

# A LEVY-AREA BETWEEN BROWNIAN MOTION AND ROUGH PATHS WITH APPLICATIONS TO ROBUST NON-LINEAR FILTERING AND RPDES

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**ABSTRACT.** We give meaning and study the regularity of differential equations with a rough path term and a Brownian noise term, that is we are interested in equations of the type

$$S_t^\eta = S_0 + \int_0^t a(S_r^\eta) dr + \int_0^t b(S_r^\eta) \circ dB_r + \int_0^t c(S_r^\eta) d\eta_r$$

where  $\eta$  is a deterministic geometric, step-2 rough path and  $B$  is a multi-dimensional Brownian motion. En passant, we give a short and direct argument that implies integrability estimates for rough differential equations with Gaussian driving signals which is of independent interest.

## 1. INTRODUCTION

The contribution of this article is twofold: firstly, we give meaning to differential equations of the type

$$(1.1) \quad S_t^\eta = S_0 + \int_0^t a(S_r^\eta) dr + \int_0^t b(S_r^\eta) \circ dB_r + \int_0^t c(S_r^\eta) d\eta_r,$$

that is, for a deterministic, step 2-rough path  $\eta$  we are looking for a stochastic process  $S^\eta$  that is adapted to  $\sigma(B)$  and study the regularity of the map  $\eta \mapsto S^\eta$ . Secondly, we take this as an opportunity to revisit the integrability estimates of solutions of rough differential equations driven by Gaussian processes.

If either  $b \equiv 0$  or  $c \equiv 0$  then rough path theory [26, 27, 29, 19, 18] or standard Itô-calculus allow (under appropriate regularity assumptions on the vector fields  $(a, b, c)$ ) to give meaning to (1.1). However, in the generic case when the vector fields  $b$  and  $c$  have a non-trivial Lie bracket, any notion of a solution (that is consistent with an Itô–Stratonovich calculus) must take into account the area swept out between the trajectories of  $B$  and  $\eta$ . A natural approach is to identify  $S^\eta$  as the RDE solution of

$$(1.2) \quad S_t = S_0 + \int_0^t (a, b, c)(S_r) d(r, \Lambda_r)$$

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where  $\Lambda$  is a joint, step-2 rough path lift between the enhanced Brownian motion  $B = (1 + B + \int B \otimes \circ dB)$  and  $\eta$ , and  $(r, \Lambda)$  is the joint rough path between the random rough path  $\Lambda$  and the bounded variation path  $r \mapsto r$ . While the existence of a joint lift between a continuous bounded variation path and any rough path is trivial (via integration by parts), the existence of a joint lift between two given step-2 rough paths is more subtle and in general not possible. More precisely, let  $\alpha \in (\frac{1}{3}, \frac{1}{2})$  and denote with  $\mathcal{C}^{0,\alpha}(\mathbb{R}^d)$  the space of geometric, step-2,  $\alpha$ -Hölder rough paths over  $\mathbb{R}^d$  (we often only write  $\mathcal{C}^{0,\alpha}$  and  $d$  is chosen according to context). Fix two geometric, step-2 rough paths  $\eta = (1 + \eta^1 + \eta^2) \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ ,  $b = (1 + b^1 + b^2) \in \mathcal{C}^{0,\alpha}(\mathbb{R}^e)$ . In general, one cannot hope to find a joint rough path lift, i.e. a geometric rough path  $\lambda = (1 + \lambda^1 + \lambda^2) \in \mathcal{C}^{0,\alpha}(\mathbb{R}^{d+e})$  such that (formally)

$$\lambda^1 = (\eta^1, b^1) \text{ and } \lambda^2 = \begin{pmatrix} \eta^2 & \int \eta \otimes db \\ \int b \otimes d\eta & b^2 \end{pmatrix}$$

since the entries on the cross-diagonal of  $\lambda^2$  are not well-defined. (What is guaranteed by the extension theorem in [28] is that there exists a weak geometric rough path  $\bar{\lambda}$  such that  $\bar{\lambda}^1 = (\eta^1, b^1)$ , however this  $\bar{\lambda}$  is highly non-unique and no consistency with  $\eta$  or  $b$  on the second level is guaranteed).

In Section 2 we show that in the case when the deterministic rough path  $b$  is replaced by enhanced Brownian motion  $B$ , there does indeed exist a stochastic process  $\Lambda$  which merits in a certain sense to be called the “canonical joint lift” of  $\eta$  and  $B$ . In Section 3 we use this lift  $\Lambda$  to give meaning to differential equations (1.1) resp. (1.2) and establish local Lipschitzness of the solution map  $\eta \mapsto S^\eta$  from the space of geometric rough paths equipped with Hölder metric into the space of stochastic processes adapted to the Brownian filtration equipped with the topology of uniform convergence in  $L^q(\Omega)$ -norm. This is exactly the type of robustness we are interested in and finally allows us to turn to our initial motivation: differential equations of the form (1.1) naturally arise in certain robustness problems and were previously treated with a flow decomposition which ultimately leads to stronger regularity assumptions on the vector fields. In Section 4 we give two such applications. One revisits Clark’s robustness problem in nonlinear filtering and provides an alternative to the recent approach via flow decomposition carried out in [6], the other one is a Feynman–Kac representation of solutions of PDEs with linear rough path noise.

Our application to stochastic filtering demands exponential integrability of differential equations driven by  $\Lambda$ . We take this as an opportunity to revisit existing results on integrability estimates for rough paths and rough differential equations in a general setup (which then even implies Gaussian integrability for differential equations driven by  $\Lambda$  that are uniform in  $\eta$ ). In Section 5 we give a surprisingly short proof of the integrability properties of RDEs driven by Gaussian rough paths by revisiting and combining the key insights from [14] and [5] in a direct and tractable way which we think is of independent interest.

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## 2. THE JOINT LIFT

As usual we denote with  $Lip^\gamma$  the set of  $\gamma$ -Lipschitz functions  $a : \mathbb{R}^{d_1} \rightarrow \mathbb{R}^{d_2}$  in the sense of E. Stein<sup>1</sup> where  $d_1$  and  $d_2$  are chosen according to the context.  $G_d^2 \cong \mathbb{R}^d \oplus so(d)$  is the free nilpotent group<sup>2</sup> of step 2 over  $\mathbb{R}^d$ . We equip the space of geometric rough paths with the non-homogeneous metric  $\rho_{\alpha-H\ddot{o}l}$  which makes it a Polish space, denoted  $\mathcal{C}^{0,\alpha}$ , and denote the associated non-homogeneous norm<sup>3</sup> on this non-linear space with  $\|\cdot\|_{\alpha-H\ddot{o}l}$ , (similarly we denote the non-separable space of weak geometric rough paths with  $\mathcal{C}^\alpha(\mathbb{R}^d)$ , cf. [18, Chapter 9.2]).

**Theorem 3.** *Let  $\alpha \in (\frac{1}{3}, \frac{1}{2})$ ,  $\boldsymbol{\eta} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$  and  $B = (B^i)_{i=1}^e$  be an  $e$ -dimensional Brownian motion carried on a probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  satisfying the usual conditions. Then for every  $\alpha' < \alpha$  there exists a  $\mathcal{C}^{0,\alpha'}(\mathbb{R}^{d+e})$ -valued random variable  $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^\eta$  on  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  which fulfills  $\mathbb{P}$ -a.s. that for every  $t \geq 0$ ,*

$$(2.1) \quad \begin{aligned} \boldsymbol{\Lambda}_t^{1;i} &= \begin{cases} \boldsymbol{\eta}_t^{1;i} & , \text{ if } i \in \{1, \dots, d\} \\ B_t^{i-d} & , \text{ if } i \in \{d+1, \dots, d+e\} \end{cases} \\ \boldsymbol{\Lambda}_t^{2;i,j} &= \begin{cases} \boldsymbol{\eta}_t^{2;i,j} & , \text{ if } i, j \in \{1, \dots, d\} \\ \int_0^t B_r^{i-d} \circ dB_r^{j-d} & , \text{ if } i, j \in \{d+1, \dots, d+e\} \end{cases} \end{aligned}$$

Moreover,

- (i)  $\boldsymbol{\Lambda}^\eta$  has Gaussian tails, locally uniform in  $\boldsymbol{\eta}$ :  $\forall r > 0 \exists \delta = \delta(\alpha', \alpha, T, r) > 0$  such that

$$(2.2) \quad \sup_{\|\boldsymbol{\eta}\|_{\alpha-H\ddot{o}l} \leq r} \mathbb{E} \exp(\delta \|\boldsymbol{\Lambda}^\eta\|_{\alpha'-H\ddot{o}l}^2) < \infty.$$

<sup>1</sup>That is bounded  $k$ -th derivative for  $k = 0, \dots, \lfloor \gamma \rfloor$  and  $(\gamma - \lfloor \gamma \rfloor)$ -Hölder continuous  $\lfloor \gamma \rfloor$ -th derivative, where  $\lfloor \gamma \rfloor$  is the largest integer strictly smaller than  $\gamma$ .

<sup>2</sup>This is the correct state space for a geometric  $1/p$ -Hölder rough path; the space of such paths subject to  $1/p$ -Hölder regularity (in rough path sense) yields a complete metric space under  $1/p$ -Hölder rough path metric. Technical details of geometric rough path spaces can be found e.g. in Section 9 of [18].

<sup>3</sup>We denote norms on linear spaces with  $|\cdot|$  and “norms” on non-linear spaces (like  $G_d^2$  or  $\mathcal{C}_d^{0,\alpha}$ ) with  $\|\cdot\|$ .

- (ii)  $\boldsymbol{\eta} \mapsto \boldsymbol{\Lambda}^\eta$  is locally Lipschitz in  $L^q$ :  $\forall r > 0$ , there exists a constant  $c_{Lip} = c_{Lip}(r, q, \alpha, \alpha')$  such that for all  $\boldsymbol{\eta}, \bar{\boldsymbol{\eta}} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$  with  $\|\boldsymbol{\eta}\|_{\alpha-H\ddot{o}l}, \|\bar{\boldsymbol{\eta}}\|_{\alpha-H\ddot{o}l} \leq r$

$$\left| \rho_{\alpha'-H\ddot{o}l}(\boldsymbol{\Lambda}^\eta, \boldsymbol{\Lambda}^{\bar{\eta}}) \right|_{L^q(\Omega; \mathbb{R})} \leq c_{Lip} \rho_{\alpha-H\ddot{o}l}(\boldsymbol{\eta}, \bar{\boldsymbol{\eta}}).$$

- (iii)  $\boldsymbol{\Lambda}$  is consistent with the Stratonovich lift for semimartingales: let  $N$  be a multidimensional continuous semimartingale carried on another probability space  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}})$  and consider the product space with  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  equipped with  $\bar{\mathbb{P}} \otimes \mathbb{P}$ . Denote with  $N$  resp.  $(N, B)$  the Stratonovich lift of the semimartingales  $N$  resp.  $(N, B)$ . Then for  $\bar{\mathbb{P}}$ -a.e.  $\bar{\omega} \in \bar{\Omega}$  we have

$$\mathbb{P} \left[ \omega : \boldsymbol{\Lambda}^{N(\bar{\omega})}(\omega) = (N, B)(\bar{\omega} \otimes \omega) \right] = 1.$$

*Proof.* Define

$$\begin{aligned} \boldsymbol{\Lambda}_t^{1;i} &:= \begin{cases} \boldsymbol{\eta}^{1;i} & , \text{ if } i \in \{1, \dots, d\} \\ B^{i-d} & , \text{ if } i \in \{d+1, \dots, d+e\} \end{cases}, \\ (2.3) \quad \boldsymbol{\Lambda}_t^{2;i,j} &:= \begin{cases} \boldsymbol{\eta}^{2;i,j} & , \text{ if } i, j \in \{1, \dots, d\} \\ \int_0^t B_r^{i-d} \circ dB_r^{j-d} & , \text{ if } i, j \in \{d+1, \dots, d+e\} \\ \int_0^t \eta_u^i dB_u^{j-d} & , \text{ if } i \in \{1, \dots, d\}, j \in \{1+d, \dots, d+e\} \\ \eta_t^j B_t^{i-d} - \int_0^t \eta_u^j dB_u^{i-d} & , \text{ if } i \in \{d+1, \dots, d+e\}, j \in \{1, \dots, d\}. \end{cases} \end{aligned}$$

Then  $\boldsymbol{\Lambda}_t = (1 + \boldsymbol{\Lambda}_t^1 + \boldsymbol{\Lambda}_t^2) \in 1 + \mathbb{R}^{d+e} + (\mathbb{R}^{d+e})^{\otimes 2}$  and a direct calculation shows that  $\boldsymbol{\Lambda}_{s,t} := \boldsymbol{\Lambda}_s^{-1} \otimes \boldsymbol{\Lambda}_t = \exp[(\eta_{s,t}, B_{s,t}) + A_{s,t}]$  with<sup>4</sup>

$$(2.4) \quad so(d) \ni A_{s,t}^{i,j} = \begin{cases} \frac{1}{2} \left( \int_s^t \eta_{s,u}^i d\eta_u^j - \int_s^t \eta_{s,u}^j d\eta_u^i \right) & , \text{ if } i, j \in \{1, \dots, d\} \\ \frac{1}{2} \left( \int_s^t B_{s,u}^{i-d} dB_u^{j-d} - \int_s^t B_{s,u}^{j-d} dB_u^{i-d} \right) & , \text{ if } i, j \in \{d+1, \dots, d+e\} \\ \left( \int_s^t \eta_{s,u}^i dB_u^{j-d} - \frac{1}{2} \eta_{s,t}^i B_{s,t}^{j-d} \right) & , \text{ if } i \in \{1, \dots, d\}, j \in \{1+d, \dots, d+e\} \\ \left( - \int_0^t \eta_u^j dB_u^{i-d} + \frac{1}{2} \eta_{s,t}^j B_{s,t}^{i-d} \right) & , \text{ if } i \in \{d+1, \dots, d+e\}, j \in \{1, \dots, d\}. \end{cases}$$

That is, (after throwing away a null-set depending on  $B$  and  $\boldsymbol{\eta}$ ) we have shown that  $t \mapsto \boldsymbol{\Lambda}_t$  is a continuous path that takes values in  $G^2(\mathbb{R}^{d+e})$ . It remains to demonstrate that  $\|\boldsymbol{\Lambda}\|_{\alpha'-H\ddot{o}l} < \infty$  for any  $\alpha' < \alpha$  which is then enough to conclude that  $\boldsymbol{\Lambda} \in \mathcal{C}^{0,\alpha'}(\mathbb{R}^{d+e})$   $\mathbb{P}$ -a.s. for  $\alpha' < \alpha$  due to the embedding of weak geometric rough paths into geometric rough paths ( $\mathcal{C}^\beta \subset \mathcal{C}^{0,\beta'}$  for  $\beta > \beta'$  follows from [17, Theorem 19]). We show this Hölder regularity by proving a stronger statement, namely that  $\|\boldsymbol{\Lambda}\|_{\alpha'}$  has Gaussian tails. It is clear that the first level of  $\boldsymbol{\Lambda}$  is  $\alpha$ -Hölder; to deal with the second level note that  $|A_{st}| \sim \sum_{i,j} |A_{st}^{ij}|$  and that  $2\alpha$ -Hölder regularity already holds for the first two cases, that is  $i, j \in \{1, \dots, d\}$  and  $i, j \in \{d+1, \dots, d+e\}$  (in fact

<sup>4</sup>We could define  $\boldsymbol{\Lambda}$  directly via (2.4) but the above way might be a bit more intuitive.

even a Gauss tail via the Fernique estimate for rough path norms [14]). Now for the remaining case we have from above definition of  $A^{i,j}$  that

$$|A_{s,t}^{i,j}| \leq \left| \int_s^t \eta_{s,u}^i dB_u^j \right| + \frac{1}{2} |\eta_{s,t}^i B_{s,t}^j|$$

and since  $|\eta_{s,t}^i B_{s,t}^j| \leq |\eta|_{\alpha-H\ddot{o}l} |B|_{\alpha-H\ddot{o}l} |t-s|^{2\alpha}$  it just remains to treat  $\left| \int_s^t \eta_{s,u}^i dB_u^j \right|$ .

But since for each  $s < t$ ,  $\int_s^t \eta_{s,u}^i dB_u^j$  has the same distribution as  $\sqrt{\int_s^t (\eta_{s,u}^i)^2 du} Z$  for some fixed  $Z \sim \mathcal{N}(0, 1)$ , we also have

$$\exp \left[ \kappa \left( \frac{\left| \int_s^t (\eta_{s,u}^i) dB_u^j \right|}{(t-s)^{2\alpha}} \right)^2 \right] \stackrel{Law}{=} \exp \left[ \kappa Z^2 \left( \frac{\sqrt{\left| \int_s^t (\eta_{s,u}^i)^2 du \right|}}{(t-s)^{2\alpha}} \right)^2 \right]$$

and by the elementary estimate  $\left| \int_s^t (\eta_{s,u}^i)^2 du \right| \leq \|\eta\|_{\alpha-H\ddot{o}l}^2 (t-s)^{2\alpha+1}$  we can conclude by taking  $\sup_{s < t} \mathbb{E}$  in above expression and using the Gaussian integrability for  $Z$ , i.e. there exists a  $\kappa > 0$  such that

$$\sup_{s < t} \mathbb{E} \exp \left[ \kappa Z^2 |\eta|_{\alpha-H\ddot{o}l}^2 (t-s)^{1-2\alpha} \right] < \infty.$$

By [18, Theorem A.19] this yields the desired  $2\alpha'$ -Hölder regularity of  $\left| \int_s^t \eta_{s,u}^i dB_u^j \right|$  for any  $\alpha' < \alpha$ . Putting everything together, we have shown that  $\|\Lambda\|_{\alpha'-H\ddot{o}l} < \infty$ ,  $\mathbb{P}$ -a.s. In fact we even have shown

$$\sup_{\|\eta\|_{\alpha-H\ddot{o}l} \leq r} \mathbb{E} \exp \left( \delta \|\Lambda^\eta\|_{\alpha'-H\ddot{o}l}^2 \right) < \infty, \quad \forall \alpha' < \alpha.$$

It remains to show the claimed Lipschitz continuity of the map  $\eta \mapsto \Lambda^\eta$ . Therefore let  $q \geq q_0(\alpha, \alpha')$ , as given in [18, Theorem A.13], take  $\eta, \bar{\eta} \in \mathcal{C}^{0,\alpha}$  with  $\|\eta\|_{\alpha-H\ddot{o}l}, \|\bar{\eta}\|_{\alpha-H\ddot{o}l} \leq r$  and denote the corresponding lifts  $\Lambda^\eta, \Lambda^{\bar{\eta}}$ . Set  $\varepsilon := \rho_{\alpha'-H\ddot{o}l}(\Lambda^\eta, \Lambda^{\bar{\eta}})$ . By (2.2) there exists a constant  $c_1 = c_1(q, r)$  such that (we denote the Carnot–Caratheodory metric on  $G_{d+e}^2$  with  $d_{CC}$ )

$$|d_{CC}(\Lambda_s^\eta, \Lambda_t^\eta)|_{L^q(\Omega; \mathbb{R})}^q, \left| d_{CC}(\Lambda_s^{\bar{\eta}}, \Lambda_t^{\bar{\eta}}) \right|_{L^q(\Omega; \mathbb{R})}^q \leq c_1 |t-s|^{\alpha q}.$$

Moreover

$$\begin{aligned} \left| \pi_1 \left( \Lambda_{s,t}^\eta - \Lambda_{s,t}^{\bar{\eta}} \right) \right|_{L^q(\Omega; \mathbb{R})}^q &= |\eta_{s,t} - \bar{\eta}_{s,t}|^q \\ &\leq \varepsilon^q |t-s|^{\alpha q}, \end{aligned}$$

and (again the constants  $c$  may only depend on  $r$  and  $q$ )

$$\begin{aligned} \left| \pi_2 \left( \Lambda_{s,t}^\eta - \Lambda_{s,t}^{\bar{\eta}} \right) \right|_{L^{q/2}(\Omega; \mathbb{R})}^{2/q} &= \left| \frac{1}{2} \pi_1 \left( \Lambda_{s,t}^\eta - \Lambda_{s,t}^{\bar{\eta}} \right) \otimes \pi_1 \left( \Lambda_{s,t}^\eta - \Lambda_{s,t}^{\bar{\eta}} \right) + A_{s,t} - \bar{A}_{s,t} \right|_{L^{q/2}(\Omega; \mathbb{R})}^{2/q} \\ &\leq c \left( \varepsilon^{q/2} |t-s|^{\alpha q} + |A_{s,t} - \bar{A}_{s,t}|_{L^{q/2}(\Omega; \mathbb{R})}^{2/q} \right) \end{aligned}$$

Also,

$$\begin{aligned} &\left| \int_s^t \eta_{s,u}^i d\eta_u^j - \int_s^t \eta_{s,u}^j d\eta_u^i - \left( \int_s^t \bar{\eta}_{s,u}^i d\bar{\eta}_u^j - \int_s^t \bar{\eta}_{s,u}^j d\bar{\eta}_u^i \right) \right|_{L^{q/2}(\Omega; \mathbb{R})}^{2/q} \\ &= \left| \int_s^t \eta_{s,u}^i d\eta_u^j - \int_s^t \eta_{s,u}^j d\eta_u^i - \left( \int_s^t \bar{\eta}_{s,u}^i d\bar{\eta}_u^j - \int_s^t \bar{\eta}_{s,u}^j d\bar{\eta}_u^i \right) \right|^{2/q} \\ &\leq \varepsilon^{q/2} |t-s|^{\alpha q}, \end{aligned}$$

and finally

$$\begin{aligned} &\left| \int_s^t \eta_{s,u}^i dB_u^{j-d} - \frac{1}{2} \eta_{s,t}^i B_{s,t}^{j-d} - \left( \int_s^t \bar{\eta}_{s,u}^i d\bar{B}_u^{j-d} - \frac{1}{2} \bar{\eta}_{s,t}^i \bar{B}_{s,t}^{j-d} \right) \right|_{L^{q/2}(\Omega; \mathbb{R})}^{2/q} \\ &\leq c \left| \int_s^t \eta_{s,u}^i - \bar{\eta}_{s,u}^i dB_u^{j-d} \right|_{L^{q/2}(\Omega; \mathbb{R})}^{2/q} + c \left| \eta_{s,t}^i B_{s,t}^{j-d} - \bar{\eta}_{s,t}^i \bar{B}_{s,t}^{j-d} \right|_{L^{q/2}(\Omega; \mathbb{R})}^{2/q} \\ &\leq c \left| \int_s^t |\eta_{s,u}^i - \bar{\eta}_{s,u}^i|^2 du \right|^{q/4} + c |\eta_{s,t}^i - \bar{\eta}_{s,t}^i|^{q/2} |B_{s,t}^{j-d}|_{L^{q/2}(\Omega; \mathbb{R})}^{2/q} \\ &\leq c \varepsilon^{q/2} |t-s|^{\alpha q/2 + q/4} + c \varepsilon^{q/2} |t-s|^{\alpha q/2} |t-s|^{q/4} \\ &\leq c \varepsilon^{q/2} |t-s|^{\alpha q}. \end{aligned}$$

Hence,

$$\left| \pi_2 \left( \Lambda_{s,t}^\eta - \Lambda_{s,t}^{\bar{\eta}} \right) \right|_{L^{q/2}(\Omega; \mathbb{R})}^{2/q} \leq c_2 \varepsilon^{q/2} |t-s|^{\alpha q}$$

and applied with  $m = m(r, q) := \max \left\{ 1, c_1^{1/q}, c_2^{1/(2q)} \right\}$  we have  $\forall q \geq 1$  that

$$\begin{aligned} \left| d_{CC} \left( \Lambda_s^\eta, \Lambda_t^{\bar{\eta}} \right) \right|_{L^q(\Omega; \mathbb{R})} &\leq m |t-s|^\alpha, \\ \left| \pi_1 \left( \Lambda_{s,t}^\eta - \Lambda_{s,t}^{\bar{\eta}} \right) \right|_{L^q(\Omega; \mathbb{R})} &\leq \varepsilon m |t-s|^\alpha, \\ \left| \pi_2 \left( \Lambda_{s,t}^\eta - \Lambda_{s,t}^{\bar{\eta}} \right) \right|_{L^{q/2}(\Omega; \mathbb{R})} &\leq \varepsilon m^2 |t-s|^{2\alpha}. \end{aligned}$$

By [18, Theorem A.13 (i)] there exists a  $q$  large enough and a constant  $k = k(\alpha, \alpha', T, q)$  such that

$$\left| \sup_{s < t} \frac{\left| \pi_1 \left( \Lambda_{s,t}^\eta - \Lambda_{s,t}^{\bar{\eta}} \right) \right|}{|t-s|^{\alpha'}} \right|_{L^q(\Omega; \mathbb{R})} \leq \varepsilon km \text{ and } \left| \sup_{s < t} \frac{\left| \pi_2 \left( \Lambda_{s,t}^\eta - \Lambda_{s,t}^{\bar{\eta}} \right) \right|}{|t-s|^{\alpha'}} \right|_{L^{q/2}(\Omega; \mathbb{R})} \leq \varepsilon (km)^2$$

Using this with  $q$  and  $2q$  we get from the definition of  $\rho_{\alpha'-H\ddot{o}l}$  that

$$|\rho_{\alpha'-H\ddot{o}l}(\Lambda^\eta, \Lambda^{\bar{\eta}})|_{L^q(\Omega; \mathbb{R})} \leq \left| \sup_{s \neq t} \frac{\left| \pi_1 \left( \Lambda_{s,t}^\eta - \Lambda_{s,t}^{\bar{\eta}} \right) \right|}{|t-s|^{\alpha'}} \right|_{L^q(\Omega; \mathbb{R})} + \left| \sup_{s \neq t} \frac{\left| \pi_2 \left( \Lambda_{s,t}^\eta - \Lambda_{s,t}^{\bar{\eta}} \right) \right|}{|t-s|^{2\alpha'}} \right|_{L^q(\Omega; \mathbb{R})} \leq c_{Lip} \varepsilon$$

In the above argument we assumed that  $q$  is large enough, but since  $L^p$  is Lipschitz continuously embedded in  $L^q$  for  $p > q$ , the result follows for all  $q$ .

Above arguments imply (i) and (ii). We now establish point (iii). Denote with  $(N, B)$  the usual Stratonovich lift of the  $(d+e)$ -dimensional, continuous semimartingale  $(N, B)$  carried on the probability space

$$(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}}) = (\bar{\Omega} \times \Omega, \bar{\mathcal{F}}_t \otimes \mathcal{F}_t, \bar{\mathcal{F}} \otimes \mathcal{F}, \bar{\mathbb{P}} \otimes \mathbb{P}).$$

We need to compare this lift for  $\bar{\mathbb{P}}-a.e.$   $\bar{\omega}$  with the process  $\Lambda^{N(\bar{\omega})}$  defined on  $(\Omega, \mathcal{F}_t, \mathcal{F}, \mathbb{P})$ . Note that we cannot use that  $\hat{\omega} = (\bar{\omega}, \omega) \mapsto \Lambda^{N(\bar{\omega})}(\omega)$  is a random variable on  $\hat{\Omega}$ , since the above argument does not imply  $\hat{\mathcal{F}}$ -measurability (i.e. joint measurability in  $(\bar{\omega}, \omega)$ ). However, for the components of the first level, this is easily seen to be true: we immediately get by the construction of  $\Lambda^N$  that  $(\bar{\omega}, \omega) \mapsto \pi_1 \left( \left( \Lambda^{N(\bar{\omega})} \right) (\omega) \right)$  is  $\hat{\mathcal{F}}$ -measurable and that it coincides with  $\pi_1((N, B))$   $\hat{\mathbb{P}}$ -a.s. It remains to consider the second level and we only discuss the case  $i \in \{1, \dots, d\}$  and  $j \in \{d+1, \dots, d+e\}$  (the other cases follow either immediately or by a similar argument). To avoid confusion about probability space on which the involved stochastic integrals are constructed, we use the notation  $d_{\hat{\mathbb{P}}}$  resp.  $d_{\mathbb{P}}$ . By definition of the Stratonovich lift,

$$(N, B)^{(2);i,j} = \int_0^\cdot N_r^i \circ d_{\hat{\mathbb{P}}} B_r^{j-d}, \hat{\mathbb{P}} - a.s.$$

and since by assumption the components of  $N$  are independent of  $B$ , above Itô-integral coincides with the Itô-version  $\int_0^\cdot N_r^i d_{\mathbb{P}} B_r^{j-d}$ . By standard results

$$\left( \int_0^t N_r^i d_{\hat{\mathbb{P}}} B_r^{j-d} \right) (\hat{\omega}) = \lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} N_{(k-1)/2^n t}^i(\bar{\omega}) \left[ B_{k/2^n t}^{j-d}(\omega) - B_{(k-1)/2^n t}^{j-d}(\omega) \right] \quad \forall t$$

holds for all  $\hat{\omega} = (\bar{\omega}, \omega)$  in some subset  $\hat{A} \subset \hat{\mathcal{F}}$  of full measure,  $\hat{\mathbb{P}}[\hat{A}] = 1$ . By a Fubini type theorem (e.g. [2, Theorem 3.4.1]), there exists a subset  $\bar{\Omega}^\circ \subset \bar{\Omega}$  of full

measure, such that for every  $\bar{\omega} \in \bar{\Omega}^\circ$  the projection  $\hat{A}_{\bar{\omega}} := \left\{ \omega \in \Omega : (\bar{\omega}, \omega) \in \hat{A} \right\}$  satisfies  $\mathbb{P} \left[ \hat{A}_{\bar{\omega}} \right] = 1$ . On the other hand for every fixed  $\bar{\omega} \in \bar{\Omega}$

$$\Lambda^{(2);i,j}(\mathbf{N}(\bar{\omega})) = \int_0^\cdot N_r^i(\bar{\omega}) d_{\mathbb{P}} B_r^{j-d} \quad \mathbb{P} - a.s.$$

and

$$\left( \int_0^t N_r^i(\bar{\omega}) d_{\mathbb{P}} B_r^{j-d} \right) (\omega) = \lim_{m \rightarrow \infty} \sum_{k=1}^{2^{n_m}} N_{(k-1)/2^{n_m}t}^i(\bar{\omega}) \left[ B_{k/2^{n_m}t}^{j-d}(\omega) - B_{(k-1)/2^{n_m}t}^{j-d}(\omega) \right]$$

for every  $\omega \in D_{\bar{\omega}}$  where  $D_{\bar{\omega}} \subset \Omega$  is a set of full measure,  $\mathbb{P}[D_{\bar{\omega}}] = 1$ . ( $D_{\bar{\omega}}$  as well as the subsequence  $(n_m)_m$  depends on  $\bar{\omega}$ ). So for  $\bar{\omega} \in \bar{\Omega}^\circ, \omega \in \hat{A}_{\bar{\omega}} \cap D_{\bar{\omega}}$  we have that

$$\begin{aligned} \left( \int_0^t N_r^i d_{\hat{\mathbb{P}}} B_r^{j-d} \right) (\bar{\omega}, \omega) &= \lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} N_{(k-1)/2^n t}^i(\bar{\omega}) \left[ B_{k/2^n t}^{j-d}(\omega) - B_{(k-1)/2^n t}^{j-d}(\omega) \right] \\ &= \lim_{m \rightarrow \infty} \sum_{k=1}^{2^{n_m}} N_{(k-1)/2^{n_m} t}^i(\bar{\omega}) \left[ B_{k/2^{n_m} t}^{j-d}(\omega) - B_{(k-1)/2^{n_m} t}^{j-d}(\omega) \right] \\ &= \left( \int_0^t N_r^i(\bar{\omega}) d_{\mathbb{P}} B_r^{j-d} \right) (\omega). \end{aligned}$$

The second equality holds since the sum converges along  $n$ , hence also along any subsequence  $(n_m)_m$ . Noting that for every  $\bar{\omega} \in \bar{\Omega}^\circ$  we have  $\mathbb{P} \left[ \hat{A}_{\bar{\omega}} \cap D_{\bar{\omega}} \right] = 1$  we can conclude that for  $\bar{\mathbb{P}}$ -a.e.  $\bar{\omega} \in \bar{\Omega}$

$$(2.5) \quad (\mathbf{N}, \mathbf{B}) = \Lambda^{(\bar{\omega})} \quad \mathbb{P}\text{-a.s.}$$

□

*Remark 4.* The null-set on which the equality (2.1) holds depends on  $\boldsymbol{\eta} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^{d+e})$  (and the version of the stochastic process  $B$ ), i.e. the map  $\boldsymbol{\eta} \mapsto \Lambda^{\boldsymbol{\eta}}$  can be quite “ugly” from a measure-theoretic point of view. However, Theorem 3 shows that after taking expectations (resp.  $L^q$  norms) this map is actually quite regular and we will see that this is sufficient for important applications (Section 3 and 4).

*Remark 5.* In the construction we use the fact that the bracket between  $N$  and  $B$  is 0. Especially, the consistency in Theorem 3 is only true for independent processes  $N$  and  $B$ .

*Remark 6.* Lyons [25] constructs a two-dimensional Gaussian process such that its marginals are Brownian motions and shows that for several different definitions of Itô and Stratonovich integrals (as limit of Riemann sums, Fourier series approach) the cross-integrals are only defined on a null-set. This does not contradict Theorem 3 due to the previous remark/the assumption of independence.<sup>5</sup>

<sup>5</sup>It is even not obvious if the process in [25] has a bracket.

*Remark 7.* The key observation for the proof of Theorem 3 is that by assuming an integration by parts formula holds, the cross integral can be implicitly defined. Especially, definition (2.3) still makes sense if we replace Stratonovich by Itô integration and one can run the above argument to arrive at a rough path lift  $\Lambda^{Itô, \eta}$  that is now a non-geometric rough path (to be specific, one only needs to slightly change the Fernique argument to account for the Itô–Stratonovich correction). The proof of consistency follows also as above. Unfortunately, for the application in non-linear filtering given in Section 4, this does not lead to better results regarding the regularity of the vector fields in the filtering problem.

### 3. ROUGH AND STOCHASTIC DIFFERENTIAL EQUATIONS (RSDEs)

Our goal is to give meaning to the differential equation

$$dS_t^\eta = a(S_t^\eta) dt + b(S_t^\eta) \circ dB_t + c(S_t^\eta) d\eta_t,$$

i.e. for a fixed rough path  $\eta$  we want to find a stochastic process  $S^\eta$  on the probability space which carries the Brownian motion  $B$ . Theorem 3 guarantees the existence of a canonical, random joint lift  $\Lambda$  of  $B$  and  $\eta$ , hence we can solve for every fixed rough path  $\eta \in \mathcal{C}_d^{0, \alpha}$  the random RDE

$$\begin{aligned} dS_t &= a(S_t) dt + (b, c)(S_t) d\Lambda_t \\ &= (a, b, c)(S_t) d(t, \Lambda_t). \end{aligned}$$

Theorem 10 shows that this is indeed the right solution in terms of consistent approximation results as well as continuity of the solution map; Theorem 11 shows consistency with usual SDE solution in the case that  $\eta$  is the rough path lift of another Brownian motion. Before we give the proofs let us introduce some standard notation.

**Definition 8.** Let  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  be a filtered probability space satisfying the usual condition. Denote with  $\mathcal{S}^0(\Omega)$  the space of adapted, continuous processes in  $\mathbb{R}^{d_S}$ , with the topology of uniform convergence in probability. For  $q \geq 1$  we denote with  $\mathcal{S}^q(\Omega)$  the space of processes  $X \in \mathcal{S}^0$  such that

$$|X|_{\mathcal{S}^q} := \left| |X|_{\infty; [0, t]} \right|_{L^q(\Omega; \mathbb{R})} = \left( \mathbb{E} \left[ \sup_{s \leq t} |X_s|^q \right] \right)^{1/q} < \infty.$$

#### 3.1. Existence and continuity of the solution map.

**Assumption 9.**  $a \in Lip^{1+\epsilon}$  for some  $\epsilon > 0$ ,  $\alpha \in (\frac{1}{3}, \frac{1}{2})$ ,  $\gamma > \frac{1}{\alpha}$  and  $b, c \in Lip^\gamma$ .

**Theorem 10.** Let  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  be a filtered probability space satisfying the usual conditions, carrying a  $e$ -dimensional Brownian motion  $B$  and a random variable  $S_0$  independent of  $B$ . Let  $(a, b, c)$  fulfill Assumption 9 for some  $\alpha \in (\frac{1}{3}, \frac{1}{2})$ . Then there exists a  $d_S$ -dimensional process  $S^\eta \in \mathcal{S}^0$  such that for every sequence  $(\eta^n)_n$ ,  $\eta^n \in C^1([0, T], \mathbb{R}^d)$

and such that  $(1 + \eta^n + \int \eta^n \otimes d\eta^n) \rightarrow_n \boldsymbol{\eta}$  in  $\rho_{\alpha-H\ddot{o}l}$ -metric for some  $\boldsymbol{\eta} \in \mathcal{C}^{0,\alpha}$ , the solutions  $(S^n)_n$  of the SDE

$$dS_t^n = a(S_t^n) dt + b(S_t^n) \circ dB_t + c(S_t^n) d\eta_t^n, \quad S^n(0) = S_0^n.$$

converge uniformly on compacts in probability to  $S^\eta$ ,

$$(3.1) \quad S^n \rightarrow_{n \rightarrow \infty} S^\eta \text{ in } \mathcal{S}^0$$

and the process  $S^\eta$  only depends on  $\boldsymbol{\eta}$  and the process  $B$  but not on the approximating sequence  $(\eta^n)_n$ . We say that  $S^\eta$  is the solution of the RSDE

$$S_t^\eta = S_0 + \int_0^t a(S_r^\eta) dr + \int_0^t b(S_r^\eta) \circ dB_r + \int_0^t c(S_r^\eta) d\boldsymbol{\eta}_r.$$

Moreover,

$$(1) \quad \forall q \geq 1, S^\eta \in \mathcal{S}^q(\Omega), \text{ the map} \\ (3.2) \quad (\mathcal{C}^{0,\alpha}, \rho_{\alpha-H\ddot{o}l}) \rightarrow (\mathcal{S}^q(\Omega), |\cdot|_{\mathcal{S}^q}), \boldsymbol{\eta} \mapsto S^\eta$$

is locally Lipschitz continuous,

- (2) If  $S_0$  has Gaussian tails then  $S$  also has Gaussian tails, locally uniform in  $\boldsymbol{\eta}$ :  
 $\forall r > 0 \exists \delta = \delta(\alpha', \alpha, T, r) > 0$  such that

$$\sup_{\|\boldsymbol{\eta}\|_{\alpha-H\ddot{o}l} \leq r} \mathbb{E} \left[ \exp \left( \delta \|S^\eta\|_{\infty; [0, T]}^2 \right) \right] < \infty.$$

*Proof.* Choose  $\alpha' < \alpha$  large enough, such that  $\gamma > 1/\alpha'$  and apply standard existence and uniqueness results<sup>6</sup> to get a solution  $S^\eta$  of the RDE

$$(3.3) \quad S_t^\eta = S_0 + \int_0^t a(S_r^\eta) dr + \int_0^t (b, c)(S_r^\eta) d\boldsymbol{\Lambda}_r^\eta.$$

(and denote with  $S^\eta$  the full RDE solution).

**Point (1) (and (3.1)).** Let  $\alpha' < \alpha$  with  $\gamma > p' := \frac{1}{\alpha'}$ . By Theorem 4 in [1] we have (see Section 5 for the definition of  $N_1(\|\boldsymbol{S}^\eta\|_{p'-var}^{p'}; [0, T])$ )

$$\|S^\eta - S^{\bar{\eta}}\|_\infty \leq C \rho_{\alpha'-H\ddot{o}l}(S^\eta, S^{\bar{\eta}}) \exp \left[ c \left( N_1 \left( \|\boldsymbol{S}^\eta\|_{p'-var}^{p'}; [0, T] \right) + N_1 \left( \|\boldsymbol{S}^{\bar{\eta}}\|_{p'-var}^{p'}; [0, T] \right) + 1 \right) \right].$$

Hence, using Theorem 3

$$\begin{aligned} \|S^\eta - S^{\bar{\eta}}\|_{L^q} &\leq C |\rho_{\alpha'-H\ddot{o}l}(S^\eta, S^{\bar{\eta}})|_{L^{2q}} \left| \exp \left[ c \left( N_1 \left( \|\boldsymbol{S}^\eta\|_{p'-var}^{p'}; [0, T] \right) + N_1 \left( \|\boldsymbol{S}^{\bar{\eta}}\|_{p'-var}^{p'}; [0, T] \right) + 1 \right) \right] \right| \\ &\leq C \rho_{\alpha-H\ddot{o}l}(\boldsymbol{\eta}, \bar{\boldsymbol{\eta}}) \left| \exp \left[ c \left( N_1 \left( \|\boldsymbol{S}^\eta\|_{p'-var}^{p'}; [0, T] \right) + N_1 \left( \|\boldsymbol{S}^{\bar{\eta}}\|_{p'-var}^{p'}; [0, T] \right) + 1 \right) \right] \right|_{L^{2q}}. \end{aligned}$$

The last  $L^{2q}$ -norm is bounded locally uniformly in  $\boldsymbol{\eta}, \bar{\boldsymbol{\eta}}$  by Corollary 25. This yields the desired local Lipschitzness of the solution map (3.2). Now apply this continuity

<sup>6</sup>We only have  $a \in Lip^{1+\epsilon}$  so we have to use results on RDEs with drift (e.g. [18, Theorem 12.6 and Theorem 12.10]) to get existence of a unique solution.

with the fact that if  $\eta$  is the lift of a smooth path  $\eta$ , then  $S^\eta$  is the standard SDE solution of the SDE

$$dS = a(S_r) dr + b(S_r) dB_r + c(S_r) d\eta_r.$$

(e.g. [18, Section 17.5]).

**Point (2).** This follows from Corollary 25 in combination with the pathwise estimates<sup>7</sup>

$$\begin{aligned} |S^\eta|_\infty &\leq |S_0^\eta| + |S^\eta|_0 \leq |S_0^\eta| + C |S^\eta|_{p'-\text{var}} \\ &\leq |S_0^\eta| + C (N_1(\Lambda^\eta; [0, T]) + 1), \end{aligned}$$

for some constant  $C$ . The last estimate follows from Lemma 4 and Corollary 3 in [16].  $\square$

### 3.2. Consistency with SDE solutions.

**Theorem 11.** Let  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ ,  $B, S_0$  and  $a, b, c$  be as in Theorem 10. Let  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}})$  be another probability space satisfying the usual conditions and carrying an  $\bar{e}$ -dimensional Brownian motion  $\bar{B}$  and denote

$$(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}}) = (\Omega \times \bar{\Omega}, \mathcal{F} \otimes \bar{\mathcal{F}}_t, \mathcal{F}_t \otimes \bar{\mathcal{F}}_t, \mathbb{P} \otimes \bar{\mathbb{P}}).$$

Let  $\hat{S}$  be the unique solution on  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}})$  of the SDE

$$(3.4) \quad \hat{S}_t = \hat{S}_0 + \int_0^t a(\hat{S}_r) dr + \int_0^t b(\hat{S}_r) \circ dB_r + \int_0^t c(\hat{S}_r) \circ d\bar{B}_r.$$

Denote with  $\bar{B}$  the Stratonovich lift of the Brownian motion  $\bar{B}$  on  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}})$ . Then for  $\bar{\mathbb{P}}$ -a.e.  $\bar{\omega} \in \bar{\Omega}$

$$(3.5) \quad \mathbb{P} \left[ \hat{S}_t(\bar{\omega}, \cdot) = S_t^{\bar{B}(\bar{\omega})}(\cdot), t \in [0, T] \right] = 1.$$

*Proof.* By Theorem 3, we know that for  $\bar{\mathbb{P}}$ -a.e.  $\bar{\omega} \in \bar{\Omega}$  we have

$$(3.6) \quad (\bar{B}, B) = \Lambda^{\bar{B}(\bar{\omega})} \text{ } \mathbb{P}\text{-a.s.}$$

Standard results in rough path theory (cf. [18, Section 17.5]), guarantee that the RDE solution to (3.3) driven by  $(\bar{B}, B)$  coincides  $\hat{\mathbb{P}}$ -a.s. with the SDE solution of (3.4). Combining this with (3.6) implies immediately (3.5).  $\square$

## 4. APPLICATIONS

In this section we show that RSDEs, as introduced in Section 3, appear naturally in robustness questions of two important applications: nonlinear filtering and stochastic/rough PDEs.

<sup>7</sup>We use the same notation as in [18],  $|S|_0 \equiv \sup_{s \neq t} |S_t - S_s|$

**4.1. Robustness in Nonlinear Filtering.** Nonlinear filtering is concerned with the estimation of a Markov process based on some observation of it; e.g. consider the classic case of a Markov process  $(X, Y)$  that takes values in  $\mathbb{R}^{d_X+d_Y}$  of the form

$$(4.1) \quad \begin{cases} dX_t = l_0(X_t, Y_t) dt + \sum_k Z_k(X_t, Y_t) dB_t^k + \sum_j L_j(X_t, Y_t) d\tilde{B}_t^j & \text{(signal)} \\ dY_t = h(X_t, Y_t) dt + d\tilde{B}_t & \text{(observation)} \end{cases}$$

with  $B$  and  $\tilde{B}$  independent, multidimensional Brownian motions. The goal is to compute for a given real-valued function  $\varphi$

$$\pi_t(\varphi) = \mathbb{E}[\varphi(X_t) | \sigma(Y_r, r \in [0, t])].$$

From basic measure theory it follows that there exists a measurable map

$$(4.2) \quad \theta_t^\varphi : C([0, T], \mathbb{R}^{d_Y}) \rightarrow \mathbb{R} \text{ such that } \theta_t^\varphi(Y|_{[0, t]}) = \pi_t(\varphi) \quad \mathbb{P} - a.s$$

In the late seventies Clark pointed out that this formulation is not sufficient from a practical point of view: it would be natural to demand that  $\theta_t^\varphi(\cdot)$  is continuous<sup>8</sup>. Clark showed that in the uncorrelated noise case (i.e.  $L \equiv 0$  in 4.1) there exists a unique

$$\bar{\theta}_t^\varphi : C([0, T], \mathbb{R}^{d_Y}) \rightarrow \mathbb{R}$$

which is continuous in uniform norm and fulfills (4.2), thus providing a “robust version” of the conditional expectation  $\pi_t(\varphi)$ . Unfortunately, in the correlated noise case this is no longer true (it is easy to construct counterexamples; see [6, Example 1]). Recently, it was shown in [6] that also in this situation robustness prevails, however only in a rough path sense, i.e. there exists a map

$$\bar{\theta}_t^\varphi : \mathcal{C}^{0, \alpha}(\mathbb{R}^{d_Y}) \rightarrow \mathbb{R} \text{ such that } \bar{\theta}_t^\varphi(\mathbf{Y}|_{[0, t]}) = \pi_t(\varphi) \quad \mathbb{P} - a.s$$

here  $\mathbf{Y}$  is the canonical rough path lift of the semimartingale  $Y$ . The argument in [6] relies on an observation of Mark Davis [7], namely that under an appropriate change of measure the observation  $Y$  is a Brownian motion independent of  $B$ , the signal satisfies the SDE

$$(4.3) \quad dX_t = \bar{l}_0(X_t, Y_t) dt + \sum_k Z_k(X_t, Y_t) dY_t^k + \sum_j L_j(X_t, Y_t) d\bar{B}_t^j$$

where  $\bar{l}_0 = l_0 + \sum_k Z_k h_k$  and that the robustness question is linked to the (rough pathwise) robustness of  $Y \mapsto X$ . To treat the resulting differential equation driven by Brownian noise  $B$  and a rough path (instead of  $Y$ ) a flow decomposition is used in [6]. We can now replace this argument by Theorem 10 and Theorem 11 which leads to different regularity assumptions on the vector fields.

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<sup>8</sup>The functional  $\phi_t^\varphi$  is only uniquely defined up to null-sets on pathspace but only discrete observations of  $Y$  are available. Moreover, the model chosen for the observation process might only be close in law to the “real-world” observation process.

**Theorem 12.** Let  $\varphi \in Lip^1$  and let  $\gamma > \frac{1}{\alpha}$  for some  $\alpha \in (\frac{1}{3}, \frac{1}{2})$  and

$$l_0 \in Lip^{1+\epsilon}, \quad h, Z, L \in Lip^\gamma$$

for some  $\epsilon > 0$ . Denote with  $(X, Y)$  be the solution of (4.1). Then there exists a continuous map

$$\theta : (\mathcal{C}^{0,\alpha}, \rho_{\alpha-H\ddot{o}l}) \rightarrow (\mathbb{R}, |\cdot|)$$

such that

$$\theta(\mathbf{Y}) = \mathbb{E}[\varphi(X_t) | \sigma(Y_r, r \in [0, t])] \quad \mathbb{P} - a.s.$$

where  $\mathbf{Y}$  denotes the Stratonovich lift of the semimartingale  $Y$  to a geometric rough path.

*Proof.* To switch the equation (4.3) to the Stratonovich formulation define

$$\begin{aligned} L_0^j(x, y) &= \bar{l}_0^j - \frac{1}{2} \sum_k \sum_i \partial_{x^i} Z_k^j(x, y) Z_k^i(x, y) - \frac{1}{2} \sum_k \partial_{y^k} Z_k^j(x, y) \\ &\quad - \frac{1}{2} \sum_k \sum_i \partial_{x^i} L_k^j(x, y) L_k^i(x, y) - \frac{1}{2} \sum_k \partial_{y^k} L_k^j(x, y). \end{aligned}$$

By Theorem 10,

$$\begin{aligned} dX_t^\eta &= L_0(X_t^\eta, Y_t^\eta) dt + Z(X_t^\eta, Y_t^\eta) d\eta_t + \sum_j L_j(X_t^\eta, Y_t^\eta) \circ d\bar{B}_t^j \\ dY_t^\eta &= d\eta_t \\ dI_t^\eta &= h(X_t^\eta, Y_t^\eta) d\eta_t - \frac{1}{2} D_k h(X_t^\eta, Y_t^\eta) dt \end{aligned}$$

has unique solution  $(X_t^\eta, Y_t^\eta, I_t^\eta) \in \mathcal{S}^2$  and following the proof of [6, Theorem 6] shows continuity of  $\theta$  (for these steps it is important to have  $\mathbb{E}[\exp(qI_t^\eta)] < \infty$  for  $q \geq 2$  as guaranteed by Theorem 10). Similarly, we can follow step-by-step [6, Theorem 7] to show the consistency  $\theta(\mathbf{Y}) = \pi_t(\varphi) \quad \mathbb{P} - a.s.$   $\square$

*Remark 13.* The regularity assumption in [6] is  $h, Z \in Lip^{4+\epsilon}, L \in Lip^1$ , i.e. above approach allows to relax the regularity of the sensor function  $h$  and  $Z$  by two degrees of regularity for the price of an additional degree of regularity of  $L = (L_i)_{i=1}^d$ .

**4.2. Feynman–Kac representation for linear RPDEs.** Over the last years there has been an increased interest in giving a (rough) pathwise meaning to stochastic partial differential equations and several approaches have emerged, see for example Gubinelli et al. [8, 20], Hairer et al. [21, 22] and Teichmann [31]. The approach we focus on in this section is related to the work of Lions and Souganidis [24] and Friz et al. [3, 4, 15, 9]. In a setting similar to the one in [15] we are able, using rough SDEs, to prove existence and uniqueness of solutions under weaker assumptions on the coefficients and give a stochastic representation for the solution.

**Definition 14.** Let  $\eta \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$  be a geometric rough path for some  $\alpha \in (0, 1]$  and  $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^{m'}$ ,  $a : \mathbb{R}^m \rightarrow \mathbb{R}^m$ ,  $G_i : \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$  and  $\phi : \mathbb{R}^m \rightarrow \mathbb{R}$  be such that

for every  $\eta \in C^1([0, T], \mathbb{R}^d)$  there exists a unique bounded, uniformly continuous, viscosity solution  $v^\eta : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}$  to

$$\begin{cases} -dv^\eta - L(x, v^\eta, Dv^\eta, D^2v^\eta) dt - \sum_{i=1}^d G_i(x, v^\eta, Dv^\eta) \dot{\eta}_t^i = 0, \\ v^\eta(T, x) = \phi(x), \end{cases}$$

where

$$L : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^m \rightarrow \mathbb{R} \text{ is given as } L(x, p, M) := \text{Tr}[\sigma(x) \sigma^T(x) M] + a(x) \cdot p.$$

We then say that a bounded, uniformly continuous function  $v : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}$  is a solution of the rough partial differential equation (RPDE)

$$\begin{cases} -dv - L(x, v, Dv, D^2v) dt - c(x) \cdot Dv d\boldsymbol{\eta}_t = 0, \\ v(T, x) = \phi(x), \end{cases}$$

(with  $c = (G_1, \dots, G_d)$ ), if for every sequence of smooth paths  $(\eta^n)_n \subset C^1([0, T], \mathbb{R}^d)$  such that  $\eta^n \rightarrow \boldsymbol{\eta}$  as  $n \rightarrow \infty$  in rough path metric we have in locally uniform convergence

$$v^{\eta^n} \rightarrow v \text{ as } n \rightarrow \infty.$$

*Remark 15.* Of course above definition is only of use if one can show the existence of a solution for an interesting family of  $(L, c, \phi)$  in the above sense (uniqueness is built into the definition by the uniqueness of the approximating solutions). This is still an area of active research but for example if  $c$  is affine linear there exists a solution in above sense (see [23, 4, 15]). The theorem below shows not only the existence of such a solution by a short proof relying on RSDEs as introduced in Section 3 but gives additionally a Feynman–Kac representation. This finally leads to lower regularity assumptions on the noise vector fields (however, in contrast to [23, 4, 15] it only applies to linear operators  $L$ ).

**Theorem 16.** *Let  $\boldsymbol{\eta} \in \mathcal{C}^{0,\alpha}$  be a geometric rough path,  $\alpha \in (0, 1]$ . Assume  $\gamma > \frac{1}{\alpha}$ ,  $\sigma, c \in \text{Lip}^\gamma$ ,  $\zeta > 1$ ,  $a \in \text{Lip}^\zeta$ . Assume  $\phi$  is bounded and uniformly continuous. Then, there exists a unique solution to the RPDE<sup>9</sup>*

$$\begin{cases} -dv - L(x, v, Dv, D^2v) dt - c(x) \cdot Dv d\boldsymbol{\eta}_t = 0, \\ v(T, x) = \phi(x), \end{cases}$$

Moreover  $v(t, x) = \mathbb{E}[\phi(S_T^{t,x})]$  where  $S^{s,x}$  denotes the solution of the RSDE

$$(4.4) \quad \begin{cases} dS_t^{s,x} = \bar{a}(S_t^{s,x}) dt + \sigma(S_t^{s,x}) \circ dB_t + c(S_t^{s,x}) d\boldsymbol{\eta}_t, \\ S_s^{s,x} = x. \end{cases}$$

where  $(\sigma^i$  denotes the  $i$ th column of  $\sigma$ )

$$\bar{a} = a - \frac{1}{2} \sum_{i=1}^d D_{\sigma^i} \sigma^i.$$

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<sup>9</sup>We use  $-$  signs to emphasize that we treat a backward equation.

*Proof.* Let  $(\eta^n)_n$  be a sequence of smooth paths converging to  $\eta$  in rough path topology. For every fixed  $n$  we have the Feynman–Kac representation (see e.g. [30, Theorem 4.13])

$$v^n(t, x) = \mathbb{E} [\phi(S_T^{n,t,x})],$$

where  $v^n$  is the unique, bounded viscosity solution to

$$\begin{aligned} -dv^n - L(x, v^n, Dv^n, D^2v^n) dt - c(x) \cdot Dv^n d\eta_t^n &= 0, \\ v^n(T, x) &= \phi(x), \end{aligned}$$

and  $S^{n,s,x}$  solves the SDE

$$\begin{cases} dS_t^{n,s,x} = \bar{a}(S_t^{n,s,x}) dt + \sigma(S_t^{n,s,x}) \circ dB_t + c(S_t^{n,s,x}) d\eta_t^n, \\ S_s^{n,s,x} = x. \end{cases}$$

Theorem 11 now gives the pointwise convergence

$$v^n(t, x) = \mathbb{E} [\phi(S_T^{n,t,x})] \rightarrow_{n \rightarrow \infty} \mathbb{E} [\phi(S_T^{t,x})] =: v(t, x).$$

To get locally uniform convergence, it suffices to show local equicontinuity of  $(v^n)_n$  (by the Arzelà–Ascoli theorem). By the same arguments as in Theorem 6 one sees that a rough SDE is also locally uniformly continuous in the initial condition  $S_0$ , uniformly over  $\eta$  in bounded sets. Moreover, it is straightforward to show, that for every  $q \geq 1$

$$\mathbb{E} [\|\Lambda^\eta\|_{p-var;[0,t]}^q] = o(1) \text{ as } t \rightarrow 0,$$

locally uniformly for  $\eta$ . Putting the above together yields the local equicontinuity of the  $(v^n)_n$ .  $\square$

*Remark 17.* Theorem 16 can be easily extended to cover equations of the type

$$\begin{aligned} -dv - L(t, x, v, Dv, D^2v) dt - c(t, x, u, Du) d\eta_t &= 0, \\ v(T, x) &= \phi(x), \end{aligned}$$

where  $c$  is affine linear in  $(u, Du)$  (as in [15]). For brevity we only treat the gradient case.

*Remark 18.* In Theorem 16 we only assume  $c \in Lip^\gamma$  in contrast to  $Lip^{\gamma+2}$  as in [4, 15] where a flow decomposition is used.

## 5. INTEGRABILITY ESTIMATES FOR GAUSSIAN ROUGH DIFFERENTIAL EQUATIONS REVISITED

A classic result of X. Fernique [10] shows that Gaussian probability measures on separable Banach spaces have Gaussian tails in the Banach norm. If one considers as Banach space an abstract Wiener space, this immediately implies Gauss tails of norms of Gaussian processes which is of uttermost importance for many applications in stochastic analysis. In rough path norms, iterated stochastic integrals additionally appear and Fernique’s theorem is no longer directly applicable. Another issue is that

the genuine rough-*pathwise* estimates<sup>10</sup> for solutions of RDEs driven by Gaussian processes do not “see” probabilistic cancellations, hence do not lead to useful probabilistic estimates (e.g.  $L^q(\Omega)$  estimates) for solutions of such RDEs.

In [14, Theorem 2] the Borell–Sudakov–Tsirelson inequality — an analogue of the Gaussian isoperimetric inequality which holds in infinite dimensional spaces — was used to prove a generalization of Fernique’s theorem. This implies for example that  $\|\mathbf{B}\|_{p-var}$  has Gauss tails for  $p > 2$  (see also our proof of Theorem 3) but combined only with pathwise estimates for RDE solutions this is not even sufficient to derive moment estimates for RDE solutions driven by Brownian motion (see footnote 10; in Itô’s stochastic calculus this is of course easy to establish). A key insight was recently made in [5] by introducing “*greedy partitions*” which allow to capture the needed probabilistic cancellations. The main result in [5] can then be seen as the verification that a certain random measure  $N$  (which is related to the norm of a Gaussian rough path along such greedy partitions, Definition 20), has exponential tails on compact sets (or even Gaussian tails in the case of Brownian motion). The proof also uses the Borell–Sudakov–Tsirelson inequality. In this section, using the isoperimetric inequality in a slightly different spirit, we give another proof of the main result in [5]. Our proof, based on a generalization of [14, Theorem 2] and the greedy partitions of [5], is surprisingly short and, as we hope, may be somewhat more instructive.

**5.1. Revisiting the generalized Fernique theorem.** We first present a generalization of [14, Theorem 2] which can be stated in a fairly general framework. Let  $E$  be a real, locally convex Hausdorff space. A measure  $\gamma$  on the Borel sets of  $E$  is called a (centered) *Gauß measure* if the push forward measure under each element of the topological dual of  $E$  is a (centered) normal random variable in  $\mathbb{R}$ . The corresponding Cameron–Martin space will be denoted by  $\mathcal{H}$ . The triplet  $(E, \mathcal{H}, \gamma)$  will be called a *Gaussian space*.  $\gamma$  is called a *Radon probability measure* on the Borel sets of  $E$  if  $\gamma(B) = \gamma_*(B)$  for every Borel set  $B$  where, for any subset  $A \subset E$ ,

$$\gamma_*(A) := \sup \{ \gamma(K) : K \text{ compact and } K \subseteq A \}.$$

**Theorem 19.** *Let  $(E, \mathcal{H}, \gamma)$  be a Gaussian space with  $\gamma$  being centered and a Radon measure<sup>11</sup>. Let  $f, g: E \rightarrow \mathbb{R} \cup \{+\infty, -\infty\}$  be measurable functions. Assume that there is a null-set  $N$  such that for every  $x$  outside  $N$  we have*

$$|f(x)| \leq |g(x - h)| + \sigma |h|_{\mathcal{H}}$$

*for every  $h \in \mathcal{H}$ . Assume further that there is an  $r_0 \geq 0$  such that*

$$\gamma \left\{ x \in E : |g(x)| \leq \frac{r_0}{2} \right\} =: a > 0.$$

<sup>10</sup>The solution  $dy = V(y) d\mathbf{x}$  is estimated  $|y_t| \leq c \cdot \exp \left( c \|\mathbf{x}\|_{p-var}^p \right)$  and this is known to be rough-*pathwise* optimal, see [13]. Applied with  $\mathbf{x} = \mathbf{B}$  and  $p > 2$  and the Gaussian tail property of  $\|\mathbf{B}\|_{p-var}$  this does not even imply the integrability of the RDE solution.

<sup>11</sup>Note that probability measures on the Borel sets of Polish spaces are Radon measures, thus Gaussian measures on separable Fréchet spaces (and therefore on Banach spaces) are always Radon measures.

Then

$$\gamma \{x \in E : |f(x)| > r\} \leq 1 - \Phi\left(\alpha + \frac{r}{2\sigma}\right)$$

for every  $r \geq r_0$  where  $\Phi$  denotes the cumulative distribution function of a standard normal random variable and  $\alpha \in \mathbb{R}$  is chosen such that  $\Phi(\alpha) \leq a$ .

*Proof.* Inspection of the proof in [14] shows that the very same argument holds when  $f(x-h)$  is replaced by  $g(x-h)$ .  $\square$

**5.2. Greedy partitions.** Recall the following definition from [5].

**Definition 20.** Let  $\omega: \{u, v \in [0, T]^2 : u \leq v\} \rightarrow \mathbb{R}_+$  be a control function [18]. Let  $[s, t] \subseteq [0, T]$  and choose  $\beta > 0$ . Define  $\{\tau_0 \leq \tau_1 \leq \dots\}$  as

$$\begin{aligned} \tau_0 &= s \\ \tau_{i+1} &= \inf \{u : \omega(\tau_i, u) \geq \beta, \tau_i < u \leq t\} \wedge t. \end{aligned}$$

Then we set  $N_\beta(\omega; [s, t]) := \sup \{n \in \mathbb{N}_0 : \tau_n < t\}$ . If  $\mathbf{x}: [0, T] \rightarrow G^{[p]}(\mathbb{R}^d)$  is a weakly geometric  $p$ -rough path and  $\|\cdot\|_{p\text{-var}}$  denotes the homogeneous  $p$ -variation norm induced by the Carnot–Caratheodory norm (cf. [18, Chapter 8]), we set  $N_\beta(\mathbf{x}; [s, t]) := N_\beta(\|\mathbf{x}\|_{p\text{-var}}^p; [s, t])$ .

**Lemma 21.** Let  $\mathbf{x}$  be a weakly geometric  $p$ -rough path and  $h$  be a path of bounded  $q$ -variation where  $1 \leq q \leq p$  and  $\frac{1}{p} + \frac{1}{q} > 1$ . Then there is an  $\beta = \beta(p, q)$  such that<sup>12</sup>

$$N_\beta(T_h(\mathbf{x}); [0, T]) \leq \|\mathbf{x}\|_{p\text{-var}}^p + \|h\|_{q\text{-var}}^q.$$

*Proof.* We have

$$\begin{aligned} N_\beta(\|T_h(\mathbf{x})\|_{p\text{-var}}^p; [0, T]) &\leq N_\beta(C_{p,q}(\|\mathbf{x}\|_{p\text{-var}}^p + \|h\|_{q\text{-var}}^p); [0, T]) \\ &= N_1(\|\mathbf{x}\|_{p\text{-var}}^p + \|h\|_{q\text{-var}}^p; [0, T]) \end{aligned}$$

with the choice  $\beta = C_{p,q}$ , using [18, Theorem 9.33]. By definition,

$$N_1(\|\mathbf{x}\|_{p\text{-var}}^p + \|h\|_{q\text{-var}}^p; [0, T]) \leq \sum_{\tau_i} \|\mathbf{x}\|_{p\text{-var}; [\tau_i, \tau_{i+1}]}^p + \|h\|_{q\text{-var}; [\tau_i, \tau_{i+1}]}^p$$

where  $(\tau_i)$  is a finite partition of  $[0, T]$  for which  $\|\mathbf{x}\|_{p\text{-var}; [\tau_i, \tau_{i+1}]}^p + \|h\|_{q\text{-var}; [\tau_i, \tau_{i+1}]}^p \leq 1$  for every  $\tau_i$ , and in particular  $\|h\|_{q\text{-var}; [\tau_i, \tau_{i+1}]}^p \leq \|h\|_{q\text{-var}; [\tau_i, \tau_{i+1}]}^q$ . Hence

$$\begin{aligned} N_1(\|\mathbf{x}\|_{p\text{-var}}^p + \|h\|_{q\text{-var}}^p; [0, T]) &\leq \sum_{\tau_i} \|\mathbf{x}\|_{p\text{-var}; [\tau_i, \tau_{i+1}]}^p + \|h\|_{q\text{-var}; [\tau_i, \tau_{i+1}]}^q \\ &\leq \|\mathbf{x}\|_{p\text{-var}; [0, T]}^p + \|h\|_{q\text{-var}; [0, T]}^q. \end{aligned}$$

$\square$

<sup>12</sup> $T_h$  denotes the usual translation operator, see [18, Chapter 9].

**5.3. Integrability estimates for rough path valued random variables.** Combining the above leads to a simple and easy proof of integrability estimates for Gaussian rough path norms.

**Theorem 22** (Integrability of rough path valued random variables). *Let  $(\Omega, \mathcal{H}, \gamma)$  be a centered Gaussian space with  $\Omega = C_0([0, T], \mathbb{R}^d)$ . Assume that there is a measurable map  $F: \Omega \rightarrow C^p$  to the space of geometric  $p$ -rough paths. Furthermore, assume that there is an embedding*

$$(5.1) \quad \iota: \mathcal{H} \hookrightarrow C^{q-\text{var}}$$

*with  $1 \leq q \leq p$  and  $\frac{1}{p} + \frac{1}{q} > 1$  and that the set*

$$(5.2) \quad \{\omega : T_h(F(\omega)) = F(\omega + h) \text{ for all } h \in \mathcal{H}\} =: \tilde{\Omega}$$

*has full measure. Then for all  $\beta > 0$ ,  $N_\beta(F; [0, T])^{\frac{1}{q}}$  has Gaussian tails. More specific, if*

$$\mathbb{P}[\|F\|_{p-\text{var}} \leq K] \geq a > 0$$

*and if  $M$  is a bound on  $\|\iota\|_{\mathcal{H} \hookrightarrow C^{q-\text{var}}}$ , there is a  $\delta = \delta(p, q, K, a, M, \beta) > 0$  such that*

$$\mathbb{E} \left[ \exp \left( \delta N_\beta(F; [0, T])^{\frac{2}{q}} \right) \right] < \frac{1}{\delta}.$$

*Proof.* Lemma 21 implies that there is a  $\beta_0$  such that

$$N_{\beta_0}(F(\omega); [0, T]) \leq \|F(\omega - h)\|_{p-\text{var}}^p + \|\iota\|_{\mathcal{H} \hookrightarrow C^q}^q |h|_{\mathcal{H}}^q$$

holds on the set  $\tilde{\Omega}$  for every  $h \in \mathcal{H}$ . Thus we may apply Theorem 19 to conclude the assertion for  $\beta_0$ . By Lemma 3 in [16],  $N_\beta$  and  $N_{\beta'}$  are comparable for all  $\beta, \beta' > 0$ . We hence get the stated result for all  $\beta > 0$ .  $\square$

If the covariance of a Gaussian process has finite  $\rho$ -variation for some  $\rho < 2$ , it can be lifted in the sense of Friz–Victoir, cf. [11]. Finite  $\rho$ -variation of the covariance also implies the embedding (5.1) with  $q = \rho$ , cf. [11, Proposition 17], which means that 5.1 is fulfilled whenever  $\rho < 3/2$ . A slightly stronger condition, so called mixed  $(1, \rho)$ -variation, was seen to imply an even sharper embedding with  $q = \frac{2}{\rho^{-1}+1}$ , cf. [12], thus condition 5.1 holds for all  $\rho < 2$ . Choosing  $q$  according to one of these embeddings, we obtain

**Corollary 23** (Integrability of Gaussian rough paths). *Let  $(\Omega, \mathcal{H}, \gamma)$  be a centered Gaussian space with  $\Omega = C_0([0, T], \mathbb{R}^d)$  and let  $X: \Omega \rightarrow C_0([0, T], \mathbb{R}^d)$  denote the coordinate process. Assume that all components of  $X$  are independent and that the 2-dimensional  $\rho$ -variation of the covariance function  $R$  of  $X$  is finite for some  $\rho < 2$ . Let  $\mathbf{X}$  denote the lift of  $X$  in the sense of Friz–Victoir. Then  $N_\beta(\mathbf{X}; [0, T])^{1/q}$  has Gaussian tails for every  $\beta > 0$ .*

*Proof.* By construction of the lift,  $\mathbf{X}$  takes values in  $C^{0,p}$  almost surely, and (5.2) holds by [18, Proposition 15.58]. We thus conclude with Theorem 22.  $\square$

**5.4. Application: Integrability of RSDE solutions.** We now apply these general results to Rough and Stochastic differential equations (RSDEs) as introduced in Section 3. First we need a Lemma.

**Lemma 24.** ,

i) For the joint lift  $\Lambda^\eta$  from Theorem 3 we have

$$\mathbb{P} [\Lambda^\eta (\omega + h) = T_h \Lambda^\eta (\omega) \ \forall h \in \mathcal{H} (\mathbb{R}^e)] = 1$$

ii) For all  $r > 0$  and  $p > \frac{1}{\alpha}$  there is a  $k$  such that

$$\inf_{\|\eta\|_{\alpha-H\ddot{o}l} < r} \mathbb{P} [\|\Lambda^\eta\|_{p-var} \leq k] \geq \frac{1}{2}.$$

*Proof.* i) Let  $D = \{0 = t_0 < \dots < t_m = T\}$  be any partition of  $[0, T]$ ,  $|D|$  denotes its mesh size. Let  $B^D$  be the piecewise linear approximation of  $B$  on the partition  $D$ . An easy calculation shows that

$$\int \eta_{0,r} dB_r^D = \sum_i \eta_{0,\bar{t}_i} B_{t_i, t_{i+1}}$$

for some deterministic  $\bar{t}_i \in [t_i, t_{i+1}]$  and where the integral on the left hand side is defined as Riemann–Stieltjes integral. As a consequence,

$$\begin{aligned} \left| \int \eta_{0,r} dB_r - \int \eta_{0,r} dB_r^D \right|_{L^2} &\leq \left| \int \eta_{0,r} dB_r - \sum_i \eta_{0,t_i} B_{t_i, t_{i+1}} \right|_{L^2} \\ &\quad + \left| \sum_i \eta_{0,t_i} B_{t_i, t_{i+1}} - \sum_i \eta_{0,\bar{t}_i} B_{t_i, t_{i+1}} \right|_{L^2}. \end{aligned}$$

Now the first term converges to zero, as  $|D| \rightarrow 0$ , by definition of the Itô integral as limit of left-point Riemann sums. Using the fact that  $\eta$  is deterministic, we dominated the second term by

$$\left( \sum_i |\eta_{t_i} - \eta_{\bar{t}_i}| |t_{i+1} - t_i| \right)^{1/2},$$

which converges to 0 as  $|D| \rightarrow 0$ , by continuity of  $\eta$ .

Using this characterization of the Itô integral as the limit of smooth integrals, we can now finish the proof using exactly the same argument as in [18, Proposition 15.58].

ii) If  $\|\eta\|_{\alpha-H\ddot{o}l} < r$ , by Markov's inequality and Theorem 3,

$$\mathbb{P} [\|\Lambda^\eta\|_{p-var} \leq k] \geq 1 - \frac{C}{k}$$

where  $C$  is a constant depending on  $r$ . Choosing  $k$  large enough gives the result.  $\square$

**Corollary 25** (Integrability of joint lift). *Let  $\Lambda^\eta$  be the joint lift from Theorem 3 with sample paths in a  $p$ -rough paths space with  $p > \frac{1}{\alpha}$ . Then  $N_\beta (\Lambda^\eta; [0, T])$  has Gaussian*

tails for every  $\beta > 0$ . More specific, for every  $r > 0$  there is a  $\delta = \delta(p, \alpha, \beta, r) > 0$  such that

$$\sup_{\eta : \|\eta\|_{\alpha-H\ddot{o}l} \leq r} \mathbb{E} \left[ \exp \left( \delta N_\beta (\Lambda^\eta; [0, T])^2 \right) \right] \leq \frac{1}{\delta}.$$

*Proof.* For the Brownian motion, (5.1) holds with  $q = 1$  and  $\|\iota\|_{\mathcal{H} \hookrightarrow C^{1-\text{var}}} \leq \sqrt{T}$ , cf. [11, Proposition 17]. The assertion follows from Theorem 22 and Lemma 24.

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