted to tL_i.

Now N^t coincides with $\Upsilon_i(tL_i)$ on P_i^t and $\Xi_i^t\bigl(\Sigma_i\times(tT,t^\tau]\bigr)$, so $\varepsilon^t\mathrm{d}V^t=\operatorname{Im}\Omega|_{\Upsilon_i(tL_i)}$ on these regions. As h^t is the restriction of g to N^t we have $|\mathrm{d}V^t|=1$, computing $|\,.\,|$ using g, so

$$\bigl|\Upsilon_i^*(\varepsilon^t)\bigr|=\bigl|\Upsilon_i^*(\operatorname{Im}\Omega)|_{tL_i}\bigr|\quad\text{on } t\bigl(K\cup\varphi_i(\Sigma_i\times(T,t^{\tau-1}]\bigr)\subset tL_i\cap B_R.\tag{43}$$

Combining (42) and (43) gives $\bigl|\varepsilon^t\bigr|=O\bigl((\Upsilon_i)_*(r)\bigr)$ on P_i^t and $\Xi_i^t\bigl(\Sigma_i\times(tT,t^\tau]\bigr)$. As $(\Upsilon_i)_*(r)=O(t)$ on P_i^t, we see that

$$\bigl|(\Xi_i^t)^*(\varepsilon^t)\bigr|(\sigma,r)=O(r)\quad\text{for } r\in(tT,t^\tau],\text{ and}\quad |\varepsilon^t|=O(t)\quad\text{on } P_i^t.\tag{44}$$

A similar argument for the derivative $\mathrm{d}\varepsilon^t$ gives

$$\bigl|(\Xi_i^t)^*(\mathrm{d}\varepsilon^t)\bigr|(\sigma,r)=O(1)\quad\text{for } r\in(tT,t^\tau],\text{ and}\quad |\mathrm{d}\varepsilon^t|=O(1)\quad\text{on } P_i^t.\tag{45}$$

In [8, Prop. 2.10] we show that if N is a compact SL m-fold in M and $\tilde N$ is a nearby Lagrangian m-fold written as the graph of a C^1 small closed 1-form α on N using Theorem 3.7, and $\operatorname{Im}\Omega|_{\tilde N}=\psi^m\sin\theta\,\mathrm{d}V$, then

$$\psi^m\sin\theta=-\mathrm{d}^*(\psi^m\alpha)+O(|\alpha|^2)+O(|\nabla\alpha|^2)\quad\text{when } |\alpha|,|\nabla\alpha| \text{ are small.}\tag{46}$$

This is extended in [8, Prop. 6.3] to the case in which X is a compact SL m-fold with conical singularities, and α a small 1-form on X'.

Now on K^t and the annuli $\Xi_i^t\bigl(\Sigma_i \times [2t^\tau, R')\bigr)$ for $i = 1,\ldots,n$, N^t is the graph of a small closed 1-form $t^2\alpha$ on X'. Abusing notation, identify K^t and $\Xi_i^t\bigl(\Sigma_i \times [2t^\tau, R')\bigr)$ with the corresponding regions in X', so that ψ, α, ρ make sense on these regions in N^t. Then [8, Prop. 6.3] shows that

$$\varepsilon^t = \psi^m \sin\theta^t = -\mathrm{d}^*(\psi^m t^2\alpha) + O(\rho^{-2}|t^2\alpha|^2) + O(|t^2\nabla\alpha|^2) \tag{47}$$

on K^t and $\Xi_i^t\bigl(\Sigma_i \times [2t^\tau, R')\bigr)$ when $\rho^{-1}|t^2\alpha|, |t^2\nabla\alpha|$ are small. Here $\rho : X' \to (0,1]$ is a *radius function*, as in Definition 3.5.

Since $|\nabla^k\alpha| = \rho^{-1-k}$ by (9) as $\alpha \in Y_{X'}$, we see that $\rho^{-1}|t^2\alpha|, |t^2\nabla\alpha|$ are $O(t^{2-2\tau})$ on K^t and $\Xi_i^t\bigl(\Sigma_i \times [2t^\tau, R')\bigr)$, so (47) holds, giving

$$\varepsilon^t = O(t^4) \text{ on } K^t, \text{ and } \bigl|(\Xi_i^t)^*(\varepsilon^t)\bigr| = O(t^4 r^{-4}) \text{ on } \Xi_i^t\bigl(\Sigma_i \times [2t^\tau, R')\bigr), \tag{48}$$

as $\mathrm{d}^*(\psi^m\alpha) = 0$. By a similar argument for derivatives we obtain

$$\mathrm{d}\varepsilon^t = O(t^4) \text{ on } K^t, \text{ and } \bigl|(\Xi_i^t)^*(\mathrm{d}\varepsilon^t)\bigr| = O(t^4 r^{-5}) \text{ on } \Xi_i^t\bigl(\Sigma_i \times [2t^\tau, R')\bigr). \tag{49}$$

On the annuli $\Xi_i^t\bigl(\Sigma_i \times (t^\tau, 2t^\tau)\bigr)$ we can apply the same argument, but we have to be rather more careful. Here N^t is not the graph of $t^2\alpha$ over X', but instead the graph of the 1-form $(\Upsilon_i \circ \phi_i)_*(\xi_i^t - \eta_i)$ on the corresponding annulus in X'. From (34) we find that

$$\begin{aligned}(\xi_i^t - \eta_i)(\sigma, r) = {}& t^2(\Upsilon_i \circ \phi_i)^*(\alpha)(\sigma, r)\\ &+ \mathrm{d}\bigl[\bigl(1 - F(t^{-\tau}t)\bigr)\bigl(t^2 E_i(\sigma, t^{-1}r) - A_i(\sigma, r) - t^2 T_i(\sigma, r)\bigr)\bigr].\end{aligned} \tag{50}$$

So applying [8, Prop. 6.3] again as in (47) gives

$$\begin{aligned}(\Xi_i^t)^*(\varepsilon^t)(\sigma, r) = {}& O\bigl(r^{-2}|\xi_i^t - \eta_i|^2\bigr) + O\bigl(|\nabla(\xi_i^t - \eta_i)|^2\bigr)\\ &- \mathrm{d}^*\bigl(\psi^m \mathrm{d}\bigl[\bigl(1 - F(t^{-\tau}t)\bigr)\bigl(t^2 E_i(\sigma, t^{-1}r) - A_i(\sigma, r) - t^2 T_i(\sigma, r)\bigr)\bigr]\bigr)\end{aligned} \tag{51}$$

on $\Sigma_i \times (t^\tau, 2t^\tau)$, provided $r^{-1}|\xi_i^t - \eta_i|$ and $|\nabla(\xi_i^t - \eta_i)|$ are small. Here $|\,.\,|$ and d^* are computed using the metric $(\Upsilon_i \circ \phi_i)^*(g)$, and we have used $\mathrm{d}^*(\psi^m\alpha) = 0$ to eliminate the term $-\mathrm{d}^*\bigl(\psi^m t^2(\Upsilon_i \circ \phi_i)^*(\alpha)\bigr)$.

It is important that there are *no linear terms* in $t^2(\Upsilon_i \circ \phi_i)^*(\alpha)$ or $t^2\pi_i^*(\gamma_i)$ in (51). If we had applied the cruder method used in [9, Prop. 6.4] there would have been such linear terms, which would have contributed $O(t^{2-2\tau})$ to $|(\Xi_i^t)^*(\varepsilon^t)|$ on $\Sigma_i \times (t^\tau, 2t^\tau)$ and $O(t^{2+\tau(m/2-1)})$ to $\|\varepsilon^t\|_{L^{2m/(m+2)}}$, which is too large for part (i) of Theorem 5.3 to hold.

Using (34) to write terms in (51) in terms of γ_i, A_i, E_i, T_i and estimating these using (14), (30) and (32), we find that $r^{-1}|\xi_i^t - \eta_i|$ and $|\nabla(\xi_i^t - \eta_i)|$ are $O(t^{2-2\tau})$ on $\Sigma_i \times (t^\tau, 2t^\tau)$, as the $t^2\pi_i^*(\gamma_i)$ term in (34) dominates, so that (51) holds for small t and on $\Sigma_i \times (t^\tau, 2t^\tau)$ we have

$$(\Xi_i^t)^*(\varepsilon^t) = O(t^{4-4\tau}) + O(t^{\tau(\mu_i - 2)}) + O(t^{(1-\tau)(2-\lambda)}) + O(t^{2+\tau(\nu_i - 2)}). \tag{52}$$

By a similar argument for derivatives we find that on $\Sigma_i \times (t^\tau, 2t^\tau)$ we have

$$(\Xi_i^t)^*(\mathrm{d}\varepsilon^t) = O(t^{4-5\tau}) + O(t^{\tau(\mu_i - 3)}) + O(t^{(1-\tau)(2-\lambda)-\tau}) + O(t^{2+\tau(\nu_i - 3)}). \tag{53}$$

Making $\delta > 0$ smaller if necessary, equations (38)–(41) now follow from (44), (45), (48), (49), (52) and (53), for some $C > 0$ independent of t. □

Now we can estimate norms of ε^t and $\mathrm{d}\varepsilon^t$, as in part (i) of Theorem 5.3.

Proposition 6.5 *There exists $C' > 0$ such that for all $t \in (0, \delta)$ we have*

$$\|\varepsilon^t\|_{L^{\frac{2m}{m+2}}} \leqslant C' t^{\tau(1+m/2)} \Big(t^{4-4\tau} + t^{(1-\tau)(2-\lambda)} + \sum_{i=1}^{n} \big(t^{\tau(\mu_i-2)} + t^{2+\tau(\nu_i-2)} \big) \Big), \tag{54}$$

$$\|\varepsilon^t\|_{C^0} \leqslant C' \Big(t^{4-4\tau} + t^{(1-\tau)(2-\lambda)} + \sum_{i=1}^{n} \big(t^{\tau(\mu_i-2)} + t^{2+\tau(\nu_i-2)} \big) \Big), \quad \textit{and} \tag{55}$$

$$\|\mathrm{d}\varepsilon^t\|_{L^{2m}} \leqslant C' t^{-\tau/2} \Big(t^{4-4\tau} + t^{(1-\tau)(2-\lambda)} + \sum_{i=1}^{n} \big(t^{\tau(\mu_i-2)} + t^{2+\tau(\nu_i-2)} \big) \Big), \tag{56}$$

computing norms with respect to the metric h^t on N^t.

Proof. The proof is similar to that of [9, Prop. 6.5]. Using (38), (40) and (41) we find that $\|\varepsilon^t\|_{L^{2m/(m+2)}}$ has contributions $O(t^{2+m/2})$ from P_i^t, $O(t^{\tau(2+m/2)})$ from $\Xi_i^t\big(\Sigma_i \times (tT, t^\tau]\big)$, and

$$O(t^{4+\tau(m-6)/2}) + O(t^{\tau(\mu_i-1+m/2)}) + O(t^{2-\lambda+\tau(\lambda-1+m/2)}) + O(t^{2+\tau(\nu_i-1+m/2)})$$

from $\Xi_i^t\big(\Sigma_i \times (t^\tau, 2t^\tau)\big)$, and $O(t^{4+\tau(m-6)/2})$ from $\Xi_i^t\big(\Sigma_i \times [2t^\tau, R')\big)$ as $m < 6$, and $O(t^4)$ from K^t, since $\mathrm{vol}(K^t) = O(1)$.

Now $O(t^{\tau(\mu_i-1+m/2)})$ dominates $O(t^{\tau(2+m/2)})$ and $O(t^{2+m/2})$ as $\mu_i < 3$ and $0 < \tau < 1$, and $O(t^{4+\tau(m-6)/2})$ dominates $O(t^4)$ as $m < 6$ and $\tau > 0$. The remaining terms give (54) for some $C' > 0$. Equations (55)–(56) follow from (38)–(41) in the same way. □

We also need to estimate $\|\pi_{W^t}(\varepsilon^t)\|_{L^1}$. Following [9, §7.2], which assumes condition (iv) of Definition 6.1, with no significant changes we prove:

Proposition 6.6 *There exists $C'' > 0$ such that for all $t \in (0, \delta)$ we have $\|\pi_{W^t}(\varepsilon^t)\|_{L^1} \leqslant C'' t^{(m+1)\tau}$, computing norms using h^t on N^t.*

For part (i) of Theorem 5.3 to hold, we want $\|\varepsilon^t\|_{L^{2m/(m+2)}} \leqslant A_2 t^{\kappa+m/2}$, $\|\varepsilon^t\|_{C^0} \leqslant A_2 t^{\kappa-1}$, $\|\mathrm{d}\varepsilon^t\|_{L^{2m}} \leqslant A_2 t^{\kappa-3/2}$ and $\|\pi_{W^t}(\varepsilon^t)\|_{L^1} \leqslant A_2 t^{\kappa+m-1}$ for some $\kappa > 1$. As $t < 1$ we see from (54) that $\|\varepsilon^t\|_{L^{2m/(m+2)}} \leqslant A_2 t^{\kappa+m/2}$ holds with $A_2 \geqslant 2(n+1)C'$ provided

$$\tau(1+m/2) + 4 - 4\tau \geqslant \kappa + m/2, \quad \tau(1+m/2) + (1-\tau)(2-\lambda) \geqslant \kappa + m/2, \tag{57}$$

$$\tau(1+m/2) + \tau(\mu_i - 2) \geqslant \kappa + m/2, \qquad \tau(1+m/2) + 2 + \tau(\nu_i - 2) \geqslant \kappa + m/2, \tag{58}$$

for all $i = 1, \ldots, n$. As $\tau \leqslant 1$ we find from (54)–(56) that (57)–(58) also imply $\|\varepsilon^t\|_{C^0} \leqslant A_2 t^{\kappa-1}$ and $\|\mathrm{d}\varepsilon^t\|_{L^{2m}} \leqslant A_2 t^{\kappa-3/2}$. Similarly, Proposition 6.6 implies that $\|\pi_{W^t}(\varepsilon^t)\|_{L^1} \leqslant A_2 t^{\kappa+m-1}$ holds with $A_2 \geqslant C''$ provided

$$(m+1)\tau \geqslant \kappa + m - 1. \tag{59}$$

Now (57) is equivalent to $\kappa\leqslant 1+(1-\tau)(6-m)/2$ and $\kappa\leqslant 1+(1-\tau)(1+m/2-\lambda)$, which admit a solution $\kappa>1$ as $\tau<1$, $m<6$ and $\lambda<-1-m/2$. Also, the conditions on τ in (33) ensure that (58) for $i=1,\ldots,n$ and (59) admit a solution $\kappa>1$, and this is the reason for (33). Taking this κ and $A_2=\max\bigl(2(n+1)C',C''\bigr)$, we have proved the analogue of [9, Th. 6.6]:

Theorem 6.7 *Making $\delta>0$ smaller if necessary, there exist $A_2>0$ and $\kappa>1$ such that the functions $\varepsilon^t=\psi^m\sin\theta^t$ on N^t satisfy $\|\varepsilon^t\|_{L^{2m/(m+2)}}\leqslant A_2t^{\kappa+m/2}$, $\|\varepsilon^t\|_{C^0}\leqslant A_2t^{\kappa-1}$, $\|d\varepsilon^t\|_{L^{2m}}\leqslant A_2t^{\kappa-3/2}$ and $\|\pi_{W^t}(\varepsilon^t)\|_{L^1}\leqslant A_2t^{\kappa+m-1}$ for all $t\in(0,\delta)$, as in part* (i) *of Theorem 5.3.*

6.3 Lagrangian neighbourhoods and bounds on $R(h^t),\delta(h^t)$

It turns out that [9, §6.3–§6.5] need only minor changes to apply to the Lagrangian m-folds N^t of Definition 6.2. Here is the analogue of [9, Def. 6.7].

Definition 6.8 Define an open neighbourhood $U_{N^t}\subset T^*N^t$ of the zero section N^t in T^*N^t and a smooth map $\Phi_{N^t}:U_{N^t}\rightarrow M$ as follows. As N^t is the disjoint union of K^t and P_i^t,Q_i^t for $i=1,\ldots,n$ we shall define U_{N^t} and Φ_{N^t} separately over K^t,P_i^t and Q_i^t.

Let $\Pi^t:K^t\rightarrow K$ be the natural projection, recalling from (35) that K^t is isomorphic to the graph $\Gamma(t^2\alpha)$ of $t^2\alpha$ over K. Then Π^t is a diffeomorphism, so $d\Pi^t:T^*K^t\rightarrow T^*K$ is a vector bundle isomorphism. Write π for both projections $\pi:T^*N^t\rightarrow N^t$ and $\pi:T^*K\rightarrow K$.

Now let $y\in T^*N^t\cap\pi^*(K^t)$, and define

$$x=\pi(y)\in K^t,\quad y'=d\Pi^t(y)\in T^*K\quad\text{and}\quad x'=\pi(y')=\Pi^t(x)\in K. \tag{60}$$

Define $U_{N^t}\cap\pi^*(K^t)$ and $\Phi_{N^t}|_{U_{N^t}\cap\pi^*(K^t)}$ by

$$\begin{gathered} y\in U_{N^t}\cap\pi^*(K^t)\text{ if and only if }y'+t^2\alpha(x')\in U_{X'},\\ \text{and then }\Phi_{N^t}(y)=\Phi_{X'}\bigl(y'+t^2\alpha(x')\bigr). \end{gathered} \tag{61}$$

For $i=1,\ldots,n$, define $U_{N^t}\cap\pi^*(P_i^t)$ and $\Phi_{N^t}|_{U_{N^t}\cap\pi^*(P_i^t)}$ by

$$\begin{gathered} U_{N^t}\cap\pi^*(P_i^t)=d(\Upsilon_i\circ t)\bigl(\{\gamma\in T^*K_i:t^{-2}\gamma\in U_{L_i}\}\bigr)\\ \text{and}\quad\Phi_{N^t}\circ d(\Upsilon_i\circ t)(\gamma)=\Upsilon_i\circ t\circ\Phi_{L_i}(t^{-2}\gamma). \end{gathered} \tag{62}$$

Here the diffeomorphism $\Upsilon_i\circ t:K_i\rightarrow P_i^t$ induces $d(\Upsilon_i\circ t):T^*K_i\rightarrow T^*P_i^t$, and $\gamma\mapsto t^{-2}\gamma$ is multiplication by t^{-2} in the vector space fibres of $T^*K_i\rightarrow K_i$.

Let F be as in Definition 6.2. Now $\Xi_i^t:\Sigma_i\times(tT,R')\rightarrow Q_i^t$ is a diffeomorphism, and induces an isomorphism $d\Xi_i^t:T^*\bigl(\Sigma_i\times(tT,R')\bigr)\rightarrow T^*Q_i^t$. As in (15)–(16) and (24)–(25), define $U_{N^t}\cap\pi^*(Q_i^t)$ and $\Phi_{N^t}|_{U_{N^t}\cap\pi^*(Q_i^t)}$ by

$$\begin{aligned} (d\Xi_i^t)^*(U_{N^t})=\bigl\{&(\sigma,r,\varsigma,u)\in T^*\bigl(\Sigma_i\times(tT,R')\bigr):\\ &\bigl|(\varsigma,u)+t^2F(t^{-\tau}r)(\Upsilon_i\circ\phi_i)^*(\alpha)(\sigma,r)\bigr|<\zeta r\bigr\}\quad\text{and} \end{aligned} \tag{63}$$

$$\Phi_{N^t}\circ d\Xi_i^t(\sigma,r,\varsigma,u)\equiv\Upsilon_i\circ\Phi_{C_i}\bigl(\sigma,r,\varsigma+\xi_i^{1,t}(\sigma,r),u+\xi_i^{2,t}(\sigma,r)\bigr) \tag{64}$$

for all $(\sigma, r, \varsigma, u) \in (\mathrm{d}\Xi_i^t)^*(U_{N^t})$, computing $\nabla, |\,.\,|$ using $\iota_i^*(g')$.

Careful consideration shows that U_{N^t} is well-defined, and Φ_{N^t} is well-defined in (61), (62) and (64) for small t, so making $\delta > 0$ smaller if necessary Φ_{N^t} is well-defined for $t \in (0, \delta)$. Clearly Φ_{N^t} is smooth on each of $U_{N^t} \cap \pi^*(K^t)$, $U_{N^t} \cap \pi^*(P_i^t)$ and $U_{N^t} \cap \pi^*(Q_i^t)$, but we must still show that Φ_{N^t} is smooth over the joins between them.

The condition $|(\varsigma, u) + t^2 F(t^{-\tau} r)(\Upsilon_i \circ \phi_i)^*(\alpha)(\sigma, r)| < \zeta r$ in (63) is chosen to ensure that the definitions of U_{N^t} over K^t, P_i^t and Q_i^t join smoothly together over ∂K^t and ∂P_i^t. Therefore U_{N^t} is an *open tubular neighbourhood* of N^t in T^*N^t. Following [9, Def. 6.4] we find that Φ_{N^t} is *smooth* on U_{N^t}, that is, the definitions over P_i^t, Q_i^t and K^t join together smoothly over ∂P_i^t and ∂K^t.

We shall show that $\Phi_{N^t}^*(\omega) = \hat{\omega}$. On $U_{N^t} \cap \pi^*(K^t)$ this holds as $\Phi_{X'}^*(\omega) = \hat{\omega}$ and α is closed. On $U_{N^t} \cap \pi^*(P_i^t)$ it follows from $\Upsilon_i^*(\omega) = \omega'$, $\Phi_{L_i}^*(\omega') = \hat{\omega}$, and the fact that the powers of t in (62) cancel out in their effect on $\Phi_{N^t}^*(\omega)$. On $U_{N^t} \cap \pi^*(Q_i^t)$ it holds as $\Upsilon_i^*(\omega) = \omega'$, $\Phi_{C_i}^*(\omega') = \hat{\omega}$ and ξ_i^t is closed.

Define an m-form β^t on U_{N^t} by $\beta^t = \Phi_{N^t}^*(\mathrm{Im}\,\Omega)$, as in Definition 5.2.

Using this Lagrangian neighbourhood map Φ_{N^t} in part (iii), we find that that parts (ii)–(v) of Theorem 5.3 hold for N^t when $t \in (0, \delta)$, as in [9, Th. 6.8].

Theorem 6.9 *Making $\delta > 0$ smaller if necessary, there exist $A_1, A_3, \ldots, A_6 > 0$ such that for all $t \in (0, \delta)$, as in parts* (ii)–(v) *of Theorem 5.3 we have*

(ii) $\psi \geqslant A_3$ *on* N^t.

(iii) *The subset $\mathcal{B}_{A_1 t} \subset T^*N^t$ of Definition 5.2 lies in U_{N^t}, and $\|\hat{\nabla}^k \beta^t\|_{C^0} \leqslant A_4 t^{-k}$ on $\mathcal{B}_{A_1 t}$ for $k = 0, 1, 2$ and 3.*

(iv) *The injectivity radius $\delta(h^t)$ satisfies $\delta(h^t) \geqslant A_5 t$.*

(v) *The Riemann curvature $R(h^t)$ satisfies $\|R(h^t)\|_{C^0} \leqslant A_6 t^{-2}$.*

Here part (iii) *uses the notation of Definition 5.2, and parts* (iv) *and* (v) *refer to the compact Riemannian manifold (N^t, h^t).*

Proof. This was proved in [9, Th. 6.8], but with N^t, Φ_{N^t} defined more simply. The changes for the new N^t, Φ_{N^t} are very minor. The main difference is that in [9, §6.3] N^t, Φ_{N^t} are independent of t over K, but here N^t, Φ_{N^t} do depend on t over K^t. Identifying K^t with K using Π^t in Definition 6.8 we have $\nabla^j(h^t - g|_K) = O(t^2)$ for $j \geqslant 0$. Therefore the contributions to $\|\hat{\nabla}^k \beta^t\|_{C^0}, \delta(h^t)$ and $\|R(h^t)\|_{C^0}$ from K^t are all $O(1)$ for small t, as in [9, §6.3]. □

6.4 Sobolev embedding inequalities on N^t

We now prove that parts (vi) and (vii) of Theorem 5.3 hold for N^t, W^t. Here is the analogue of [9, Th. 7.8], which gives part (vi) of Theorem 5.3 for N^t, W^t.

Theorem 6.10 *Making $\delta > 0$ smaller if necessary, there exists $A_7 > 0$ such that for all $t \in (0, \delta)$, if $v \in L^2_1(N^t)$ with $\int_{N^t} vw\,\mathrm{d}V^t = 0$ for all $w \in W^t$ then $v \in L^{2m/(m-2)}(N^t)$ and $\|v\|_{L^{2m/(m-2)}} \leqslant A_7 \|\mathrm{d}v\|_{L^2}$.*

Proof. This was proved in [9, Th. 6.12] for X' connected and [9, Th. 7.8] for general X', but with N^t, Φ_{N^t} defined more simply. Here is how to modify the proofs. First consider the case of [9, §6.4], when X' is connected.

In [9, §6.4] we define a partition of unity function F^t on N^t. The new definition is as follows. Choose $a, b \in \mathbb{R}$ with $0 < a < b < \tau$. Making $\delta > 0$ smaller if necessary we have $2t^\tau < t^b < t^a < \min(1, R')$ for all $t \in (0, \delta)$. Let $G : (0, \infty) \to [0, 1]$ be a smooth, decreasing function with $G(s) = 1$ for $s \in (0, a]$ and $G(s) = 0$ for $s \in [b, \infty)$. For $t \in (0, \delta)$, define a function $F^t : N^t \to [0, 1]$ by

$$F^t(x) = \begin{cases} 1, & x \in K^t, \\ G\bigl((\log r)/(\log t)\bigr), & x = \Xi^t_i(\sigma, r) \in Q^t_i, \quad i = 1, \ldots, n, \\ 0, & x \in P^t_i, \quad i = 1, \ldots, n. \end{cases} \tag{65}$$

The main point we have to deal with is that in [9, §6.4] the function F^t was supported on $N^t \cap X'$, and so if $v \in C^1(N^t)$ then we could treat $F^t v$ as a compactly-supported function on X'. But for the new N^t this no longer holds. Instead, on the support of F^t we have $N^t = \Phi_{X'}\bigl(\Gamma(t^2\alpha)\bigr)$, where $\Gamma(t^2\alpha)$ is the graph of $t^2\alpha$ in T^*X'.

Identifying $\Gamma(t^2\alpha)$ with X' using $\pi : T^*X' \to X'$ and N^t with $\Gamma(t^2\alpha)$ using $\Phi_{X'}$ defines an identification between N^t and X' on the support of F^t. Thus, if $v \in C^1(N^t)$ we can regard $F^t v$ as a compactly-supported function on X'. Then [9, Prop. 6.11] gives

$$\|F^t v\|_{L^{2m/(m-2)}} \leqslant D_2\bigl(\bigl\|\mathrm{d}(F^t v)\bigr\|_{L^2} + \bigl|\textstyle\int_{X'} F^t v\,\mathrm{d}V_g\bigr|\bigr), \tag{66}$$

computing norms using g on X', where $D_2 > 0$ is independent of v, t.

For the proof of [9, Th. 6.12] to work for the new N^t, we need (66) to hold computed using h^t on N^t. Since h^t is the restriction of h^t to $\Gamma(t^2\alpha)$ and $\alpha = O(\rho^{-1})$, $\nabla\alpha = O(\rho^{-2})$ we find that

$$h^t = g|_{X'} + O(t^2\rho^{-2}) = g|_{X'} + O(t^{2-2\tau}) \tag{67}$$

on the support of F^t, identifying N^t and X' there as above. Thus h^t and $g|_{X'}$ are uniformly equivalent on the support of F^t for small t, so increasing D_2 we see that (66) holds with $\|\,.\,\|_{L^{2m/(m-2)}}$, $\|\,.\,\|_{L^2}$ computed using h^t.

From (67) we have $\mathrm{d}V_g = \bigl(1 + O(t^{2-2\tau})\bigr)\mathrm{d}V^t$, so that

$$\begin{aligned} \bigl|\textstyle\int_{X'} F^t v\,\mathrm{d}V_g\bigr| &\leqslant \bigl|\textstyle\int_{N^t} F^t v\,\mathrm{d}V^t\bigr| + Ct^{2-2\tau}\textstyle\int_{N^t} |F^t v|\,\mathrm{d}V^t \\ &\leqslant \bigl|\textstyle\int_{N^t} F^t v\,\mathrm{d}V^t\bigr| + C' t^{2-2\tau}\|F^t v\|_{L^{2m/(m-2)}}, \end{aligned} \tag{68}$$

for small t and $C, C' > 0$ independent of t. Here norms are computed using h^t, and in the second line we use $\|F^t v\|_{L^1} \leqslant \mathrm{vol}(N^t)^{(m+2)/2m}\|F^t\|_{L^{2m/(m-2)}}$, and that $\mathrm{vol}(N^t)$ is bounded independently of t.

Substituting (68) into (66) gives

$$(1 - C'D_2 t^{2-2\tau})\|F^t v\|_{L^{2m/(m-2)}} \leqslant D_2\big(\big\| \mathrm{d}(F^t v)\big\|_{L^2} + \big| \textstyle\int_{N^t} F^t v \,\mathrm{d}V^t \big| \big),$$

computing norms using h^t. For small t we have $C'D_2 t^{2-2\tau} \leqslant \frac{1}{2}$, and so

$$\|F^t v\|_{L^{2m/(m-2)}} \leqslant 2D_2\big(\big\| \mathrm{d}(F^t v)\big\|_{L^2} + \big| \textstyle\int_{N^t} F^t v \,\mathrm{d}V^t \big| \big).$$

This is the first line of [9, eq. (79)], with $2D_2$ in place of D_2. The rest of the proof of [9, Th. 6.12] then follows. This proves the theorem when X' is connected. For general X' we follow the proof of [9, Th. 7.8], but with F^t defined as in (65). The modifications above extend easily, and the theorem follows. □

The proof of [9, Th. 7.9] also holds for the new N^t, W^t with no significant changes. This proves part (vii) of Theorem 5.3 for N^t, W^t.

Theorem 6.11 *Making $\delta > 0$ smaller if necessary, for all $t \in (0,\delta)$ and $w \in W^t$ we have $\|\mathrm{d}^*\mathrm{d}w\|_{L^{2m/(m+2)}} \leqslant \frac{1}{2}A_7^{-1}\|\mathrm{d}w\|_{L^2}$, where $A_7 > 0$ is as in Theorem 6.10. Also there exists $A_8 > 0$ such that for all $t \in (0,\delta)$ and $w \in W^t$ with $\int_{N^t} w \,\mathrm{d}V^t = 0$ we have $\|w\|_{C^0} \leqslant A_8 t^{1-m/2}\|\mathrm{d}w\|_{L^2}$.*

6.5 The main result when $Y(L_i) \neq 0$ and $\lambda_i = 0$

Here is the first of the main results of this paper, the analogue of Theorem 5.5 but allowing $\lambda_i = 0$ and $Y(L_i) \neq 0$.

Theorem 6.12 *Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold for $2 < m < 6$, and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Define $\psi : M \to (0,\infty)$ as in (3). Let $L_1, \ldots, L_n$ be Asymptotically Conical SL m-folds in $\mathbb{C}^m$ with cones $C_1, \ldots, C_n$ and rates $\lambda_1, \ldots, \lambda_n$. Write $X' = X \setminus \{x_1, \ldots, x_n\}$ and $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$. Suppose that $\lambda_i \leqslant 0$ for $i = 1, \ldots, n$, and that there exists $\varrho \in H^1(X', \mathbb{R})$ such that $\big(Y(L_1), \ldots, Y(L_n)\big)$ is the image of ϱ under the map $H^1(X', \mathbb{R}) \to \bigoplus_{i=1}^n H^1(\Sigma_i, \mathbb{R})$ in (6).*

Set $q = b^0(X')$, and let $X'_1, \ldots, X'_q$ be the connected components of X'. For $i = 1, \ldots, n$ let $l_i = b^0(\Sigma_i)$, and let $\Sigma_i^1, \ldots, \Sigma_i^{l_i}$ be the connected components of Σ_i. Define $k(i,j) = 1, \ldots, q$ by $\Upsilon_i \circ \varphi_i\big(\Sigma_i^j \times (0, R')\big) \subset X'_{k(i,j)}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$. Suppose that

$$\sum_{\substack{1\leqslant i\leqslant n,\ 1\leqslant j\leqslant l_i:\\ k(i,j)=k}} \psi(x_i)^m Z(L_i) \cdot [\Sigma_i^j] = 0 \quad \textit{for all } k = 1, \ldots, q. \tag{69}$$

Suppose also that the compact m-manifold N obtained by gluing L_i into X' at x_i for $i = 1, \ldots, n$ is connected. A sufficient condition for this to hold is that X and L_i for $i = 1, \ldots, n$ are connected.

Then there exists $\epsilon > 0$ and a smooth family $\big\{\tilde N^t : t \in (0,\epsilon]\big\}$ of compact, nonsingular SL m-folds in (M, J, ω, Ω) diffeomorphic to N, such that $\tilde N^t$ is constructed by gluing tL_i into X at x_i for $i = 1, \ldots, n$. In the sense of currents in Geometric Measure Theory, $\tilde N^t \to X$ as $t \to 0$.

Proof. The hypotheses of the theorem imply that Definition 6.1 holds. Let $\delta > 0$ and N^t, W^t for $t \in (0,\delta)$ be as in Definitions 6.2 and 6.3, and make $\delta > 0$ smaller if necessary so that Theorems 6.7, 6.9, 6.10 and 6.11 apply. Theorem 6.7 gives constants $\kappa > 1$ and $A_2 > 0$ such that part (i) of Theorem 5.3 holds for N^t for all $t \in (0,\delta)$, replacing N, W, θ by N^t, W^t, θ^t respectively.

Let the Lagrangian neighbourhood $\Phi_{N^t} : U_{N^t} \to M$ and the m-form β^t on U_{N^t} be as in Definition 6.8. Then Theorem 6.9 gives constants $A_1, A_3, \ldots, A_6 > 0$ such that parts (ii)–(v) of Theorem 5.3 hold for N^t for all $t \in (0,\delta)$, replacing N, β, h by N^t, β^t, h^t respectively. Theorems 6.10 and 6.11 give $A_7, A_8 > 0$ such that parts (vi) and (vii) of Theorem 5.3 hold for N^t for all $t \in (0,\delta)$, replacing N, W by N^t, W^t respectively.

We have not yet shown that the inequality $\cos\theta^t \geqslant \frac{1}{2}$ in Definition 5.2 holds. From parts (i) and (ii) of Theorem 5.3 we see that $|\sin\theta^t| \leqslant A_2 A_3^{-m} t^{\kappa-1}$ on N^t. Thus for small $t \in (0,\delta)$ we have $|\sin\theta^t| \leqslant \frac{\sqrt{3}}{2}$ as $\kappa > 1$, so that $|\cos\theta^t| \geqslant \frac{1}{2}$. As N^t is approximately special Lagrangian on K^t we have $e^{i\theta^t} \approx 1$ on K^t, so $\cos\theta^t \geqslant \frac{1}{2}$ on N^t as $\cos\theta^t$ is continuous and N^t connected.

Let $\epsilon, K > 0$ be as given in Theorem 5.3 depending on $\kappa, A_1, \ldots, A_8$ and m, and make $\epsilon > 0$ smaller if necessary to ensure that $\epsilon < \delta$ and $\cos\theta^t \geqslant \frac{1}{2}$ on N^t for $t \leqslant \epsilon$. Then Theorem 5.3 shows that for all $t \in (0,\epsilon]$ we can deform N^t to a nearby compact, nonsingular SL m-fold $\tilde N^t$, given by $\tilde N^t = (\Phi_{N^t})_*\bigl(\Gamma(\mathrm{d}f^t)\bigr)$ for some $f^t \in C^\infty(N^t)$ with $\|\mathrm{d}f^t\|_{C^0} \leqslant Kt^\kappa < A_1 t$.

Since N^t and Φ_{N^t} depend smoothly on t, we see that f^t is the locally unique solution of a nonlinear elliptic p.d.e. on N^t depending smoothly on t. It quickly follows from general theory that f^t depends smoothly on t, and so $\tilde N^t$ does. We show that $\tilde N^t \to X$ as currents as $t \to 0$ as in [9, §6.5]. □

When $m = 6$, the proof of Theorem 6.7 shows that the bounds in part (i) of Theorem 5.3 hold with $\kappa = 1$ rather than $\kappa > 1$. Thus the proof *only just* fails when $m = 6$, and with some improvements to the material of §6.1–§6.2 Theorem 6.12 can probably be proved for $m = 6$. However, the author does not know how to extend Theorem 6.12 to the case $m > 6$.

If X' is connected, so that $q = 1$, then $k(i,j) \equiv 1$ and (69) becomes

$$\sum_{i=1}^{n} \psi(x_i)^m Z(L_i) \cdot \sum_{j=1}^{l_i} [\Sigma_i^j] = 0.$$

But $\sum_{j=1}^{l_i} [\Sigma_i^j] = [\Sigma_i]$, and $Z(L_i) \cdot [\Sigma_i] = 0$ as $Z(L_i)$ is the image of a class in $H^{m-1}(L_i, \mathbb{R})$ by Definition 4.2, and $\Sigma_i = \partial L_i$, so $[\Sigma_i]$ maps to zero in $H_{m-1}(L_i, \mathbb{R})$. Therefore (69) holds automatically when X' is connected. Also, N is automatically connected. Thus Theorem 6.12 simplifies in this case to give an analogue of Theorem 5.4:

Theorem 6.13 *Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold for $2 < m < 6$, and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Let $L_1, \ldots, L_n$ be Asymptotically Conical SL m-folds in $\mathbb{C}^m$ with cones $C_1, \ldots, C_n$ and rates $\lambda_1, \ldots, \lambda_n$. Suppose that $\lambda_i \leqslant 0$ for*

$i = 1, \ldots, n$, that $X' = X \setminus \{x_1, \ldots, x_n\}$ is connected, and that there exists $\varrho \in H^1(X', \mathbb{R})$ such that $\big(Y(L_1), \ldots, Y(L_n)\big)$ is the image of ϱ under the map $H^1(X', \mathbb{R}) \to \bigoplus_{i=1}^n H^1(\Sigma_i, \mathbb{R})$ in (6), where $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$.

Then there exists $\epsilon > 0$ and a smooth family $\big\{\tilde N^t : t \in (0, \epsilon]\big\}$ of compact, nonsingular SL m-folds in (M, J, ω, Ω), such that $\tilde N^t$ is constructed by gluing tL_i into X at x_i for $i = 1, \ldots, n$. In the sense of currents, $\tilde N^t \to X$ as $t \to 0$.

7 Desingularizing in families when $Y(L_i) = 0$

Next we consider a different generalization of Theorems 5.4 and 5.5, to *families* of almost Calabi–Yau m-folds $(M, J^s, \omega^s, \Omega^s)$. Let $\mathcal{F} \subset \mathbb{R}^d$ be open with $0 \in \mathcal{F}$, and $\big\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\big\}$ be a smooth family of almost Calabi–Yau m-folds with $(M, J^0, \omega^0, \Omega^0) = (M, J, \omega, \Omega)$.

Suppose X is an SL m-fold in (M, J, ω, Ω) with conical singularities at $x_1, \ldots, x_n$ and cones C_i, and that $L_1, \ldots, L_n$ are AC SL m-folds in $\mathbb{C}^m$ with cones C_i and rates λ_i. In the rest of the paper we shall construct special Lagrangian desingularizations $\tilde N^{s,t}$ of X not just in (M, J, ω, Ω), as in [9] and §6, but in $(M, J^s, \omega^s, \Omega^s)$ for small $s \in \mathcal{F}$.

In this section, as in Theorems 5.4 and 5.5 but not as in §6, we shall assume that $\lambda_i < 0$, so that $Y(L_i) = 0$ by Proposition 4.3. Then §8 combines the new material of §6 and this section to study desingularization in families when $Y(L_i) \neq 0$.

The advantage of desingularizing in families $(M, J^s, \omega^s, \Omega^s)$, rather than a single almost Calabi–Yau m-fold (M, J, ω, Ω), is that by varying the cohomology classes $[\omega^s]$ and $[\operatorname{Im} \Omega^s]$ we can overcome obstructions to the existence of SL desingularizations $\tilde N^t$ in (M, J, ω, Ω). In this section, by varying $[\operatorname{Im} \Omega^s]$ we show how to relax equation (28) in Theorem 5.5.

7.1 Setting up the problem

The next three definitions are analogues of Definitions 6.1–6.3. We consider the following situation, modifying [9, Def. 6.1 & Def. 7.1] to the families case.

Definition 7.1 Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold with metric g, and define $\psi : M \to (0, \infty)$ as in (3). Let X be a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$ with identifications υ_i, cones C_i and rates μ_i. Define $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$ for $i = 1, \ldots, n$. Let $L_1, \ldots, L_n$ be AC SL m-folds in $\mathbb{C}^m$, where L_i has cone C_i and rate λ_i for $i = 1, \ldots, n$. As in Definition 2.7, let $\big\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\big\}$ be a *smooth family of deformations* of (M, J, ω, Ω), with base space $\mathcal{F} \subset \mathbb{R}^d$. Suppose that

(i) L_i has rate $\lambda_i < 0$ for $i = 1, \ldots, n$, so that $Y(L_i) = 0$ by Proposition 4.3.

(ii) $[\omega^s] \cdot \iota_*(\gamma) = 0$ for all $s \in \mathcal{F}$ and $\gamma \in H_2(X, \mathbb{R})$, where $\iota_* : H_2(X, \mathbb{R}) \to H_2(M, \mathbb{R})$ is the natural inclusion.

(iii) Let N be the compact m-manifold obtained by gluing L_i into X' at x_i for $i = 1, \ldots, n$ in the obvious way. Suppose N is connected. A sufficient condition for this to hold is that X and L_i for $i = 1, \ldots, n$ are connected.

We use the following notation:

- Let R, B_R, X' and ι_i, Υ_i for $i = 1, \ldots, n$ be as in Definition 3.4.
- Let ζ and U_{C_i}, Φ_{C_i} for $i = 1, \ldots, n$ be as in Theorem 3.8.
- Let R' and $\phi_i, \eta_i, \eta_i^1, \eta_i^2$ for $i = 1, \ldots, n$ be as in Theorem 3.9.
- Let $q = b^0(X')$, and write the connected components of X' as $X'_1, \ldots, X'_q$.
- For $i = 1, \ldots, n$ let $l_i = b^0(\Sigma_i)$, and write the connected components of Σ_i as $\Sigma_i^1, \ldots, \Sigma_i^{l_i}$.
- Then $\Upsilon_i \circ \varphi_i\bigl(\Sigma_i^j \times (0, R')\bigr)$ is a connected subset of X', and so lies in exactly one X'_k. Define $k(i, j)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$ by $\Upsilon_i \circ \varphi_i\bigl(\Sigma_i^j \times (0, R')\bigr) \subset X'_{k(i,j)}$.
- Let $U_{X'}, \Phi_{X'}$ be as in Theorem 3.11.
- Let A_i be as in Theorem 3.10 for $i = 1, \ldots, n$, so that $\eta_i = \mathrm{d}A_i$.
- Apply Theorem 4.5 to L_i with $\zeta, U_{C_i}, \Phi_{C_i}$ as above, for $i = 1, \ldots, n$. Let $T > 0$ be as in the theorem, the same for all i. Let the subset $K_i \subset L_i$, the diffeomorphism $\varphi_i : \Sigma_i \times (T, \infty) \to L_i \setminus K_i$ and the 1-form χ_i on $\Sigma_i \times (T, \infty)$ with components χ_i^1, χ_i^2 be as in Theorem 4.5.
- Let U_{L_i}, Φ_{L_i} be as in Theorem 4.8 for $i = 1, \ldots, n$.
- Let $E_i \in C^\infty\bigl(\Sigma_i \times (T, \infty)\bigr)$ be as in Theorem 4.6 for $i = 1, \ldots, n$.

  Note that as $Y(L_i) = 0$ by assumption we have $\gamma_i = 0$ in Theorem 4.6, and therefore $\chi_i = \mathrm{d}E_i$ for $i = 1, \ldots, n$.
- Apply Theorems 3.12 and 3.13 to X and $\bigl\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\bigr\}$. By Theorem 3.13, part (ii) above implies that $[\nu^s] = 0$, so we can take $\nu^s \equiv 0$.

  Then Theorem 3.12 gives an open $\mathcal{F}' \subseteq \mathcal{F}$ with $0 \in \mathcal{F}'$ and $\psi^s, \upsilon_i^s, \Upsilon_i^s, \Phi_{X'}^s$ for $s \in \mathcal{F}'$ depending smoothly on s, satisfying (17) and

$$\begin{gathered} \upsilon_i^0 = \upsilon_i, \;\; \Upsilon_i^0 = \Upsilon_i, \;\; \Phi_{X'}^0 = \Phi_{X'}, \;\; (\upsilon_i^s)^*(\Omega) = \psi^s(x_i)^m \Omega', \\ \Upsilon_i^s(0) = x_i, \;\; \mathrm{d}\Upsilon_i^s|_0 = \upsilon_i^s, \;\; (\Upsilon_i^s)^*(\omega^s) = \omega', \;\text{ and }\; (\Phi_{X'}^s)^*(\omega^s) = \hat{\omega}. \end{gathered} \tag{70}$$

Instead of just defining N^t for $t \in (0, \delta)$ as in Definition 6.2, we define a larger family $\bigl\{N^{s,t} : s \in \mathcal{F}', \, t \in (0, \delta)\bigr\}$, where $N^{s,t}$ is Lagrangian in (M, ω^s).

Definition 7.2 In the situation of Definition 7.1, choose a smooth, increasing function $F : (0, \infty) \to [0, 1]$ with $F(r) \equiv 0$ for $r \in (0, 1)$ and $F(r) \equiv 1$ for $r > 2$. Write F' for $\mathrm{d}F/\mathrm{d}r$. Let $\tau \in (0, 1)$ satisfy

$$0 < \max_{i=1,\ldots,n} \left\{ \frac{m}{m+1}, \frac{m+2}{2\mu_i + m - 2} \right\} < \tau < 1, \tag{71}$$

which is possible as $\mu_i > 2$ implies $(m+2)/(2\mu_i+m-2) < 1$.

For $i=1,\ldots,n$, $s\in\mathcal{F}'$ and small $t>0$, define $P_i^{s,t}=\Upsilon_i^s(tK_i)$. This is well-defined if $tK_i\subset B_R\subset\mathbb{C}^m$, and is a compact submanifold of M with boundary, diffeomorphic to K_i. As K_i is Lagrangian in $(\mathbb{C}^m,\omega')$ and $(\Upsilon_i^s)^*(\omega^s)=\omega'$, we see that $P_i^{s,t}$ is *Lagrangian* in (M,ω^s).

For $i=1,\ldots,n$ and $t>0$ with $tT<t^\tau<2t^\tau<R'$, define a 1-form ξ_i^t on $\Sigma_i\times(tT,R')$ by

$$\begin{aligned}\xi_i^t(\sigma,r)&=\mathrm{d}\bigl[F(t^{-\tau}r)A_i(\sigma,r)+t^2(1-F(t^{-\tau}r))E_i(\sigma,t^{-1}r)\bigr]\\&=F(t^{-\tau}r)\eta_i(\sigma,r)+t^{-\tau}F'(t^{-\tau}r)A_i(\sigma,r)\mathrm{d}r\\&\quad+t^2(1-F(t^{-\tau}r))\chi_i(\sigma,t^{-1}r)-t^{2-\tau}F'(t^{-\tau}r)E_i(\sigma,t^{-1}r)\mathrm{d}r.\end{aligned}\tag{72}$$

Let $\xi_i^{1,t},\xi_i^{2,t}$ be the components of ξ_i^t in $T^*\Sigma$ and $\mathbb{R}$, as for η_i,χ_i in Theorems 3.9 and 4.5. Note that when $r\geqslant 2t^\tau$ we have $F(t^{-\tau}r)\equiv 1$ so that $\xi_i^t(\sigma,r)=\eta_i(\sigma,r)$, and when $r\leqslant t^\tau$ we have $F(t^{-\tau}r)\equiv 0$, so that $\xi_i^t(\sigma,r)=t^2\chi_i(\sigma,t^{-1}r)$. Thus ξ_i^t is an exact 1-form which interpolates between $\eta_i(\sigma,r)$ near $r=R'$ and $t^2\chi_i(\sigma,t^{-1}r)$ near $r=tT$.

Choose $\delta\in(0,1]$ with $\delta T\leqslant\delta^\tau<2\delta^\tau\leqslant R'$ and $\delta K_i\subset B_R\subset\mathbb{C}^m$ and

$$\bigl|\xi_i^t(\sigma,r)\bigr|<\zeta r\quad\text{on }\Sigma_i\times(tT,R')\text{ for all }i=1,\ldots,n\text{ and }t\in(0,\delta).\tag{73}$$

This is possible as in [9, Def. 6.2]. For $i=1,\ldots,n$, $s\in\mathcal{F}'$ and $t\in(0,\delta)$ define $\Xi_i^{s,t}:\Sigma_i\times(tT,R')\to M$ by

$$\Xi_i^{s,t}(\sigma,r)=\Upsilon_i^s\circ\Phi_{C_i}\bigl(\sigma,r,\xi_i^{1,t}(\sigma,r),\xi_i^{2,t}(\sigma,r)\bigr).\tag{74}$$

Define $Q_i^{s,t}=\Xi_i^{s,t}\bigl(\Sigma_i\times(tT,R')\bigr)$ for $i=1,\ldots,n$, $s\in\mathcal{F}'$ and $t\in(0,\delta)$. As $(\Upsilon_i^s)^*(\omega^s)=\omega'$, $\Phi_{C_i}^*(\omega')=\hat\omega$ and ξ_i^t is closed we see that $(\Xi_i^{s,t})^*(\omega^s)\equiv 0$. Thus $Q_i^{s,t}$ is *Lagrangian* in (M,ω^s), and is a noncompact embedded submanifold diffeomorphic to $\Sigma_i\times(tT,R')$.

Recall that $K\subset X'$, and X' is embedded as the zero section in $U_{X'}\subset T^*X'$. Thus $K\subset U_{X'}$, and $\Phi_{X'}^s$ maps $U_{X'}\to M$. For $s\in\mathcal{F}'$, define $K^s=\Phi_{X'}^s(K)$. Then K^s is Lagrangian in (M,ω^s), as $(\Phi_{X'}^s)^*(\omega^s)=\hat\omega$, and $K\subset X'$ is Lagrangian in $(U_{X'},\hat\omega)$. For $s\in\mathcal{F}'$ and $t\in(0,\delta)$, define $N^{s,t}$ to be the disjoint union of K^s, $P_1^{s,t},\ldots,P_n^{s,t}$ and $Q_1^{s,t},\ldots,Q_n^{s,t}$. Then $N^{s,t}$ is *Lagrangian* in (M,ω^s), as $K^s,P_i^{s,t}$ and $Q_i^{s,t}$ are.

Moreover, $N^{s,t}$ is a compact, smooth submanifold of M *without boundary*. The proof of this follows [9, Def. 6.2], with simple changes. In particular, $\partial P_i^{s,t}$ joins smoothly onto the $\Sigma_i\times\{tT\}$ end of $Q_i^{s,t}$ as $\xi_i^t(\sigma,r)\equiv t^2\chi_i(\sigma,t^{-1}r)$ near $r=tT$ on $\Sigma_i\times(tT,R')$, and both $P_i^{s,t},Q_i^{s,t}$ are defined using Υ_i^s. Similarly, the $\Sigma_i\times\{R'\}$ end of $Q_i^{s,t}$ joins smoothly onto the appropriate component of ∂K^s as $\xi_i^t\equiv\eta_i$ near $r=R'$ on $\Sigma_i\times(tT,R')$, and because of the compatibility (17) between Υ_i^s, used to define Q_i^s, and $\Phi_{X'}^s$, used to define K^s.

Note that $N^{s,t}$ depends smoothly on s,t, since $\Upsilon_i^s,\Phi_{X'}^s$ depend smoothly on s. Also, when $s=0$ we have $\Upsilon_i^0=\Upsilon_i$ and $\Phi_{X'}^0=\Phi_{X'}$ by (70), so we see that $P_i^{0,t}=P_i^t$, $Q_i^{0,t}=Q_i^t$, $K^0=K$ and $N^{0,t}=N^t$, where P_i^t,Q_i^t and N^t are as in [9, Def. 6.2].

Let $h^{s,t}$ be the restriction of g to $N^{s,t}$ for $s \in \mathcal{F}'$ and $t \in (0,\delta)$, so that $(N^{s,t}, h^{s,t})$ is a compact Riemannian manifold, which is naturally oriented. Let $\mathrm{d}V^{s,t}$ be the volume form on $N^{s,t}$. As in (27) we may write $\Omega|_{N^{s,t}} = (\psi^s)^m \mathrm{e}^{i\theta^{s,t}}\,\mathrm{d}V^{s,t}$ for some phase function $\mathrm{e}^{i\theta^{s,t}}$ on $N^{s,t}$. Write $\varepsilon^{s,t} = (\psi^s)^m \sin\theta^{s,t}$, so that $\operatorname{Im}\Omega|_{N^{s,t}} = \varepsilon^{s,t}\,\mathrm{d}V^{s,t}$. Then $\varepsilon^{s,t}$ depends smoothly on s, as Ω^s, Υ_i^s and $\Phi_{X'}^s$ do.

Note that Definition 7.2 does not include the terms in γ_i, T_i and α added in Definition 6.2, which were there to allow $Y(L_i) \neq 0$. Here is the analogue of Definition 6.3, defining vector spaces $W^{s,t}$ which will be W in Definition 5.2.

Definition 7.3 In the situation of Definitions 7.1 and 7.2 define vector spaces V_i of bounded harmonic functions on L_i and $v_i^{\mathbf{c}_i} \in V_i$ for $\mathbf{c}_i \in \mathbb{R}^{l_i}$ satisfying (36) for $i = 1,\ldots,n$, as in Definition 6.3. We shall define a vector subspace $W^{s,t} \subset C^\infty(N^{s,t})$ for $s \in \mathcal{F}'$ and $t \in (0,\delta)$, with an isomorphism $W^{s,t} \cong \mathbb{R}^q$.

Fix $\mathbf{d} = (d_1,\ldots,d_q) \in \mathbb{R}^q$, and as in Definition 6.3 define $\mathbf{c}_i \in \mathbb{R}^{l_i}$ for $i = 1,\ldots,n$ depending linearly on $\mathbf{d}$. Let $F : (0,\infty) \to [0,1]$ and $\tau \in (0,1)$ be as in Definition 7.2. Make $\delta > 0$ smaller if necessary so that $tT < \frac{1}{2}t^\tau$ for all $t \in (0,\delta)$. For $s \in \mathcal{F}'$ and $t \in (0,\delta)$, define $w_{\mathbf{d}}^{s,t} \in C^\infty(N^{s,t})$ as follows:

(i) The subset $K^s \subset N^{s,t}$ has q connected components $\Phi_{X'}^s(K \cap X_k')$. Define $w_{\mathbf{d}}^{s,t} \equiv d_k$ on $\Phi_{X'}^s(K \cap X_k')$ for $k = 1,\ldots,q$.

(ii) Define $w_{\mathbf{d}}^{s,t}$ on $P_i^{s,t} \subset N^{s,t}$ by $(\Upsilon_i^s \circ t \circ \varphi_i)^*(w_{\mathbf{d}}^{s,t}) \equiv v_i^{\mathbf{c}_i}$ on K_i.

(iii) Define $w_{\mathbf{d}}^{s,t}$ on $Q_i^{s,t} \subset N^{s,t}$ by

$$(\Xi_i^{s,t})^*(w_{\mathbf{d}}^{s,t})(\sigma, r) = \bigl(1 - F(2t^{-\tau} r)\bigr)\varphi_i^*(v_i^{\mathbf{c}_i})(\sigma, t^{-1}r) + F(2t^{-\tau} r)c_i^j \qquad (75)$$

on $\Sigma_i^j \times (tT, R')$, for $i = 1,\ldots,n$ and $j = 1,\ldots,l_i$.

The argument of Definition 6.3 shows that $w_{\mathbf{d}}^{s,t}$ is smooth, and linear in $\mathbf{d}$. Define $W^{s,t} = \{w_{\mathbf{d}}^{s,t} : \mathbf{d} \in \mathbb{R}^q\}$. Then $W^{s,t} \subset C^\infty(N^{s,t})$ is a vector subspace isomorphic to $\mathbb{R}^q$, and $1 \in W^{s,t}$. Define $\pi_{W^{s,t}} : L^2(N^{s,t}) \to W^{s,t}$ to be projection onto $W^{s,t}$ using the L^2-inner product.

7.2 Estimating $\operatorname{Im}\Omega^s|_{N^{s,t}}$

We now estimate $\operatorname{Im}\Omega|_{N^{s,t}}$ as in §6.2, to show that $N^{s,t}, W^{s,t}$ satisfy part (i) of Theorem 5.3. Following Proposition 6.4 we bound $\varepsilon^{s,t}$ at each point in $N^{s,t}$.

Proposition 7.4 *In the situation above, making $\mathcal{F}'$ and $\delta > 0$ smaller if nec-*

essary, there exists $C > 0$ such that for all $s \in \mathcal{F}'$ and $t \in (0, \delta)$ we have

$$|\varepsilon^{s,t}| \leqslant C|s|, \quad |\mathrm{d}\varepsilon^{s,t}| \leqslant C|s| \quad \textit{on } K^s, \tag{76}$$

$$\big|(\Xi_i^{s,t})^*(\varepsilon^{s,t})\big|(\sigma, r) \leqslant \begin{cases} Cr, & r \in (tT, t^\tau], \\ Ct^{\tau(\mu_i - 2)} + Ct^{(1-\tau)(2-\lambda_i)}, & r \in (t^\tau, 2t^\tau), \\ Cr|s|, & r \in [2t^\tau, R'), \end{cases} \tag{77}$$

$$\big|(\Xi_i^{s,t})^*(\mathrm{d}\varepsilon^{s,t})\big|(\sigma, r) \leqslant \begin{cases} C, & r \in (tT, t^\tau], \\ Ct^{\tau(\mu_i - 3)} + Ct^{(1-\tau)(2-\lambda_i)-\tau}, & r \in (t^\tau, 2t^\tau), \\ C|s|, & r \in [2t^\tau, R'), \end{cases} \tag{78}$$

$$\textit{and} \quad |\varepsilon^{s,t}| \leqslant Ct, \quad |\mathrm{d}\varepsilon^{s,t}| \leqslant C \quad \textit{on } P_i^{s,t} \textit{ for all } i = 1, \dots, n. \tag{79}$$

Here $|\,.\,|$ is computed using $h^{s,t}$ or $(\Xi_i^{s,t})^*(h^{s,t})$.

Proof. When $s = 0$ we have $N^{0,t} = N^t$, where N^t is as in [9, Def. 6.2]. Therefore [9, Prop. 6.4] proves (76)–(79) when $s = 0$. Now $K^s = \Phi_{X'}^s(K)$ is independent of t, and so are $h^{s,t}|_{K^s}$ and $\varepsilon^{s,t}|_{K^s}$. Also $\varepsilon^{0,t}|_{K^0} \equiv 0$ as $K^0 = K$ is special Lagrangian in $(M, J^0, \omega^0, \Omega^0)$, and $\varepsilon^{s,t}$ depends smoothly on s. Therefore by Taylor's Theorem and compactness of $K^s \cong K$ we see that

$$|\varepsilon^{s,t}| = O\big(|s|\big), \quad |\mathrm{d}\varepsilon^{s,t}| = O\big(|s|\big) \quad \text{on } K^s, \text{ for small } s \in \mathcal{F}'. \tag{80}$$

Consider the m-form $\psi^s(x_i)^{-m}(\Xi_i^{s,t})^*(\varepsilon^{s,t}\mathrm{d}V^{s,t}) - \psi(x_i)^{-m}(\Xi_i^{0,t})^*(\varepsilon^{0,t}\mathrm{d}V^{0,t})$ on $\Sigma_i \times (tT, R')$. By (74) this is the pull-back of $\psi^s(x_i)^{-m}(\Upsilon_i^s)^*(\operatorname{Im}\Omega^s) - \psi(x_i)^{-m}\Upsilon_i^*(\operatorname{Im}\Omega)$ on B_R under the map $(\sigma, r) \mapsto \Phi_{C_i}\big(\sigma, r, \xi_i^{1,t}(\sigma, r), \xi_i^{2,t}(\sigma, r)\big)$. But since $(\upsilon_i^s)^*(\Omega) = \psi^s(x_i)^m\Omega'$, by (70) we see that

$$\psi^s(x_i)^{-m}(\Upsilon_i^s)^*(\operatorname{Im}\Omega^s) - \psi(x_i)^{-m}\Upsilon_i^*(\operatorname{Im}\Omega) = O\big(r|s|\big) \quad \text{on } B_R. \tag{81}$$

Combining all these facts we see that

$$\big|(\Xi_i^{s,t})^*(\varepsilon^{s,t})\big| = \big(1 + O(|s|)\big) \cdot \big|(\Xi_i^{0,t})^*(\varepsilon^{0,t})\big| + O\big(r|s|\big) \quad \text{on } \Sigma_i \times (tT, R'). \tag{82}$$

A similar proof shows that

$$\big|(\Xi_i^{s,t})^*(\mathrm{d}\varepsilon^{s,t})\big| = \big(1 + O(|s|)\big) \cdot \big|(\Xi_i^{0,t})^*(\mathrm{d}\varepsilon^{0,t})\big| + O\big(|s|\big) \quad \text{on } \Sigma_i \times (tT, R'). \tag{83}$$

Since $\big|(\Upsilon_i^s)^*(\operatorname{Im}\Omega^s)|_{tL_i}\big| = O(r)$ on $tL_i \cap B_R$ for small $s \in \mathcal{F}'$, as in (42), the proofs of (44) and (45) show that for $i = 1, \dots, n$ we have

$$|\varepsilon^{s,t}| = O(t), \quad |\mathrm{d}\varepsilon^{s,t}| = O(1) \quad \text{on } P_i^{s,t}, \text{ for small } s \in \mathcal{F}'. \tag{84}$$

Finally, making $\mathcal{F}'$ and $\delta > 0$ smaller if necessary, for some $C > 0$ we see that equation (76) follows from (80), equation (77) from (77) with $s = 0$ and (82), equation (78) from (78) with $s = 0$ and (83), and equation (79) from (84). $\square$

Proposition 6.5 then immediately generalizes to give:

Proposition 7.5 *For some $C'>0$ and all $s\in\mathcal{F}'$ and $t\in(0,\delta)$ we have*

$$\|\varepsilon^{s,t}\|_{L^{2m/(m+2)}}\leqslant C'|s|+C't^{\tau(1+m/2)}\sum_{i=1}^n\bigl(t^{\tau(\mu_i-2)}+t^{(1-\tau)(2-\lambda_i)}\bigr),\qquad(85)$$

$$\|\varepsilon^{s,t}\|_{C^0}\leqslant C'|s|+C'\sum_{i=1}^n\bigl(t^{\tau(\mu_i-2)}+t^{(1-\tau)(2-\lambda_i)}\bigr),\qquad(86)$$

$$\text{and}\quad\|\mathrm{d}\varepsilon^{s,t}\|_{L^{2m}}\leqslant C'|s|+C't^{-\tau/2}\sum_{i=1}^n\bigl(t^{\tau(\mu_i-2)}+t^{(1-\tau)(2-\lambda_i)}\bigr),\qquad(87)$$

computing norms with respect to the metric $h^{s,t}$ on $N^{s,t}$.

7.3 Bounding $\|\pi_{W^{s,t}}(\varepsilon^{s,t})\|_{L^1}$ under conditions on s,t

Next we estimate $\|\pi_{W^{s,t}}(\varepsilon^{s,t})\|_{L^1}$, using the method of [9, §7.2]. First we bound $\int_{N^{s,t}}w^{s,t}_{\mathbf{d}}\varepsilon^{s,t}\,\mathrm{d}V^{s,t}$ for all $\mathbf{d}\in\mathbb{R}^q$. As $w^{s,t}_{\mathbf{d}}\equiv d_k$ on $\Phi^s_{X'}(K\cap X'_k)$ and $\Xi^{s,t}_i(\Sigma^j_i\times[t^\tau,R'))$ when $k(i,j)=k$, we see that

$$\begin{aligned}\int_{N^{s,t}}w^{s,t}_{\mathbf{d}}\varepsilon^{s,t}\,\mathrm{d}V^{s,t}&=\sum_{i=1}^n\biggl(\int_{P^{s,t}_i}w^{s,t}_{\mathbf{d}}\varepsilon^{s,t}\,\mathrm{d}V^{s,t}+\int_{\Xi^{s,t}_i(\Sigma_i\times(tT,t^\tau))}w^{s,t}_{\mathbf{d}}\varepsilon^{s,t}\,\mathrm{d}V^{s,t}\biggr)\\&+\sum_{k=1}^q d_k\biggl(\int_{\Phi^s_{X'}(K\cap X'_k)}\varepsilon^{s,t}\,\mathrm{d}V^{s,t}+\sum_{\substack{1\leqslant i\leqslant n,\,1\leqslant j\leqslant l_i:\\k(i,j)=k}}\int_{\Xi^{s,t}_i(\Sigma^j_i\times[t^\tau,R'))}\varepsilon^{s,t}\,\mathrm{d}V^{s,t}\biggr).\end{aligned}\qquad(88)$$

Following the proof of [9, Prop. 7.4] with trivial modifications, we find:

Proposition 7.6 *For all $s\in\mathcal{F}'$, $t\in(0,\delta)$, $\mathbf{d}\in\mathbb{R}^q$ and $i=1,\ldots,n$ we have*

$$\int_{P^{s,t}_i}w^{s,t}_{\mathbf{d}}\varepsilon^{s,t}\,\mathrm{d}V^{s,t}+\int_{\Xi^{s,t}_i(\Sigma_i\times(tT,t^\tau))}w^{s,t}_{\mathbf{d}}\varepsilon^{s,t}\,\mathrm{d}V^{s,t}=O\bigl(|\mathbf{d}|t^{(m+1)\tau}\bigr).\qquad(89)$$

Observe that the closure $\overline{X'_k}$ of X'_k in M is an m-chain in M *without boundary*, and thus defines an integral homology class $[\overline{X'_k}]\in H_m(M,\mathbb{Z})$, with $[X]=\sum_{k=1}^q[\overline{X'_k}]$. Using this we can state the analogue of [9, Prop. 7.5].

Proposition 7.7 *Making $\mathcal{F}'$ smaller if necessary, for all $s\in\mathcal{F}'$, $t\in(0,\delta)$, $\mathbf{d}\in\mathbb{R}^q$ and $k=1,\ldots,q$ we have*

$$\begin{aligned}\int_{\Phi^s_{X'}(K\cap X'_k)}\varepsilon^{s,t}\,\mathrm{d}V^{s,t}+\sum_{\substack{1\leqslant i\leqslant n,\,1\leqslant j\leqslant l_i:\\k(i,j)=k}}\int_{\Xi^{s,t}_i(\Sigma^j_i\times[t^\tau,R'))}\varepsilon^{s,t}\,\mathrm{d}V^{s,t}=\bigl[\operatorname{Im}\Omega^s\bigr]\cdot\bigl[\overline{X'_k}\bigr]\\-t^m\sum_{\substack{1\leqslant i\leqslant n,\,1\leqslant j\leqslant l_i:\\k(i,j)=k}}\psi(x_i)^mZ(L_i)\cdot[\Sigma^j_i]+O\bigl(t^{(m+1)\tau}\bigr)+O\bigl(|s|t^m\bigr).\end{aligned}\qquad(90)$$

Here $[\operatorname{Im}\Omega^s]\in H^m(M,\mathbb{R})$, $[\overline{X'_k}]\in H_m(M,\mathbb{Z})$, $Z(L_i)\in H^{m-1}(\Sigma_i,\mathbb{R})$ is as in §4.1, and $[\Sigma^j_i]\in H_{m-1}(\Sigma_i,\mathbb{Z})$.

Proof. As $\varepsilon^{s,t}\mathrm{d}V^{s,t} = \operatorname{Im}\Omega^s|_{N^{s,t}}$, the left hand side of (90) is the integral of $\operatorname{Im}\Omega^s$ over the m-chain

$$Z_k = \Phi^s_{X'}(K \cap X'_k) + \sum_{\substack{1\leqslant i\leqslant n,\ 1\leqslant j\leqslant l_i:\\ k(i,j)=k}} \Xi^{s,t}_i\bigl(\Sigma^j_i \times [t^\tau, R')\bigr)$$

for $k = 1, \ldots, q$, which is a closed subset of $N^{s,t}$, with boundary $(m-1)$-chain

$$\partial Z_k = -\sum_{\substack{1\leqslant i\leqslant n,\ 1\leqslant j\leqslant l_i:\\ k(i,j)=k}} \Xi^{s,t}_i\bigl(\Sigma^j_i \times \{t^\tau\}\bigr). \tag{91}$$

For each $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$, define an m-chain A^j_i in B_R to be the image of $\Sigma^j_i \times [0, 1]$ under the map $\Sigma^j_i \times [0, 1] \to B_R$ given by

$$(\sigma, r) \longmapsto r\Phi_{C_i}\bigl(\sigma, t^\tau, t^2\chi^1_i(\sigma, t^{\tau-1}), t^2\chi^2_i(\sigma, t^{\tau-1})\bigr).$$

As $\Upsilon^s_i \circ \Phi_{C_i}\bigl(\sigma, t^\tau, t^2\chi^1_i(\sigma, t^{\tau-1}), t^2\chi^2_i(\sigma, t^{\tau-1})\bigr) \equiv \Xi^{s,t}_i(\sigma, t^\tau)$ for $\sigma \in \Sigma_i$ by Definition 7.2, we see that

$$\partial\bigl(\Upsilon^s_i(A^j_i)\bigr) = \Xi^{s,t}_i\bigl(\Sigma^j_i \times \{t^\tau\}\bigr), \tag{92}$$

regarding $\Upsilon^s_i(A^j_i)$ as an m-chain in M.

Define another m-chain Z'_k for $k = 1, \ldots, q$ to be

$$Z'_k = \overline{X'_k} - \sum_{\substack{1\leqslant i\leqslant n,\ 1\leqslant j\leqslant l_i:\\ k(i,j)=k}} \Upsilon^s_i(A^j_i).$$

As $\overline{X'_k}$ is an m-chain without boundary, we see from (91) and (92) that $\partial Z'_k = \partial Z_k$, and in fact it is easy to see that Z'_k and Z_k are homologous in M. Since $\operatorname{Im}\Omega^s$ is a closed m-form on M, this implies that $\int_{Z'_k}\operatorname{Im}\Omega^s = \int_{Z_k}\operatorname{Im}\Omega^s$.

From Theorem 3.12 we have $(\upsilon^s_i)^*(\Omega^s) = \psi^s(x_i)^m\Omega'$, where Ω' is as in (1). Thus as $(\Upsilon^s_i)^*(\Omega^s)$ is smooth on B_R, Taylor's theorem gives

$$(\Upsilon^s_i)^*(\Omega^s) = \psi^s(x_i)^m\Omega' + O(r) \quad \text{on } B_R. \tag{93}$$

Making $\mathcal{F}'$ smaller if necessary, this holds *uniformly in* s for $s \in \mathcal{F}'$.

Now A^j_i is an m-chain in $B_R \subset \mathbb{C}^m$ with boundary in the AC SL m-fold tL_i, and $[\partial A^j_i] \in H_{m-1}(tL_i, \mathbb{R})$ is the image of $[\Sigma^j_i] \in H_{m-1}(\Sigma_i, \mathbb{R})$ under the map $H_{m-1}(\Sigma_i, \mathbb{R}) \to H_{m-1}(L_i, \mathbb{R})$ dual to the map $H^{m-1}(L_i, \mathbb{R}) \to H^{m-1}(\Sigma_i, \mathbb{R})$ of (19). It then follows easily from Definition 4.2 and Lemma 4.4 that

$$\int_{A^j_i}\operatorname{Im}\Omega' = Z(tL_i)\cdot[\Sigma^j_i] = t^m Z(L_i)\cdot[\Sigma^j_i]. \tag{94}$$

But as $r = O(t^\tau)$ on A^j_i and $\operatorname{vol}(A^j_i) = O(t^{m\tau})$ we see from (93) that

$$\int_{A^j_i}\bigl((\Upsilon^s_i)^*(\operatorname{Im}\Omega^s) - \psi^s(x_i)^m\operatorname{Im}\Omega'\bigr) = O(t^{(m+1)\tau}), \tag{95}$$

which again holds uniformly in $s \in \mathcal{F}'$. As the left hand side of (90) is $\int_{Z_k} \operatorname{Im} \Omega^s = \int_{Z'_k} \operatorname{Im} \Omega^s$, equation (90) follows from $\int_{X'_k} \operatorname{Im} \Omega^s = [\operatorname{Im} \Omega^s] \cdot [\overline{X'_k}]$, equations (94), (95) and $\psi^s(x_i) = \psi(x_i) + O(|s|)$. This completes the proof. □

Now we can estimate $\|\pi_{W^{s,t}}(\varepsilon^{s,t})\|_{L^1}$ under conditions on s, t.

Proposition 7.8 *Making $\mathcal{F}', \delta$ smaller if necessary, there exists $C'' > 0$ such that for all $s \in \mathcal{F}'$ and $t \in (0, \delta)$ satisfying*

$$\big[\operatorname{Im} \Omega^s\big] \cdot \big[\overline{X'_k}\big] = t^m \sum_{\substack{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l_i: \\ k(i,j)=k}} \psi(x_i)^m Z(L_i) \cdot [\Sigma_i^j] \quad \textit{for } k = 1, \ldots, q, \tag{96}$$

we have $\|\pi_{W^{s,t}}(\varepsilon^{s,t})\|_{L^1} \leqslant C'' t^{(m+1)\tau} + C''|s|t^m$.

Proof. Let $s \in \mathcal{F}'$ and $t \in (0, \delta)$ satisfy (96). Combining equations (88), (89) (90) and (96) gives

$$\int_{N^{s,t}} w_{\mathbf{d}}^{s,t} \varepsilon^{s,t} \, \mathrm{d}V^{s,t} = O\big(|\mathbf{d}| t^{(m+1)\tau}\big) + O\big(|\mathbf{d}||s|t^m\big) \quad \text{for all } \mathbf{d} \in \mathbb{R}^q. \tag{97}$$

One can show from Definition 7.3 that $\|w_{\mathbf{d}}^{s,t}\|_{L^2} \geqslant C|\mathbf{d}|$ for some $C > 0$ and all $\mathbf{d}, s, t$. This and (97) imply that $\|\pi_{W^{s,t}}(\varepsilon^{s,t})\|_{L^2} = O(t^{(m+1)\tau}) + O(|s|t^m)$. But $\|\pi_{W^{s,t}}(\varepsilon^{s,t})\|_{L^1} \leqslant \operatorname{vol}(N^{s,t})^{1/2} \|\pi_{W^{s,t}}(\varepsilon^{s,t})\|_{L^2}$, and $\operatorname{vol}(N^{s,t}) = O(1)$. Therefore $\|\pi_{W^{s,t}}(\varepsilon^{s,t})\|_{L^1} = O(t^{(m+1)\tau}) + O(|s|t^m)$. So making $\mathcal{F}', \delta$ smaller if necessary, there exists $C'' > 0$ such that the proposition holds. □

If we do not impose equation (96) then the t^m term in (90) means that we expect $\|\pi_{W^{s,t}}(\varepsilon^{s,t})\|_{L^1} = O(t^m)$, which is too big for part (i) of Theorem 5.3 to hold for $N^{s,t}, W^{s,t}$. We can now prove the analogue of Theorem 6.7.

Theorem 7.9 *In the situation of Definitions 7.1–7.3, making $\mathcal{F}'$ and $\delta > 0$ smaller if necessary, there exist $A_2 > 0$ and $\kappa > 1$ such that for all $s \in \mathcal{F}'$ and $t \in (0, \delta)$ with $|s| \leqslant t^{\kappa + m/2}$ and*

$$\big[\operatorname{Im} \Omega^s\big] \cdot \big[\overline{X'_k}\big] = t^m \sum_{\substack{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l_i: \\ k(i,j)=k}} \psi(x_i)^m Z(L_i) \cdot [\Sigma_i^j] \quad \textit{for } k = 1, \ldots, q, \tag{98}$$

the functions $\varepsilon^{s,t} = (\psi^s)^m \sin \theta^{s,t}$ on $N^{s,t}$ satisfy $\|\varepsilon^{s,t}\|_{L^{2m/(m+2)}} \leqslant A_2 t^{\kappa + m/2}$, $\|\varepsilon^{s,t}\|_{C^0} \leqslant A_2 t^{\kappa - 1}$, $\|\mathrm{d}\varepsilon^{s,t}\|_{L^{2m}} \leqslant A_2 t^{\kappa - 3/2}$ and $\|\pi_{W^{s,t}}(\varepsilon^{s,t})\|_{L^1} \leqslant A_2 t^{\kappa + m - 1}$, as in part (i) *of Theorem 5.3.*

Proof. Make $\mathcal{F}', \delta$ smaller if necessary so that Propositions 7.4–7.8 hold. Let C', C'' be as in Propositions 7.5 and 7.8, and set $A_2 = \max\big((2n+1)C', 2C''\big) > 0$. Let $s \in \mathcal{F}'$ and $t \in (0, \delta)$ satisfy $|s| \leqslant t^{\kappa + m/2}$ and (98). We seek $\kappa > 1$ so that the four bounds on $\varepsilon^{s,t}$ hold.

As $t<1$, $|s|\leqslant t^{\kappa+m/2}$ and $A_2\geqslant(2n+1)C'$, equations (85)–(87) imply that $\|\varepsilon^{s,t}\|_{L^{2m/(m+2)}}\leqslant A_2t^{\kappa+m/2}$, $\|\varepsilon^{s,t}\|_{C^0}\leqslant A_2t^{\kappa-1}$ and $\|\mathrm{d}\varepsilon^{s,t}\|_{L^{2m}}\leqslant A_2t^{\kappa-3/2}$ if

$$\tau(1+m/2)+\tau(\mu_i-2)\geqslant\kappa+m/2, \qquad \tau(1+m/2)+(1-\tau)(2-\lambda_i)\geqslant\kappa+m/2, \tag{99}$$

$$\tau(\mu_i-2)\geqslant\kappa-1, \qquad (1-\tau)(2-\lambda_i)\geqslant\kappa-1, \tag{100}$$

$$-\tau/2+\tau(\mu_i-2)\geqslant\kappa-3/2, \quad\text{and}\quad -\tau/2+(1-\tau)(2-\lambda_i)\geqslant\kappa-3/2 \tag{101}$$

for all $i=1,\ldots,n$. As $A_2\geqslant 2C''$ and $t<1$, by Proposition 7.8 we have $\|\pi_{W^{s,t}}(\varepsilon^{s,t})\|_{L^1}\leqslant A_2t^{\kappa+m-1}$ provided

$$(m+1)\tau\geqslant\kappa+m-1. \tag{102}$$

Since $0<\tau<1$ and $\mu_i>2$ the first equation of (99) admits a solution $\kappa>1$ provided $\tau>(2+m)/(2\mu_i-2+m)$, which holds by (71). Since $\lambda_i<0$ by Definition 7.1, we can suppose $\lambda_i<\frac{1}{2}(2-m)$ by part (a) of Theorem 4.7 applied to L_i, and then the second equation of (99) admits a solution $\kappa>1$. Also (99) implies (100) and (101) as $\tau\leqslant 1$. Equation (102) admits a solution $\kappa>1$ as $\tau>\frac{m}{m+1}$ by (71). Thus (having decreased λ_i if necessary) we can choose $\kappa>1$ satisfying (99)–(102), and the theorem is proved. □

Here we have restricted to pairs $s\in\mathcal{F}'$ and $t\in(0,\delta)$ with $|s|\leqslant t^{\kappa+m/2}$. This is so that the contributions $C'|s|$ in (85)–(87) can be absorbed into $A_2t^{\kappa+m/2}$, $A_2t^{\kappa-1}$ and $A_2t^{\kappa-3/2}$. Basically, Theorem 7.9 says that if s is not too large compared to t, and (98) holds, then part (i) of Theorem 5.3 holds for $N^{s,t},W^{s,t}$.

7.4 Parts (ii)–(vii) of Theorem 5.3

Next we carry out the programme of §6.3 and §6.4 for families. Here is the analogue of Definition 6.8 and [9, Def. 6.7].

Definition 7.10 For each $s\in\mathcal{F}'$ and $t\in(0,\delta)$, we define an open neighbourhood $U_{N^{s,t}}\subset T^*N^{s,t}$ of the zero section $N^{s,t}$ in $T^*N^{s,t}$, and a smooth map $\Phi_{N^{s,t}}:U_{N^{s,t}}\rightarrow M$. Let $\pi:T^*N^{s,t}\rightarrow N^{s,t}$ be the projection. Define

$$\begin{gathered}U_{N^{s,t}}\cap\pi^*(K^s)=\mathrm{d}(\Phi^s_{X'}|_K)\bigl(U_{X'}\cap\pi^*(K)\bigr)\quad\text{and}\\ \Phi_{N^{s,t}}|_{U_{N^{s,t}}\cap\pi^*(K^s)}\circ\mathrm{d}(\Phi^s_{X'}|_K)=\Phi^s_{X'}|_{U_{X'}\cap\pi^*(K)},\end{gathered} \tag{103}$$

where $\Phi^s_{X'}|_K:K\rightarrow K^s$ is a diffeomorphism and $\mathrm{d}(\Phi^s_{X'}|_K):T^*K\rightarrow T^*K^s$ the induced isomorphism.

Following (62), define $U_{N^{s,t}}\cap\pi^*(P_i^{s,t})$ and $\Phi_{N^{s,t}}|_{U_{N^{s,t}}\cap\pi^*(P_i^{s,t})}$ by

$$\begin{gathered}U_{N^{s,t}}\cap\pi^*(P_i^{s,t})=\mathrm{d}(\Upsilon_i^s\circ t)\bigl(\{\alpha\in T^*K_i:t^{-2}\alpha\in U_{L_i}\}\bigr)\\ \text{and}\quad \Phi_{N^{s,t}}\circ\mathrm{d}(\Upsilon_i^s\circ t)(\alpha)=\Upsilon_i^s\circ t\circ\Phi_{L_i}(t^{-2}\alpha).\end{gathered} \tag{104}$$

Modifying (63) and (64), define $U_{N^{s,t}}\cap\pi^*(Q_i^{s,t})$ and $\Phi_{N^{s,t}}|_{U_{N^{s,t}}\cap\pi^*(Q_i^{s,t})}$ by

$$(\mathrm{d}\Xi_i^{s,t})^*(U_{N^{s,t}})=\bigl\{(\sigma,r,\varsigma,u)\in T^*\bigl(\Sigma_i\times(tT,R')\bigr):\bigl|(\varsigma,u)\bigr|<\zeta r\bigr\}\quad\text{and} \tag{105}$$

$$\Phi_{N^{s,t}}\circ\mathrm{d}\Xi_i^{s,t}(\sigma,r,\varsigma,u)\equiv\Upsilon_i^s\circ\Phi_{C_i}\bigl(\sigma,r,\varsigma+\xi_i^{1,t}(\sigma,r),u+\xi_i^{2,t}(\sigma,r)\bigr). \tag{106}$$

The proof in [9, Def. 6.7] shows that making $\mathcal{F}', \delta$ smaller if necessary, $U_{N^{s,t}}$ and $\Phi_{N^{s,t}}$ are well-defined for all $s \in \mathcal{F}'$ and $t \in (0,\delta)$, and $U_{N^{s,t}}$ is an open tubular neighbourhood of $N^{s,t}$ in $T^*N^{s,t}$. Since $(\Upsilon_i^s)^*(\omega^s) = \omega'$ and $(\Phi_{X'}^s)^*(\omega^s) = \hat{\omega}$ we also find that $\Phi_{N^{s,t}}^*(\omega^s) = \hat{\omega}$. Define an m-form $\beta^{s,t}$ on $U_{N^{s,t}}$ by $\beta^{s,t} = \Phi_{N^{s,t}}^*(\operatorname{Im}\Omega^s)$.

Here is the analogue of Theorem 6.9.

Theorem 7.11 *Making $\mathcal{F}', \delta$ smaller if necessary, there exist $A_1, A_3, \ldots, A_6 > 0$ such that for all $s \in \mathcal{F}'$ and $t \in (0,\delta)$, as in* (ii)–(v) *of Theorem 5.3 we have*

- (ii) *$\psi^s \geqslant A_3$ on $N^{s,t}$.*
- (iii) *The subset $\mathcal{B}_{A_1 t} \subset T^*N^{s,t}$ of Definition 5.2 lies in $U_{N^{s,t}}$, and $\|\hat{\nabla}^k \beta^{s,t}\|_{C^0} \leqslant A_4 t^{-k}$ on $\mathcal{B}_{A_1 t}$ for $k = 0, 1, 2$ and 3.*
- (iv) *The injectivity radius $\delta(h^{s,t})$ satisfies $\delta(h^{s,t}) \geqslant A_5 t$.*
- (v) *The Riemann curvature $R(h^{s,t})$ satisfies $\|R(h^{s,t})\|_{C^0} \leqslant A_6 t^{-2}$.*

Here part (iii) *uses the notation of Definition 5.2, and parts* (iv) *and* (v) *refer to the compact Riemannian manifold $(N^{s,t}, h^{s,t})$.*

Note that here we do not require s to be small compared to t, as we did in Theorem 7.9, but instead parts (ii)–(v) hold uniformly over $s \in \mathcal{F}'$. The proof is a straightforward extension of that of Theorem 6.9 to families. For any given $s \in \mathcal{F}'$ the previous proof gives $\delta, A_1, A_3, \ldots, A_6 > 0$, which can be chosen to depend continuously on s. Then making $\mathcal{F}'$ smaller, we can take $\delta, A_1, A_3, \ldots, A_6$ to be independent of s.

Making $\mathcal{F}'$ smaller if necessary, we find that all of [9, §6.4 & §7.3] can be extended to $(M, J^s, \omega^s, \Omega^s)$ for $s \in \mathcal{F}'$, and applies uniformly for $s \in \mathcal{F}'$. Thus with only minor modifications we prove versions of Theorems 6.10 and 6.11:

Theorem 7.12 *Making $\mathcal{F}'$ and $\delta > 0$ smaller if necessary, there exists $A_7 > 0$ such that for all $s \in \mathcal{F}'$ and $t \in (0,\delta)$, if $v \in L_1^2(N^{s,t})$ with $\int_{N^{s,t}} vw \,\mathrm{d}V^{s,t} = 0$ for all $w \in W^{s,t}$ then $v \in L^{2m/(m-2)}(N^{s,t})$ and $\|v\|_{L^{2m/(m-2)}} \leqslant A_7 \|\mathrm{d}v\|_{L^2}$.*

Theorem 7.13 *Making $\mathcal{F}'$ and $\delta > 0$ smaller if necessary, for all $s \in \mathcal{F}'$, $t \in (0,\delta)$ and $w \in W^{s,t}$ we have $\|\mathrm{d}^*\mathrm{d}w\|_{L^{2m/(m+2)}} \leqslant \frac{1}{2} A_7^{-1} \|\mathrm{d}w\|_{L^2}$, where $A_7 > 0$ is as in Theorem 7.12. Also there exists $A_8 > 0$ such that for all $s \in \mathcal{F}'$, $t \in (0,\delta)$ and $w \in W^{s,t}$ with $\int_{N^{s,t}} w \,\mathrm{d}V^{s,t} = 0$ we have $\|w\|_{C^0} \leqslant A_8 t^{1-m/2} \|\mathrm{d}w\|_{L^2}$.*

7.5 The main result for families when $Y(L_i) = 0$

Here is our first main result for desingularization in families, the analogue of Theorem 5.5, when we do not assume that X' is connected.

Theorem 7.14 *Suppose (M,J,ω,Ω) is an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1,\ldots,x_n$ and cones $C_1,\ldots,C_n$. Define $\psi:M\rightarrow(0,\infty)$ as in (3). Let $L_1,\ldots,L_n$ be Asymptotically Conical SL m-folds in $\mathbb{C}^m$ with cones $C_1,\ldots,C_n$ and rates $\lambda_1,\ldots,\lambda_n$. Suppose $\lambda_i<0$ for $i=1,\ldots,n$. Write $X'=X\setminus\{x_1,\ldots,x_n\}$ and $\Sigma_i=C_i\cap\mathcal{S}^{2m-1}$.*

Set $q=b^0(X')$, and let $X'_1,\ldots,X'_q$ be the connected components of X'. For $i=1,\ldots,n$ let $l_i=b^0(\Sigma_i)$, and let $\Sigma_i^1,\ldots,\Sigma_i^{l_i}$ be the connected components of Σ_i. Define $k(i,j)=1,\ldots,q$ by $\Upsilon_i\circ\varphi_i\bigl(\Sigma_i^j\times(0,R')\bigr)\subset X'_{k(i,j)}$ for $i=1,\ldots,n$ and $j=1,\ldots,l_i$. Suppose the compact m-manifold N obtained by gluing L_i into X' at x_i for $i=1,\ldots,n$ is connected. A sufficient condition for this to hold is that X and L_i for $i=1,\ldots,n$ are connected.

Suppose $\bigl\{(M,J^s,\omega^s,\Omega^s):s\in\mathcal{F}\bigr\}$ is a smooth family of deformations of (M,J,ω,Ω), with base space $\mathcal{F}\subset\mathbb{R}^d$. Let $\iota_*:H_2(X,\mathbb{R})\rightarrow H_2(M,\mathbb{R})$ be the natural inclusion. Suppose that

$$[\omega^s]\cdot\iota_*(\gamma)=0\quad\textit{for all } s\in\mathcal{F}\textit{ and }\gamma\in H_2(X,\mathbb{R}). \tag{107}$$

Define $\mathcal{G}\subseteq\mathcal{F}\times(0,1)$ to be the subset of $(s,t)\in\mathcal{F}\times(0,1)$ with

$$\bigl[\operatorname{Im}\Omega^s\bigr]\cdot\bigl[\overline{X'_k}\bigr]=t^m\sum_{\substack{1\leqslant i\leqslant n,\;1\leqslant j\leqslant l_i:\\ k(i,j)=k}}\psi(x_i)^m Z(L_i)\cdot[\Sigma_i^j\,]\quad\textit{for } k=1,\ldots,q. \tag{108}$$

Then there exist $\epsilon\in(0,1)$ and $\kappa>1$ and a smooth family

$$\bigl\{\tilde N^{s,t}:(s,t)\in\mathcal{G},\quad t\in(0,\epsilon],\quad |s|\leqslant t^{\kappa+m/2}\bigr\}, \tag{109}$$

such that $\tilde N^{s,t}$ is a compact, nonsingular SL m-fold in $(M,J^s,\omega^s,\Omega^s)$ diffeomorphic to N, which is constructed by gluing tL_i into X at x_i for $i=1,\ldots,n$. In the sense of currents in Geometric Measure Theory, $\tilde N^{s,t}\rightarrow X$ as $s,t\rightarrow 0$.

Proof. The hypotheses of the theorem imply that conditions (i)–(iii) of Definition 7.1 hold. Let $\mathcal{F}',\delta$ and $N^{s,t}$ for $s\in\mathcal{F}'$ and $t\in(0,\delta)$ be as in Definition 7.2 and $W^{s,t}$ as in Definition 7.3, and make $\mathcal{F}'$ and $\delta>0$ smaller if necessary so that Theorems 7.9, 7.11, 7.12 and 7.13 apply. Theorems 7.9, 7.11, 7.12 and 7.13 now give $\kappa>1$ and $A_1,\ldots,A_8>0$. We show that $\cos\theta^{s,t}\geqslant\frac{1}{2}$ on $N^{s,t}$ for small s,t as in the proof of Theorem 6.12.

Let $\epsilon,K>0$ be as given in Theorem 5.3 depending on $\kappa,A_1,\ldots,A_8$ and m, and make $\epsilon>0$ smaller if necessary to ensure that $\epsilon<\delta$, and that if $s\in\mathcal{F}$ with $|s|\leqslant\epsilon^{\kappa+m/2}$ then $s\in\mathcal{F}'$ and $\cos\theta^{s,t}\geqslant\frac{1}{2}$ on $N^{s,t}$ for $t\in(0,\epsilon]$. Let $\mathcal{G}$ be as in the theorem, and suppose $(s,t)\in\mathcal{G}$ with $t\in(0,\epsilon]$ and $|s|\leqslant t^{\kappa+m/2}$. Then $s\in\mathcal{F}'$, as $|s|\leqslant t^{\kappa+m/2}\leqslant\epsilon^{\kappa+m/2}$.

Theorem 7.9 shows that part (i) of Theorem 5.3 holds for $N^{s,t},W^{s,t}$, as $|s|\leqslant t^{\kappa+m/2}$ and (98) holds by choice of s,t. Theorems 7.11, 7.12 and 7.13 show that parts (ii)–(v), (vi) and (vii) of Theorem 5.3 hold for $N^{s,t},W^{s,t}$. Thus as $t\leqslant\epsilon$, Theorem 5.3 shows that there exists a nearby compact, nonsingular SL

m-fold $\tilde{N}^{s,t}$ in $(M, J^s, \omega^s, \Omega^s)$, as in (109). The remaining conclusions follow as for Theorem 6.12. $\square$

Putting $d = 0$ and $\mathcal{F} = \{0\} = \mathbb{R}^d$, Theorem 7.14 reduces to Theorem 5.5. Equation (107) is a necessary condition for the existence of any Lagrangian m-fold $\tilde{N}^{s,t}$ in $(M, J^s, \omega^s, \Omega^s)$ desingularizing X. However, (108) cannot be justified in the same way, and actually it's sufficient for $[\operatorname{Im}\Omega^s]\cdot[X] = 0$ to hold exactly (this follows from the sum of (108) over $k = 1, \ldots, q$) and for (108) to hold only approximately, that is, up to terms of order $O(t^{\kappa+m-1})$.

If X' is connected, so that $q = 1$, then as for Theorem 6.13 the right hand side of (108) is zero automatically. Then the theorem simplifies to give:

Theorem 7.15 *Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Let $L_1, \ldots, L_n$ be Asymptotically Conical SL m-folds in $\mathbb{C}^m$ with cones $C_1, \ldots, C_n$ and rates $\lambda_1, \ldots, \lambda_n$. Suppose $\lambda_i < 0$ for $i = 1, \ldots, n$, and $X' = X \setminus \{x_1, \ldots, x_n\}$ is connected.*

Suppose $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ is a smooth family of deformations of (M, J, ω, Ω), with base space $\mathcal{F} \subset \mathbb{R}^d$. Let $\iota_* : H_2(X, \mathbb{R}) \to H_2(M, \mathbb{R})$ be the natural inclusion. Suppose that

$$\begin{aligned} [\omega^s]\cdot\iota_*(\gamma) &= 0 \quad \text{for all } s \in \mathcal{F} \text{ and } \gamma \in H_2(X, \mathbb{R}), \text{ and} \\ [\operatorname{Im}\Omega^s]\cdot[X] &= 0 \quad \text{for all } s \in \mathcal{F}, \text{ where } [X] \in H_m(M, \mathbb{R}). \end{aligned} \tag{110}$$

Then there exist $\epsilon > 0$ and $\kappa > 1$ and a smooth family

$$\bigl\{\tilde{N}^{s,t} : s \in \mathcal{F}, \quad t \in (0, \epsilon], \quad |s| \leqslant t^{\kappa+m/2}\bigr\}, \tag{111}$$

such that $\tilde{N}^{s,t}$ is a compact, nonsingular SL m-fold in $(M, J^s, \omega^s, \Omega^s)$, which is constructed by gluing tL_i into X at x_i for $i = 1, \ldots, n$. In the sense of currents in Geometric Measure Theory, $\tilde{N}^{s,t} \to X$ as $s, t \to 0$.

Putting $d = 0$ and $\mathcal{F} = \{0\} = \mathbb{R}^d$, Theorem 7.15 reduces to Theorem 5.4. When $s = 0$ the SL m-fold $\tilde{N}^{0,t}$ in Theorem 7.15 coincides with the SL m-fold $\tilde{N}^t$ in (M, J, ω, Ω) constructed in Theorem 5.4. Now given the compact, nonsingular SL m-folds $\tilde{N}^t$ in Theorem 5.4, we can apply Theorem 2.9 to prove the existence of SL m-folds $\tilde{N}^{s,t}$ in $(M, J^s, \omega^s, \Omega^s)$ for s small compared to t.

Thus, much of Theorem 7.15 follows from Theorems 2.9 and 5.4. The new feature is that when t is small $|s| \leqslant t^{\kappa+m/2}$ is sufficient for the existence of $\tilde{N}^{s,t}$, whereas Theorems 2.9 and 5.4 give no quantitative restrictions on s, t.

However, Theorem 7.14 does *not* follow from Theorems 2.9 and 5.5. This is because Theorem 7.14 can prove the existence of desingularizations $\tilde{N}^{s,t}$ in $(M, J^s, \omega^s, \Omega^s)$ for $s \neq 0$ in cases when there do not exist desingularizations $\tilde{N}^t$ in (M, J, ω, Ω) because (28) does not hold, so Theorem 5.5 does not apply. Effectively, by deforming (M, J, ω, Ω) in a family $\mathcal{F}$ we can overcome obstructions to desingularizing X in the single almost Calabi–Yau m-fold (M, J, ω, Ω).

8 Desingularizing in families when $Y(L_i) \neq 0$

Finally we extend the material of §6 to families of almost Calabi–Yau m-folds $(M, J^s, \omega^s, \Omega^s)$ for $s \in \mathcal{F}$. There are *topological obstructions* to defining a Lagrangian m-fold $N^{s,t}$ in (M, ω^s) by gluing tL_i into X at x_i, as $[\omega^s|_{N^{s,t}}]$ may be nonzero in $H^2(N^{s,t}, \mathbb{R})$. Because of this, we can only define $N^{s,t}$ as a Lagrangian m-fold for (s,t) in a subset $\mathcal{G}$ of $\mathcal{F} \times (0,1)$, satisfying an equation (120) involving $[\omega^s]$, t and $Y(L_i)$ for $i = 1, \ldots, n$.

8.1 Setting up the problem

We shall consider the following situation, combining Definitions 6.1 and 7.1.

Definition 8.1 Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold, and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$ with identifications υ_i, cones C_i and rates μ_i. Let $L_1, \ldots, L_n$ be AC SL m-folds in $\mathbb{C}^m$, where L_i has cone C_i and rate λ_i for $i = 1, \ldots, n$. As in Definition 2.7, let $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ be a *smooth family of deformations* of (M, J, ω, Ω), with base space $\mathcal{F} \subset \mathbb{R}^d$.

Set $q = b^0(X')$, so that X' has q connected components, and number them $X'_1, \ldots, X'_q$. For $i = 1, \ldots, n$ let $l_i = b^0(\Sigma_i)$, so that Σ_i has l_i connected components, and number them $\Sigma_i^1, \ldots, \Sigma_i^{l_i}$. If $\Upsilon_i, \varphi_i, S_i$ are as in Definition 3.4, then $\Upsilon_i \circ \varphi_i$ is a diffeomorphism $\Sigma_i \times (0, R') \to S_i \subset X'$. For each $j = 1, \ldots, l_i$, $\Upsilon_i \circ \varphi_i\bigl(\Sigma_i^j \times (0, R')\bigr)$ is a connected subset of X', and so lies in exactly one of the X'_k for $k = 1, \ldots, q$. Define numbers $k(i,j) = 1, \ldots, q$ for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$. by $\Upsilon_i \circ \varphi_i(\Sigma_i^j \times (0, R')) \subset X'_{k(i,j)}$. Suppose that:

(i) The dimension m satisfies $2 < m < 6$, and

(ii) The AC SL m-fold L_i has rate $\lambda_i \leqslant 0$ for $i = 1, \ldots, n$.

(iii) $[\operatorname{Im} \Omega^s] \cdot [\overline{X_k}] = 0$ for all $s \in \mathcal{F}$ and $k = 1, \ldots, q$, where $[\overline{X_k}] \in H_m(M, \mathbb{R})$.

(iv) $\displaystyle\sum_{\substack{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l_i: \\ k(i,j) = k}} \psi(x_i)^m Z(L_i) \cdot [\Sigma_i^j] = 0$ for all $k = 1, \ldots, q$.

(v) Let N be the compact m-manifold obtained by gluing L_i into X' at x_i for $i = 1, \ldots, n$ in the obvious way. Suppose N is connected. A sufficient condition for this to hold is that X and L_i for $i = 1, \ldots, n$ are connected.

Let $g, \psi, R, B_R, \Sigma_i, \iota_i, \Upsilon_i, \zeta, U_{C_i}, \Phi_{C_i}, R', K, \phi_i, \eta_i, \eta_i^1, \eta_i^2, S_i, U_{X'}, \Phi_{X'}, A_i, \gamma_i, \pi_i, T, K_i, \varphi_i, \chi_i, \chi_i^1, \chi_i^2, U_{L_i}, \Phi_{L_i}, E_i$ and λ be as in Definition 6.1. Define

- Let $V \cong H^2_{\mathrm{cs}}(X', \mathbb{R})$ be a vector space of smooth closed 2-forms on X' supported in K representing $H^2_{\mathrm{cs}}(X', \mathbb{R})$.
- Let $Y(L_i) \in H^1(\Sigma_i, \mathbb{R})$ be as in Definition 4.2. Then by Hodge theory there exists a unique $\gamma_i \in C^\infty(T^*\Sigma_i)$ with $\mathrm{d}\gamma_i = \mathrm{d}^*\gamma_i = 0$ and $[\gamma_i] = Y(L_i) \in H^1(\Sigma_i, \mathbb{R})$. Let $\pi_i : \Sigma_i \times (0, R') \to \Sigma_i$ be the projection.

- Define $\varpi \in H^2_{\rm cs}(X',\mathbb{R})$ to be the image of $\bigl(Y(L_1),\ldots,Y(L_n)\bigr)$ under the map $\bigoplus_{i=1}^n H^1(\Sigma_i,\mathbb{R}) \rightarrow H^2_{\rm cs}(X',\mathbb{R})$ in (6). Let $\beta \in V$ be the unique element with $[\beta]=\varpi$.
- Choose $\nu_i \in (0,\mu_i-2)$ with $(0,\nu_i]\cap\mathcal{D}_{\Sigma_i}=\emptyset$ for $i=1,\ldots,n$.
- Apply part (b) of Theorem 3.6 with γ_i and β as above. This gives $\alpha \in C^\infty(T^*X')$ with $\mathrm{d}\alpha=\beta$, $\mathrm{d}^*(\psi^m\alpha)=0$ and $|\nabla^j\alpha|=O(\rho^{-1-j})$ for $k\geqslant 0$, and $T_i\in C^\infty\bigl(\Sigma_i\times(0,R')\bigr)$ for $i=1,\ldots,n$ with

$$(\Upsilon_i\circ\phi_i)^*(\alpha)=\pi_i^*(\gamma_i)+\mathrm{d}T_i \quad \text{on } \Sigma_i\times(0,R') \text{ for } i=1,\ldots,n, \text{ and} \tag{112}$$

$$\nabla^jT_i(\sigma,r)=O(r^{\nu_i-j}) \quad \text{as } r\rightarrow 0, \text{ for all } j\geqslant 0. \tag{113}$$

- Apply Theorems 3.12 and 3.13 to X and $\bigl\{(M,J^s,\omega^s,\Omega^s):s\in\mathcal{F}\bigr\}$ with V as above. This gives an open set $\mathcal{F}'\subseteq\mathcal{F}$ with $0\in\mathcal{F}'$ and $\psi^s,\upsilon_i^s,\Upsilon_i^s,\nu^s,\Phi^s_{X'}$ for $s\in\mathcal{F}'$ with $\nu^s\in V$ depending smoothly on s, satisfying (17) and

$$\begin{gathered}\upsilon_i^0=\upsilon_i,\;\; \Upsilon_i^0=\Upsilon_i,\;\; \nu^0=0,\;\; \Phi^0_{X'}=\Phi_{X'},\;\; (\upsilon_i^s)^*(\Omega)=\psi^s(x_i)^m\Omega',\\ \Upsilon_i^s(0)=x_i,\;\; \mathrm{d}\Upsilon_i^s|_0=\upsilon_i^s,\;\; (\Upsilon_i^s)^*(\omega^s)=\omega',\;\; (\Phi^s_{X'})^*(\omega^s)=\hat\omega+\pi^*(\nu^s).\end{gathered} \tag{114}$$

- Define a function $f^s\in C^\infty(X')$ for $s\in\mathcal{F}'$ by $(\Phi^s_{X'})^*(\operatorname{Im}\Omega^s)|_{X'}=f^s\,\mathrm{d}V$. As in (81), we can show that $\nabla^jf^s=O\bigl(\rho^{1+j}|s|\bigr)$ for all $j\geqslant 0$, where $\rho:X'\rightarrow(0,1]$ is a *radius function*, as in Definition 3.5. Part (iii) above implies that $\int_{X_k'}f^s\,\mathrm{d}V=0$ for all $s\in\mathcal{F}'$ and $k=1,\ldots,q$.
- Using $\nabla^jf^s=O\bigl(\rho^{1+j}|s|\bigr)$ and $\int_{X_k'}f^s\,\mathrm{d}V=0$ we can apply part (c) of Theorem 3.6 to f^s on X_k', for $s\in\mathcal{F}'$ and $k=1,\ldots,q$. This gives exact 1-forms on X_k' and functions on $\Sigma_i^j\times(0,R')$ for $k(i,j)=k$.

  Put these 1-forms and functions together for all k to give an exact 1-form ϱ^s on X' with $\mathrm{d}^*(\psi^m\varrho_k^s)=f^s$ and functions $Z_i^s\in C^\infty\bigl(\Sigma_i\times(0,R')\bigr)$ with $(\Upsilon_i\circ\phi_i)^*(\varrho^s)=\mathrm{d}Z_i^s$ on $\Sigma_i\times(0,R')$ for $i=1,\ldots,n$. As $\nabla^jf^s=O\bigl(\rho^{1+j}|s|\bigr)$, we find that

$$|\nabla^j\varrho^s|=O\bigl(\rho^{-1-j}|s|\bigr) \quad\text{and}\quad \nabla^jZ_i^s(\sigma,r)=O\bigl(r^{\nu_i-j}|s|\bigr) \quad \text{for all } j\geqslant 0. \tag{115}$$

Observe that parts (iii) and (iv) above are equivalent to requiring that both sides of (108) are zero. To prove a more general result, we would prefer to replace (iii) and (iv) with the single condition (108). However, part (iii) is necessary for the existence of ϱ^s and Z_i^s above, and this together with (108) implies part (iv). We will discuss this further after Theorem 8.9.

Next we define m-submanifolds $N^{s,t}$ in M for $s\in\mathcal{F}'$ and $t\in(0,\delta)$, adapting Definitions 6.2 and 7.2. It will turn out that $N^{s,t}$ is only *Lagrangian* in (M,ω^s) if $[\omega^s]\cdot\iota_*(\gamma)=t^2\varpi\cdot\gamma$ for all $\gamma\in H_2(X,\mathbb{R})$. However, we still define $N^{s,t}$ for all s,t, as it will be useful that $N^{s,t}$ depends smoothly on s,t.

Definition 8.2 We work in the situation of Definition 8.1. For $i = 1, \ldots, n$, $s \in \mathcal{F}'$ and small $t > 0$, define $P_i^{s,t} = \Upsilon_i^s(tK_i)$. Then $P_i^{s,t}$ is Lagrangian in (M, ω^s), as in Definition 7.2. Let F and τ be as in Definition 6.2. Modifying (34), for $i = 1, \ldots, n$ and $t > 0$ with $tT < t^\tau < 2t^\tau < R'$, define a closed 1-form $\xi_i^{s,t}$ on $\Sigma_i \times (tT, R')$ by

$$\begin{aligned}
\xi_i^{s,t}(\sigma, r) = {} & \mathrm{d}\big[F(t^{-\tau}r)A_i(\sigma, r) + t^2(1 - F(t^{-\tau}r))E_i(\sigma, t^{-1}r)\big] \\
& + t^2\pi_i^*(\gamma_i) + t^2\mathrm{d}\big[F(t^{-\tau}r)T_i(\sigma, r)\big] + \mathrm{d}\big[F(t^{-\tau}r)Z_i^s(\sigma, r)\big] \\
= {} & F(t^{-\tau}r)\eta_i(\sigma, r) + t^{-\tau}F'(t^{-\tau}r)A_i(\sigma, r)\mathrm{d}r \\
& + t^2(1-F(t^{-\tau}r))\chi_i(\sigma, t^{-1}r) - t^{2-\tau}F'(t^{-\tau}r)E_i(\sigma, t^{-1}r)\mathrm{d}r \\
& + t^2F(t^{-\tau}r)(\Upsilon_i \circ \phi_i)^*(\alpha)(\sigma, r) + t^{2-\tau}F'(t^{-\tau}r)T_i(\sigma, r)\mathrm{d}r \\
& + F(t^{-\tau}r)(\Upsilon_i \circ \phi_i)^*(\varrho^s)(\sigma, r) + t^{-\tau}F'(t^{-\tau}r)Z_i^s(\sigma, r)\mathrm{d}r.
\end{aligned} \tag{116}$$

Let $\xi_i^{1,s,t}, \xi_i^{2,s,t}$ be the components of $\xi_i^{s,t}$ in $T^*\Sigma$ and $\mathbb{R}$. Then when $r \geqslant 2t^\tau$ we have $\xi_i^{s,t} \equiv \eta_i + t^2(\Upsilon_i \circ \phi_i)^*(\alpha) + (\Upsilon_i \circ \phi_i)^*(\varrho^s)$, and when $r \leqslant t^\tau$ we have $\xi_i^{s,t}(\sigma, r) = t^2\chi_i(\sigma, t^{-1}r)$. Choose $\delta \in (0, 1]$ with $\delta T \leqslant \delta^\tau < 2\delta^\tau \leqslant R'$, $\delta K_i \subset B_R \subset \mathbb{C}^m$ and $\big|\xi_i^{s,t}(\sigma, r)\big| < \zeta r$ on $\Sigma_i \times (tT, R')$ for all $i = 1, \ldots, n$, $s \in \mathcal{F}'$ and $t \in (0, \delta)$. Making $\mathcal{F}'$ smaller if necessary, this is possible. Following (74), for $i = 1, \ldots, n$, $s \in \mathcal{F}'$ and $t \in (0, \delta)$ define $\Xi_i^{s,t} : \Sigma_i \times (tT, R') \to M$ by

$$\Xi_i^{s,t}(\sigma, r) = \Upsilon_i^s \circ \Phi_{C_i}\big(\sigma, r, \xi_i^{1,s,t}(\sigma, r), \xi_i^{2,s,t}(\sigma, r)\big). \tag{117}$$

Define $Q_i^{s,t} = \Xi_i^{s,t}\big(\Sigma_i \times (tT, R')\big)$ for $i = 1, \ldots, n$, $s \in \mathcal{F}'$ and $t \in (0, \delta)$. As in Definition 7.2, we find that $Q_i^{s,t}$ is Lagrangian in (M, ω^s).

Modifying Definition 6.2, let $\Gamma(t^2\alpha + \varrho^s)$ be the graph of the 1-form $t^2\alpha + \varrho^s$ in T^*X'. Then $\Gamma(t^2\alpha + \varrho^s) \cap \pi^*(K) \subset T^*K$ is the graph of $(t^2\alpha + \varrho^s)|_K$. By compactness of K, making $\mathcal{F}'$ and $\delta > 0$ smaller if necessary, we can suppose that $\Gamma(t^2\alpha + \varrho^s) \cap \pi^*(K) \subset U_{X'}$ for all $s \in \mathcal{F}'$ and $t \in (0, \delta)$. Define

$$K^{s,t} = \Phi_{X'}^s\big(\Gamma(t^2\alpha + \varrho^s) \cap \pi^*(K)\big) \quad \text{for } s \in \mathcal{F}' \text{ and } t \in (0, \delta). \tag{118}$$

Then $K^{s,t}$ is a submanifold of M with boundary, diffeomorphic to K.

We calculate when $K^{s,t}$ is Lagrangian in (M, ω^s). Write $\iota^{s,t}$ for the natural diffeomorphism $K \to K^{s,t}$ given by $\Phi_{X'}^s \circ (t^2\alpha + \varrho^s)|_K$. Now $(\Phi_{X'}^s)^*(\omega^s) = \hat\omega + \pi^*(\nu^s)$ by (114). Pushing down by $\pi : \Gamma(t^2\alpha + \varrho^s) \to X'$ gives

$$(\iota^{s,t})^*(\omega^s) = \pi_*\big(\hat\omega|_{\Gamma(t^2\alpha+\varrho^s)}\big) + \nu^s = -\mathrm{d}(t^2\alpha + \varrho^s) + \nu^s = -t^2\beta + \nu^s, \tag{119}$$

by some standard symplectic geometry, and as $\mathrm{d}\alpha = \beta$ and ϱ^s is exact.

Hence $K^{s,t}$ is Lagrangian in (M, ω^s) if and only if $\nu^s = t^2\beta$. But $\nu^s, \beta \in V$ and $V \cong H^2_{\mathrm{cs}}(X', \mathbb{R})$ by Definition 8.1, so $K^{s,t}$ is Lagrangian if and only if $[\nu^s] = t^2\varpi$ in $H^2_{\mathrm{cs}}(X', \mathbb{R})$, as $[\beta] = \varpi$. Now Theorem 3.13 identifies $[\nu^s]$ as the unique class in $H^2_{\mathrm{cs}}(X', \mathbb{R})$ with $[\nu^s] \cdot \gamma = [\omega^s] \cdot \iota_*(\gamma)$ for all $\gamma \in H_2(X, \mathbb{R})$, where $\iota_* : H_2(X, \mathbb{R}) \to H_2(M, \mathbb{R})$ is induced by the inclusion $\iota : X \to M$. Therefore $K^{s,t}$ is Lagrangian in (M, ω^s) if and only if

$$[\omega^s] \cdot \iota_*(\gamma) = t^2\varpi \cdot \gamma \quad \text{for all } \gamma \in H_2(X, \mathbb{R}). \tag{120}$$

Define $\mathcal{G} \subseteq \mathcal{F}' \times (0,\delta)$ to be the subset of (s,t) satisfying (120). Then $K^{s,t}$ is Lagrangian in (M,ω^s) if and only if $(s,t) \in \mathcal{G}$. For $s \in \mathcal{F}'$ and $t \in (0,\delta)$, define $N^{s,t}$ to be the disjoint union of $K^{s,t}$, $P_1^{s,t},\ldots,P_n^{s,t}$ and $Q_1^{s,t},\ldots,Q_n^{s,t}$. Then $N^{s,t}$ is a compact, smooth m-submanifold of M *without boundary*, which depends smoothly on s,t. The proof of this follows Definitions 6.2 and 7.2, with simple changes. Also $N^{s,t}$ is *Lagrangian* in (M,ω^s) if and only if $(s,t) \in \mathcal{G}$.

Let $h^{s,t}$, $\mathrm{d}V^{s,t}$, $\mathrm{e}^{i\theta^{s,t}}$ and $\varepsilon^{s,t} = (\psi^s)^m \sin\theta^{s,t}$ be as in Definition 7.2.

Here equation (120) is a weakening of the combination of part (iii) of Definition 6.1 and part (ii) of Definition 7.1. For by exactness in (6) and the definition of ϖ, we see that part (iii) of Definition 6.1 admits a solution ϱ if and only if $\varpi = 0$. But if $\varpi = 0$ then (120) becomes part (ii) of Definition 7.1. Thus part (iii) of Definition 6.1 and part (ii) of Definition 7.1 together imply (120), but not vice versa.

It is not difficult to see that (120) is a *necessary condition* for the existence of a *Lagrangian* m-fold $N^{s,t}$ in (M,ω^s) made by gluing tL_i into X at x_i for $i = 1,\ldots,n$, just as part (iii) of Definition 6.1 and part (ii) of Definition 7.1 were necessary for the existence of Lagrangian m-folds N^t and $N^{s,t}$ in §6 and §7. The definition shows that (120) is also a *sufficient condition* for small s,t.

In §7.2 we saw that $\mathrm{Im}\,\Omega^s|_{N^{s,t}}$ is $O(|s|)$ on K^s. The condition $|s| \leqslant t^{\kappa+m/2}$ in Theorem 7.15 is there to control these $O(|s|)$ error terms. However, in this section we will need to allow $|s| = O(t^2)$, so we cannot require $|s| \leqslant t^{\kappa+m/2}$. Therefore, an $O(|s|)$ contribution to $\mathrm{Im}\,\Omega^s|_{N^{s,t}}$ on $K^{s,t}$ is unacceptably large in this section.

This is the reason for using the exact 1-form ϱ^s and functions Z_i^s to construct $N^{s,t}$. They will have the effect of cancelling out the $O(|s|)$ contributions, so that $\mathrm{Im}\,\Omega^s|_{N^{s,t}} = O(|s|^2)$ on $K^{s,t}$. We could have used ϱ^s and Z_i^s in the same way in §7, and so relaxed the condition $|s| \leqslant t^{\kappa+m/2}$ in Theorem 7.15.

Here is the analogue of Definitions 6.3 and 7.3.

Definition 8.3 In the situation of Definitions 8.1 and 8.2 define vector spaces $W^{s,t} \subset C^\infty(N^{s,t})$ for $s \in \mathcal{F}'$ and $t \in (0,\delta)$, elements $w_{\mathbf{d}}^{s,t} \in W^{s,t}$ for $\mathbf{d} \in \mathbb{R}^q$, and the projection $\pi_{W^{s,t}}$, exactly as in Definition 7.3, except that in (i) we replace K^s by $K^{s,t}$, and define $w_{\mathbf{d}}^{s,t} \equiv d_k$ on the k^{th} connected component of $K^{s,t}$ diffeomorphic to $K \cap X_k'$.

8.2 Estimating $\mathrm{Im}\,\Omega^s|_{N^{s,t}}$

We now bound $\varepsilon^{s,t}$ on $N^{s,t}$, which yields estimates of $\mathrm{Im}\,\Omega^s|_{N^{s,t}}$ as $\mathrm{Im}\,\Omega^s|_{N^{s,t}} = \varepsilon^{s,t}\,\mathrm{d}V^{s,t}$. As this still makes sense if $N^{s,t}$ is not Lagrangian, we do it for all $s \in \mathcal{F}'$ and $t \in (0,\delta)$, rather than for $(s,t) \in \mathcal{G}$. Then we can exploit the smooth dependence of $N^{s,t}$ on s,t, including at $(0,0)$ in a certain sense. Here is the analogue of Propositions 6.4 and 7.4.

Proposition 8.4 *In the situation above, making $\mathcal{F}'$ and $\delta > 0$ smaller if nec-*

essary, there exists $C > 0$ such that for all $s \in \mathcal{F}'$ and $t \in (0,\delta)$ we have

$$|\varepsilon^{s,t}| \leqslant C|s|^2 + Ct^4, \quad |\mathrm{d}\varepsilon^{s,t}| \leqslant C|s|^2 + Ct^4 \quad \textit{on } K^{s,t}, \tag{121}$$

$$\bigl|(\Xi_i^{s,t})^*(\varepsilon^{s,t})\bigr|(\sigma,r) \leqslant \begin{cases} Cr, & r \in (tT, t^\tau], \\ C|s|t^{\tau(\nu_i-2)} + Ct^{4-4\tau} + Ct^{\tau(\mu_i-2)} \\ \quad + Ct^{(1-\tau)(2-\lambda)} + Ct^{2+\tau(\nu_i-2)}, & r \in (t^\tau, 2t^\tau), \\ C|s|^2 r^{-4} + Ct^4 r^{-4}, & r \in [2t^\tau, R'), \end{cases} \tag{122}$$

$$\bigl|(\Xi_i^{s,t})^*(\mathrm{d}\varepsilon^{s,t})\bigr|(\sigma,r) \leqslant \begin{cases} C, & r \in (tT, t^\tau], \\ C|s|t^{\tau(\nu_i-3)} + Ct^{4-5\tau} + Ct^{\tau(\mu_i-3)} \\ \quad + Ct^{(1-\tau)(2-\lambda)-\tau} + Ct^{2+\tau(\nu_i-3)}, & r \in (t^\tau, 2t^\tau), \\ C|s|^2 r^{-5} + Ct^4 r^{-5}, & r \in [2t^\tau, R'), \end{cases} \tag{123}$$

$$\textit{and} \quad |\varepsilon^{s,t}| \leqslant Ct, \quad |\mathrm{d}\varepsilon^{s,t}| \leqslant C \quad \textit{on } P_i^{s,t} \textit{ for all } i = 1,\dots,n. \tag{124}$$

Here $|\,.\,|$ is computed using $h^{s,t}$ or $(\Xi_i^{s,t})^*(h^{s,t})$.

Proof. The proof combines those of Propositions 6.4 and 7.4, so we will be brief. Identify $K^{s,t}$ and $\Xi_i^{s,t}\bigl(\Sigma_i \times [2t^\tau, R')\bigr)$ with the corresponding regions in X' in the natural way, so that we can regard f^s as a function, α, ϱ^s as 1-forms and h as a metric on these regions in $N^{s,t}$ rather than X'. Generalizing the proof of (46) and (47) we find that on $K^{s,t}$ and $\Xi_i^{s,t}\bigl(\Sigma_i \times [2t^\tau, R')\bigr)$ we have

$$\begin{aligned} \varepsilon^{s,t} &= f^s - \mathrm{d}^*_{s,t}\bigl((\psi^s)^m(t^2\alpha + \varrho^s)\bigr) + O\bigl(\rho^{-2}|t^2\alpha + \varrho^s|^2\bigr) + O\bigl(|t^2\nabla\alpha + \nabla\varrho^s|^2\bigr) \\ &= f^s - \mathrm{d}^*_{s,t}\bigl((\psi^s)^m(t^2\alpha + \varrho^s)\bigr) + O\bigl(\rho^{-4}|s|^2 + \rho^{-4}t^4\bigr) \end{aligned} \tag{125}$$

when $\rho^{-1}|t^2\alpha + \varrho^s|, |t^2\nabla\alpha + \nabla\varrho^s|$ are small. Here $\mathrm{d}^*_{s,t}$ is d^* calculated using $h^{s,t}$, and we use $|\nabla^j\alpha| = O(\rho^{-1-j})$ and $|\nabla^j\varrho^s| = O(\rho^{-1-j}|s|)$ in the second line.

Now $h^{s,t} = h + O\bigl(t^2\rho^{-2} + |s|\rho^{-2}\bigr)$ and $\nabla(h^{s,t} - h) = O\bigl(t^2\rho^{-3} + |s|\rho^{-3}\bigr)$ in these regions, and $\psi^s = \psi + O\bigl(|s|\bigr)$, $\nabla(\psi^s - \psi) = O\bigl(\rho^{-1}|s|\bigr)$, so we find that

$$\begin{aligned} \mathrm{d}^*_{s,t}\bigl((\psi^s)^m(t^2\alpha + \varrho^s)\bigr) &= \mathrm{d}^*\bigl(\psi^m(t^2\alpha + \varrho^s)\bigr) + O\bigl(\rho^{-3}(|s| + t^2)|t^2\alpha + \varrho^s|\bigr) \\ &\qquad + O\bigl(\rho^{-2}(|s| + t^2)|\nabla(t^2\alpha + \varrho^s)|\bigr) \\ &= f^s + O\bigl(\rho^{-4}|s|^2 + \rho^{-4}t^4\bigr), \end{aligned} \tag{126}$$

estimating as in (125). Here d^* is calculated w.r.t. h, and we use $\mathrm{d}^*(\psi^m\alpha) = 0$ and $\mathrm{d}^*(\psi^m\varrho^s) = f^s$ on X'. Combining (125) and (126) gives

$$\varepsilon^{s,t} = O\bigl(\rho^{-4}|s|^2 + \rho^{-4}t^4\bigr) \quad \text{on } K^{s,t} \text{ and } \Xi_i^{s,t}\bigl(\Sigma_i \times [2t^\tau, R')\bigr). \tag{127}$$

A similar proof for derivatives shows that

$$|\mathrm{d}\varepsilon^{s,t}| = O\bigl(\rho^{-5}|s|^2 + \rho^{-5}t^4\bigr) \quad \text{on } K^{s,t} \text{ and } \Xi_i^{s,t}\bigl(\Sigma_i \times [2t^\tau, R')\bigr). \tag{128}$$

As $\rho^{-1} = O(1)$ on $K^{s,t}$, equations (127) and (128) imply (121) for some $C > 0$, making $\mathcal{F}', \delta$ smaller if necessary. Also, (127) and (128) prove the bottom lines of (122) and (123) for some $C>0$. On $P_i^{s,t}$ and $\Xi_i^{s,t}\bigl(\Sigma_i\times(tT,t^\tau]\bigr)$ the definition of $N^{s,t}$ coincides with Definition 7.2. Thus, equation (124) and the top lines of (122) and (123) follow from Proposition 7.4.

The middle lines of (122) and (123) are proved as in Proposition 6.4, adapted to the families case as in Proposition 7.4. The additional terms $C|s|t^{\tau(\nu_i-2)}$ and $C|s|t^{\tau(\nu_i-3)}$ which do not appear in (38) and (39) are there to bound terms in $\varepsilon^{s,t}$ coming from terms in Z_i^s in (116), in particular $t^{-\tau}F'(t^{-\tau}r)Z_i^s(\sigma,r)\mathrm{d}r$ and its derivatives. This completes the proof. □

The purpose of including the Z_i^s and ϱ^s terms in Definition 8.2 was to arrange for the f^s terms in (125) and (126) to cancel, so that the estimate (127) has no $O\bigl(|s|\bigr)$ terms, but only $O\bigl(|s|^2\bigr)$ terms. Propositions 6.5 and 7.5 immediately generalize to give:

Proposition 8.5 *For some $C'>0$ and all $s\in\mathcal{F}'$ and $t\in(0,\delta)$ we have*

$$\begin{aligned}\|\varepsilon^{s,t}\|_{L^{2m/(m+2)}}\leqslant C't^{\tau(1+m/2)}\Bigl(&|s|^2t^{-4\tau}+t^{4-4\tau}+t^{(1-\tau)(2-\lambda)}\\&+\sum_{i=1}^n\bigl(t^{\tau(\mu_i-2)}+t^{2+\tau(\nu_i-2)}+|s|t^{\tau(\nu_i-2)}\bigr)\Bigr),\end{aligned}\tag{129}$$

$$\begin{aligned}\|\varepsilon^{s,t}\|_{C^0}\leqslant C'\Bigl(&|s|^2t^{-4\tau}+t^{4-4\tau}+t^{(1-\tau)(2-\lambda)}\\&+\sum_{i=1}^n\bigl(t^{\tau(\mu_i-2)}+t^{2+\tau(\nu_i-2)}+|s|t^{\tau(\nu_i-2)}\bigr)\Bigr),\end{aligned}\tag{130}$$

and
$$\begin{aligned}\|\mathrm{d}\varepsilon^{s,t}\|_{L^{2m}}\leqslant C't^{-\tau/2}\Bigl(&|s|^2t^{-4\tau}+t^{4-4\tau}+t^{(1-\tau)(2-\lambda)}\\&+\sum_{i=1}^n\bigl(t^{\tau(\mu_i-2)}+t^{2+\tau(\nu_i-2)}+|s|t^{\tau(\nu_i-2)}\bigr)\Bigr),\end{aligned}\tag{131}$$

computing norms with respect to the metric $h^{s,t}$ on $N^{s,t}$.

Note that putting $s=0$ in Propositions 8.4 and 8.5 gives Propositions 6.4 and 6.5. Parts (iii) and (iv) of Definition 8.1 imply that (96) holds, and therefore Proposition 7.8 generalizes to give:

Proposition 8.6 *Making $\mathcal{F}',\delta$ smaller if necessary, there exists $C''>0$ such that for all $s\in\mathcal{F}'$ and $t\in(0,\delta)$ we have $\|\pi_{W^{s,t}}(\varepsilon^{s,t})\|_{L^1}\leqslant C''t^{(m+1)\tau}+C''|s|t^m$.*

Here is the analogue of Theorems 6.7 and 7.9.

Theorem 8.7 *Making $\mathcal{F}'$ and $\delta>0$ smaller if necessary, there exist $A_2>0$, $\kappa>1$ and $\vartheta\in(0,2)$ such that for all $s\in\mathcal{F}'$ and $t\in(0,\delta)$ with $|s|\leqslant t^\vartheta$, the functions $\varepsilon^{s,t}=(\psi^s)^m\sin\theta^{s,t}$ on $N^{s,t}$ satisfy $\|\varepsilon^{s,t}\|_{L^{2m/(m+2)}}\leqslant A_2t^{\kappa+m/2}$, $\|\varepsilon^{s,t}\|_{C^0}\leqslant A_2t^{\kappa-1}$, $\|\mathrm{d}\varepsilon^{s,t}\|_{L^{2m}}\leqslant A_2t^{\kappa-3/2}$ and $\|\pi_{W^{s,t}}(\varepsilon^{s,t})\|_{L^1}\leqslant A_2t^{\kappa+m-1}$, as in part* (i) *of Theorem 5.3.*

Proof. Choose $\kappa > 1$ as in the proof of Theorem 6.7, but requiring that strict inequality hold in (57) and (58). This is clearly possible. Then all the terms in (129), (130), (131) not involving $|s|$ are bounded by multiples of $t^{\kappa+m/2}$, $t^{\kappa-1}$, $t^{\kappa-3/2}$ respectively, as in the proof of Theorem 6.7.

For the term $C' t^{\tau(1+m/2)} \cdot |s|^2 t^{-4\tau}$ in (129) to be bounded by a multiple of $t^{\kappa+m/2}$ when $|s| \leqslant t^{\vartheta}$, it is enough that $t^{\tau(1+m/2)} \cdot t^{2\vartheta} \cdot t^{-4\tau} \leqslant t^{\kappa+m/2}$. Since $t \in (0,1)$, this holds if $\tau(1+m/2) + 2\vartheta - 4\tau \geqslant \kappa + m/2$. In the same way, for all the terms in (129), (130), (131) involving $|s|$ to be bounded by multiples of $t^{\kappa+m/2}$, $t^{\kappa-1}$, $t^{\kappa-3/2}$ respectively, it is enough that for $i = 1,\ldots,n$ we have

$$\tau(1+m/2)+2\vartheta-4\tau \geqslant \kappa+m/2, \quad \tau(1+m/2)+\vartheta+\tau(\nu_i-2) \geqslant \kappa+m/2, \tag{132}$$

$$2\vartheta - 4\tau \geqslant \kappa - 1, \quad \vartheta + \tau(\nu_i - 2) \geqslant \kappa - 1, \tag{133}$$

$$-\tau/2 + 2\vartheta - 4\tau \geqslant \kappa - 3/2, \quad -\tau/2 + \vartheta + \tau(\nu_i - 2) \geqslant \kappa - 3/2. \tag{134}$$

As $\tau < 1$, equations (133) and (134) follow from (132). Thus, we need to choose $\vartheta \in (0,2)$ such that (132) holds for $i = 1,\ldots,n$. This is possible as κ was defined to satisfy the first equation of (57) and the second equation of (58) with strict inequalities. With these values of κ and ϑ, the theorem follows from Propositions 8.5 and 8.6 with $A_2 = \max\bigl(3(n+1)C', 2C''\bigr)$. □

The inequality $\vartheta < 2$ in Theorem 8.7 is important for the following reason. If $\varpi \neq 0$ then (120) implies that $t^2 \leqslant D|s|$ for some $D > 0$ and all small $(s,t) \in \mathcal{G}$. Combining this with $|s| \leqslant t^{\vartheta}$ gives $|s| \leqslant D^{\vartheta/2}|s|^{\vartheta/2}$. If $\vartheta > 2$, or $\vartheta = 2$ and $D < 1$, then this fails for all small nonzero s.

That is, if we had $\vartheta \geqslant 2$ in Theorem 8.7 and $\varpi \neq 0$ then the condition $|s| \leqslant t^{\vartheta}$ would have *excluded* all small $(s,t) \in \mathcal{G}$, so the construction would produce *no* compact SL m-folds $\tilde N^{s,t}$. But as $\vartheta < 2$ the condition $|s| \leqslant t^{\vartheta}$ includes all small $(s,t) \in \mathcal{G}$ with $|s| = O(t^2)$, which will in general be a nonempty subset of $\mathcal{G}$.

We may extend Definition 7.10 and Theorems 7.11–7.13 to the situation of this section in a straightforward way, by including the modifications introduced in Definition 6.8 and Theorems 6.9–6.11, and there are no significant new issues. Thus we prove:

Theorem 8.8 *Theorems 7.11–7.13 hold in the situation of Definitions 8.1–8.3.*

8.3 The main result for families when $Y(L_i) \neq 0$

Here is our main result, the analogue of Theorem 6.12 for families.

Theorem 8.9 *Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold for $2<m<6$, and X a compact SL m-fold in M with conical singularities at $x_1,\ldots,x_n$ and cones $C_1,\ldots,C_n$. Define $\psi : M \rightarrow (0,\infty)$ as in (3). Let $L_1,\ldots,L_n$ be Asymptotically Conical SL m-folds in $\mathbb{C}^m$ with cones $C_1,\ldots,C_n$ and rates $\lambda_1,\ldots,\lambda_n$. Suppose $\lambda_i \leqslant 0$ for $i = 1,\ldots,n$. Write $X' = X \setminus \{x_1,\ldots,x_n\}$ and $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$.*

Set $q = b^0(X')$, and let $X'_1, \ldots, X'_q$ be the connected components of X'. For $i = 1, \ldots, n$ let $l_i = b^0(\Sigma_i)$, and let $\Sigma_i^1, \ldots, \Sigma_i^{l_i}$ be the connected components of Σ_i. Define $k(i,j) = 1, \ldots, q$ by $\Upsilon_i \circ \varphi_i\bigl(\Sigma_i^j \times (0, R')\bigr) \subset X'_{k(i,j)}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$. Suppose that

$$\sum_{\substack{1 \leqslant i \leqslant n,\, 1 \leqslant j \leqslant l_i: \\ k(i,j) = k}} \psi(x_i)^m Z(L_i) \cdot [\Sigma_i^j\,] = 0 \quad \textit{for all } k = 1, \ldots, q. \tag{135}$$

Suppose also that the compact m-manifold N obtained by gluing L_i into X' at x_i for $i = 1, \ldots, n$ is connected. A sufficient condition for this to hold is that X and L_i for $i = 1, \ldots, n$ are connected.

Suppose $\bigl\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\bigr\}$ is a smooth family of deformations of (M, J, ω, Ω), with base space $\mathcal{F} \subset \mathbb{R}^d$, satisfying

$$[\operatorname{Im}\Omega^s] \cdot [\overline{X'_k}] = 0 \quad \textit{for all } s \in \mathcal{F} \textit{ and } k = 1, \ldots, q. \tag{136}$$

Define $\varpi \in H^2_{\rm cs}(X', \mathbb{R})$ to be the image of $\bigl(Y(L_1), \ldots, Y(L_n)\bigr)$ under the map $\bigoplus_{i=1}^n H^1(\Sigma_i, \mathbb{R}) \rightarrow H^2_{\rm cs}(X', \mathbb{R})$ in (6). Define $\mathcal{G} \subseteq \mathcal{F} \times (0,1)$ to be

$$\mathcal{G} = \bigl\{(s,t) \in \mathcal{F} \times (0,1) : [\omega^s] \cdot \iota_*(\gamma) = t^2 \varpi \cdot \gamma \textit{ for all } \gamma \in H_2(X, \mathbb{R})\bigr\}, \tag{137}$$

where $\iota_* : H_2(X, \mathbb{R}) \rightarrow H_2(M, \mathbb{R})$ is the natural inclusion.

Then there exist $\epsilon \in (0,1)$, $\kappa > 1$ and $\vartheta \in (0,2)$ and a smooth family

$$\bigl\{\tilde{N}^{s,t} : (s,t) \in \mathcal{G}, \quad t \in (0, \epsilon], \quad |s| \leqslant t^\vartheta\bigr\}, \tag{138}$$

such that $\tilde{N}^{s,t}$ is a compact, nonsingular SL m-fold in $(M, J^s, \omega^s, \Omega^s)$ diffeomorphic to N, which is constructed by gluing tL_i into X at x_i for $i = 1, \ldots, n$. In the sense of currents in Geometric Measure Theory, $\tilde{N}^{s,t} \rightarrow X$ as $s, t \rightarrow 0$.

The proof follows that of Theorem 7.15, but using Theorems 8.7 and 8.8 to prove parts (i)–(vii) of Theorem 5.3 for $N^{s,t}$ and $W^{s,t}$, for $(s,t) \in \mathcal{G}$. Here the restriction to $(s,t) \in \mathcal{G}$ is because $N^{s,t}$ is only Lagrangian in (M, ω^s) if (120) holds, from Definition 8.2, and Theorem 5.3 applies only when $N^{s,t}$ is Lagrangian.

When $\varpi = 0$ the SL m-fold $\tilde{N}^{0,t}$ above coincides with the SL m-fold $\tilde{N}^t$ of Theorem 6.12. More generally, as in §7.5, when $\varpi = 0$ much of Theorem 8.9 follows from Theorems 2.9 and 6.12. However, when $\varpi \neq 0$ there exist *no* desingularizations $\tilde{N}^{0,t}$ of X in (M, J, ω, Ω), but Theorem 8.9 can still give desingularizations $\tilde{N}^{s,t}$ in $(M, J^s, \omega^s, \Omega^s)$ for $s \neq 0$. That is, by deforming (M, J, ω, Ω) to $(M, J^s, \omega^s, \Omega^s)$ we can overcome the topological obstructions to desingularizing X in (M, J, ω, Ω).

Theorem 8.9 is essentially a combination of Theorems 6.12 and 7.14. However, Theorem 7.14 assumes the $[\operatorname{Im}\Omega^s]$ and $Z(L_i)$ satisfy (108), whereas Theorem 8.9 assumes the stronger (135) and (136), which together imply (108). It is natural to ask whether Theorem 8.9 still holds with the weaker assumption (108) in place of (135) and (136).

The answer to this ought really to be yes, as (136) was a technical condition introduced to ensure that ϱ^s and Z_i^s exist in Definition 8.1, and with some more work one should to be able to do without it. However, such a revised theorem would suffer from the following problem.

Suppose s,t satisfy both (108) and equation $[\omega^s]\cdot\iota_*(\gamma)=t^2\varpi\cdot\gamma$ in (137). If $\varpi\neq 0$ then (137) implies that $t^2=O\bigl(|s|\bigr)$, and we expect $|s|\approx t^2$. If the right hand side of (108) is nonzero then it gives $t^m=O\bigl(|s|\bigr)$, and we expect $|s|\approx t^m$. These conditions are not compatible, as $m>2$. The two kinds of obstruction need to be resolved at different length scales.

Actually this is not a serious problem, provided the family $\mathcal{F}$ has large enough dimension. There could still exist a family of solutions (s,t) to (108) and (137) with $|s|\approx t^2$ and $[\operatorname{Im}\Omega^s]\cdot[\overline{X_k'}]=O(|s|^{m/2})=O(t^m)$ for $k=1,\ldots,q$.

If X' is connected then (135) holds automatically as in §6.5, and (136) simplifies to $[\operatorname{Im}\Omega^s]\cdot[X]=0$, giving an analogue of Theorem 6.13 for families:

Theorem 8.10 *Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold for $2<m<6$, and X a compact SL m-fold in M with conical singularities at $x_1,\ldots,x_n$ and cones $C_1,\ldots,C_n$. Let $L_1,\ldots,L_n$ be Asymptotically Conical SL m-folds in $\mathbb{C}^m$ with cones $C_1,\ldots,C_n$ and rates $\lambda_1,\ldots,\lambda_n$. Suppose $\lambda_i\leqslant 0$ for $i=1,\ldots,n$, and $X'=X\setminus\{x_1,\ldots,x_n\}$ is connected.*

Suppose $\bigl\{(M,J^s,\omega^s,\Omega^s):s\in\mathcal{F}\bigr\}$ is a smooth family of deformations of (M,J,ω,Ω), with base space $\mathcal{F}\subset\mathbb{R}^d$, satisfying

$$[\operatorname{Im}\Omega^s]\cdot[X]=0\quad\text{for all } s\in\mathcal{F},\text{ where } [X]\in H_m(M,\mathbb{R}).\tag{139}$$

Define $\varpi\in H^2_{\rm cs}(X',\mathbb{R})$ to be the image of $\bigl(Y(L_1),\ldots,Y(L_n)\bigr)$ under the map $\bigoplus_{i=1}^n H^1(\Sigma_i,\mathbb{R})\rightarrow H^2_{\rm cs}(X',\mathbb{R})$ in (6). Define $\mathcal{G}\subseteq\mathcal{F}\times(0,1)$ to be

$$\mathcal{G}=\bigl\{(s,t)\in\mathcal{F}\times(0,1):[\omega^s]\cdot\iota_*(\gamma)=t^2\varpi\cdot\gamma\ \text{for all } \gamma\in H_2(X,\mathbb{R})\bigr\},\tag{140}$$

where $\iota_*:H_2(X,\mathbb{R})\rightarrow H_2(M,\mathbb{R})$ is the natural inclusion.

Then there exist $\epsilon\in(0,1)$, $\kappa>1$ and $\vartheta\in(0,2)$ and a smooth family

$$\bigl\{\tilde N^{s,t}:(s,t)\in\mathcal{G},\quad t\in(0,\epsilon],\quad |s|\leqslant t^\vartheta\bigr\},\tag{141}$$

such that $\tilde N^{s,t}$ is a compact, nonsingular SL m-fold in $(M,J^s,\omega^s,\Omega^s)$, which is constructed by gluing tL_i into X at x_i for $i=1,\ldots,n$. In the sense of currents in Geometric Measure Theory, $\tilde N^{s,t}\rightarrow X$ as $s,t\rightarrow 0$.

When $d=0$ and $\mathcal{F}=\{0\}=\mathbb{R}^d$ equations (136) and (139) hold automatically, and $\mathcal{G}$ in (137) and (140) is nonempty if and only if $\varpi=0$. But by exactness in (6), $\varpi=0$ is the necessary and sufficient condition for the existence of ϱ in Theorems 6.12 and 6.13. Therefore, when $d=0$ Theorems 8.9 and 8.10 reduce to Theorems 6.12 and 6.13 respectively.

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